EXTREMAL RAYS AND AUTOMORPHISMS OF HOLOMORPHIC SYMPLECTIC VARIETIES

BRENDAN HASSETT AND YURI TSCHINKEL

1. Introduction

For last fifteen years, numerous authors have studied the birational geometry of projective irreducible holomorphic symplectic varieties X, seeking to relate extremal contractions $X \to X'$ to properties of the Hodge structures on $H^2(X,\mathbb{Z})$ and $H_2(X,\mathbb{Z})$, regarded as lattices under the Beauville-Bogomolov form. Significant contributions have been made by Huybrechts, Markman, O'Grady, Verbitsky, and many others [Huy99], [Mar08], [O'G99], [Ver13], see also [Huy11].

The introduction of Bridgeland stability conditions by Bayer and Macrì provided a conceptual framework for understanding birational contractions and their centers [BM14a, BM14b]. In particular, one obtains a transparent classification of extremal birational contractions, up to the action of monodromy, for varieties of K3 type [BHT13].

Here we elaborate the Bayer-Macrì machinery through concrete examples and applications. We start by stating the key theorem and organizing the resulting extremal rays in lattice-theoretic terms; see Sections 2 and 3. We describe exceptional loci in small-dimensional cases in Sections 4 and 5. Finding concrete examples for each ray in the classification can be computationally involved; we provide a general mechanism for writing down Hilbert schemes with prescribed contractions in Section 6. Then we turn to applications. Section 7 addresses a question of Oguiso and Sarti on automorphisms of Hilbert schemes. Finally, we show that the ample cone of a polarized variety (X,h) of K3 type cannot be read off from the Hodge structure on $H^2(X,\mathbb{Z})$ in Section 8; this resolves a question of Huybrechts.

Acknowledgments: The first author was supported by NSF grants 1148609 and 1401764. The second author was supported by NSF grant 1160859. We are grateful to B. Bakker, D. Huybrechts, E. Macrì, and A. Sarti for helpful discussions; the manuscript benefited from suggestions by K. Hulek and the referee. G. Mongardi has informed us

Date: July 18, 2015.

of joint work with Knutsen and Lelli-Chiesa addressing closely related questions. We especially thank A. Bayer for allowing us to use material arising out of our collaboration and explaining subtle aspects of his work with Macrì.

2. Recollection of general theorems

Let X be deformation equivalent to the Hilbert scheme of length $n \geq 2$ subschemes of a K3 surface. Markman [Mar11, Cor. 9.5] describes an extension of lattices $H^2(X,\mathbb{Z}) \subset \widetilde{\Lambda}$ and weight-two Hodge structures $H^2(X,\mathbb{C}) \subset \widetilde{\Lambda}_{\mathbb{C}}$ characterized as follows:

- the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$ has rank one, and is generated by a primitive vector of square $2n-2$;
- as a lattice $\widetilde{\Lambda} \simeq U^4 \oplus (-E_8)^2$ where U is the hyperbolic lattice and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram;
- there is a natural extension of the monodromy action on $H^2(X,\mathbb{Z})$ to $\widetilde{\Lambda}$; the induced action on $\widetilde{\Lambda}/H^2(X,\mathbb{Z})$ is encoded by a character *cov* (see [Mar08, Sec. 4.1]);
- we have the following Torelli-type statement: X_1 and X_2 are birational if and only if there is a Hodge isometry
$$\widetilde{\Lambda}_1 \simeq \widetilde{\Lambda}_2$$
taking $H^2(X_1,\mathbb{Z})$ isomorphically to $H^2(X_2,\mathbb{Z})$;
- if X is a moduli space $M_v(S)$ of sheaves over a K3 surface S with Mukai vector v then there is an isomorphism from $\widetilde{\Lambda}$ to the Mukai lattice of S taking $H^2(X,\mathbb{Z})$ to $v^\perp$.

Generally, we use v to denote a primitive generator for the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$. Note that $v^2 = (v,v) = 2n-2$. When $X \simeq M_v(S)$ we may take the Mukai vector v as the generator.

Example 1. Suppose that $X = S^{[n]}$ for a K3 surface S so that
$$\widetilde{\Lambda} = U \oplus H^2(S,\mathbb{Z})$$
with v in the first summand. Then we can write
$$H^2(S^{[n]},\mathbb{Z}) = \mathbb{Z}\delta \oplus H^2(S,\mathbb{Z})$$
where δ generates $v^\perp \subset U$ and satisfies $(\delta,\delta) = -2(n-1)$.

There is a canonical homomorphism
$$\theta^\vee : \widetilde{\Lambda} \twoheadrightarrow H_2(X,\mathbb{Z})$$

which restricts to an inclusion

$$H^2(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$$

of finite index. By extension, it induces a $\mathbb{Q}$-valued Beauville-Bogomolov form on $H_2(X,\mathbb{Z})$.

Example 2. Retaining the notation of Example 1: Let $\delta^\vee \in H_2(X,\mathbb{Z})$ be the class orthogonal to $H^2(S,\mathbb{Z})$ such that $\delta\cdot\delta^\vee = -1$. We have $\theta^\vee(\delta) = 2(n-1)\delta^\vee$.

Assume X is projective. Let $H^2(X)_{alg} \subset H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}_{alg} \subset \widetilde{\Lambda}$ denote the algebraic classes, i.e., the integral classes of type $(1,1)$. The Beauville-Bogomolov form on $H^2(X)_{alg}$ has signature $(1,\rho(X)-1)$, where $\rho(X) = \dim(H^2_{alg}(X))$. The *Mori cone* of X is defined as the closed cone in $H_2(X,\mathbb{R})_{alg}$ containing the classes of algebraic curves in X. The *positive cone* (or more accurately, non-negative cone) in $H^2(X,\mathbb{R})_{alg}$ is the closure of the connected component of the cone

$$\{D \in H^2(X,\mathbb{R})_{alg} : D^2 > 0\}$$

containing an ample class. The dual of the positive cone in $H^2(X,\mathbb{R})_{alg}$ is the positive cone.

Theorem 3. [BHT13] *Let (X,h) be a polarized holomorphic symplectic manifold as above. The Mori cone in $H_2(X,\mathbb{R})_{alg}$ is generated by classes in the positive cone and the images under $\theta^\vee$ of the following:*

$$\{a \in \widetilde{\Lambda}_{alg} : a^2 \geq -2, |(a,v)| \leq v^2/2, (h,a) > 0\}. \tag{2.1}$$

3. Formal remarks on Theorem 3

(1) For a as enumerated in (2.1) write $R := \theta^\vee(a) \in H_2(X,\mathbb{Z})$. Recall that $(R,R) < 0$ and R is extremal in the cone described in Theorem 3 if and only if R generates the extremal ray of the birational contraction $X \to X'$ associated with the corresponding wall [BM14a, §5,12].

(2) As discussed in [BM14a, Th. 12.1], the walls in Theorem 3 also admit a natural one-to-one correspondence with

$$\{\hat{a} \in \widetilde{\Lambda}_{alg} : \hat{a}^2 \geq -2, 0 \leq (\hat{a},v) \leq v^2/2, (h,\hat{a}) > 0 \text{ if } (\hat{a},v) = 0\}. \tag{3.1}$$

Indeed, in cases of (2.1) where $(a,v) < 0$ we take $\hat{a} = -a$. From now on, we utilize these representatives of the walls.

(3) The saturation $\mathcal{H}$ of the lattice $\langle v,a\rangle$ is the fundamental invariant of each case. Observe that $\mathcal{H}$ has signature $(1,1)$ if and only if $(R,R) < 0$.

(4) Suppose $\mathcal{H}$ has signature $(1,1)$. Since h is a polarization on X, we have

$$(h,h) > 0, \quad (h,v) = 0$$

and $\langle h,a,v\rangle$ is a lattice of signature $(2,1)$.

(5) $\mathcal{H}$ has signature $(1,1)$ if and only if

$$(a,a)\,(v,v) < (a,v)^2\,;$$

this is automatic if $(a,a) = -2$, or $(a,a) = 0$ and $(a,v) \neq 0$. Since $(a,v) \leq (v,v)/2$ we necessarily have

$$(a,a) < (a,v)/2 \leq (v,v)/4, \tag{3.2}$$

and

$$(v,v) > (v-a,v-a) \geq (a,a) \geq -2. \tag{3.3}$$

Moreover, we also find

$$(a,v-a) \geq 1. \tag{3.4}$$

If $(a,a) < 0$ this follows from $(a,v) \geq 0$. If $(a,a) = 0$ we deduce $(a,v-a) \geq 0$ but equality would violate our assumption on the signature of $\mathcal{H}$. If $(a,a) > 0$ then

$$(a,v) > \sqrt{(a,a)\,(v,v)} > 2\,(a,a)\,.$$

Moreover, equality holds in (3.4) precisely when

$$(a,a) = 0, \quad (v,a) = 1. \tag{3.5}$$

(6) It is possible for $\mathcal{H} \supsetneq \langle v,a\rangle$; however, we claim $\mathcal{H} = \langle v,a'\rangle$ for some other a' satisfying the conditions in (2.1).

We prove this: Consider the parallelogram P with corners $\{0,a,v-a,v\}$. The mapping $p \mapsto v-p$ preserves P and $0 \leq (v,p) \leq (v,v)$ for each $p \in P$. If $\mathcal{H} \supsetneq \langle v,a\rangle$ then there exists an $a' \in \mathcal{H} \cap P$ such that $\mathcal{H} = \langle a',v\rangle$. Replacing a' with $v-a'$ if necessary, we may assume that $(a',v) \leq (v,v)/2$. It remains to show that $(a',a') \geq -2$. We analyze the form $(,)$ on P; it assumes its maximal and minimal values on ∂P. If $(a,a) = -2$ then $(,)\,|\partial P \geq -2$; this is clear for the sides containing a, and since $(v-a,v-a) \geq -2$ the same reasoning applies to the sides containing $v-a$. If $(a,a) \geq 0$ then a similar argument implies $(,)\,|P \geq 0$. Either way, we have $(a',a') \geq -2$.

We shall rely on these observations in Section 5 to streamline our enumeration of cases.

4. Description of the exceptional loci

We describe the exceptional loci E of extremal contractions $X \to X'$ mentioned in Section 3, up to birational equivalence. Our analysis follows [BM14a, BM14b] provided $X = M_v(T)$ for some K3 surface T, or a moduli space of twisted sheaves over X. We expect this is valid generally, however. Indeed, generically this follows from the monodromy classification of extremal rays of [BHT13].

Let v be a Mukai vector with $(v, v) > 0$ and fix $\mathcal{H} \ni v$ of signature $(1, 1)$ as in Section 3. We use bounds (3.2), (3.3), and (3.4) freely.

We define the *effective classes* of $\mathcal{H}$ to be the monoid generated by 0 and the $D \in \mathcal{H}$ satisfying $(D, D) \geq -2$ and $(v, D) > 0$. If $\mathcal{H}$ admits a spherical class s (with $(s, s) = -2$) orthogonal to v then we take one of $\{s, -s\}$ to be effective, the one meeting the polarization h positively. This differs from the definition of [BM14a, 5.5] in that it does not depend on the choice of a stability condition. An *irreducible* spherical class is one that is indecomposable in the effective monoid. An effective element D is *reducible* if one of the following holds:

- D is spherical but not irreducible;
- D is not spherical and there exists an irreducible spherical class s such that $(s, D) < 0$;
- D is isotropic but not primitive.

Every effective D with $(D, D) < -2$ is necessarily reducible.

Remark 4. Suppose $\mathcal{H} = \mathrm{Pic}(S)$ for a K3 surface S with v big and nef (hence semiample) on S. Then the reducible effective classes D are those for which $|mD|$ admits no irreducible divisors for any $m > 0$.

The lattice

$$\mathcal{H} = \begin{array}{c|cc} & v & a \\ \hline v & 2n-2 & 1 \\ a & 1 & 0 \end{array} \simeq \begin{array}{c|cc} & v-na & a \\ \hline v-na & -2 & 1 \\ a & 1 & 0 \end{array}$$

has unusual properties. It corresponds to elliptic K3 surfaces with section; the union of the section with at least two fibers is ample with the section as a fixed component. All ample divisors on K3 surfaces with base locus arise in this way [SD74].

Definition 5. A *Hilbert-Chow decomposition* of v is an expression

$$v = a + b \quad a, b \in \mathcal{H} \text{ effective},$$

where $(a, a) = 0$ and $(v, a) = 1$. An *irreducible decomposition* takes the form

$$v = a + b, \quad a, b \in \mathcal{H} \text{ irreducible effective}.$$

Hilbert-Chow and irreducible decompositions are collectively called *basic decompositions*. A basic decomposition is *distinguished* if $\mathcal{H} = \langle a, b \rangle$. We do not keep track of the order of a and b.

Note that either $(h, a) > 0$ or $(h, b) > 0$. Moreover, (v, a) or (v, b) is $\leq (v, v)/2$, and $(a, a), (b, b) \geq -2$. Thus basic decompositions are instances of the walls indexed by (2.1) and (3.1) such that $\mathcal{H}$ has indefinite Mukai form. Recall these index extremal rays of birational contractions.

Each $\mathcal{H}$ associated with an extremal ray gives rise to a distinguished decomposition; some $\mathcal{H}$ admit multiple basic decompositions. We will explain below why the isotropic vectors require extra care.

Question 6. Let $\mathcal{H}$ arise from an extremal ray of $X \to X'$ as above. Is there a bijection between basic decompositions of v and irreducible components of the exceptional locus of the contraction?

Bayer and Macrì [BM14a, §14] describe a more encompassing correspondence between strata of the exceptional locus and decompositions of v, under the assumption that $\mathcal{H}$ admits no isotropic or spherical vectors, i.e., elements w with $(w, w) = 0, -2$. This assumption is never satisfied in small dimensions, however. Based on this evidence and the examples we have computed, we assume Question 6 has a positive answer in the analysis below.

Let M_a denote the moduli space of stable objects of type a and M_{v-a} the moduli space of stable objects of type $v - a$. We only care about these up to birational equivalence so we need not specify the precise stability condition. A typical element of E corresponds to an extension

$$0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$$

where $\mathcal{A} \in M_a, \mathcal{B} \in M_{v-a}$, and $\mathcal{E}$ represents an element of $\mathbb{P}(\mathrm{Ext}^1(\mathcal{B}, \mathcal{A}))$. (Generally, one has exact triangles rather than extensions but our goal here is only to sketch representative examples for each monodromy orbit.) These have expected dimension

$$\dim(M_a) + \dim(M_{v-a}) + \dim(\mathrm{Ext}^1(\mathcal{B}, \mathcal{A}))) - 1 = (v, v) - (a, v - a) + 3$$

which equals

$$\dim(X) + 1 - (a, v - a),$$

i.e., the expected codimension of E is

$$(a, v - a) - 1 \geq 0. \tag{4.1}$$

When there is strict equality the geometry can be encapsulated by the diagram:

$$\begin{array}{ccccc} \mathbb{P}^{(a,v-a)-1} & \longrightarrow & E & \longrightarrow & M_v(T) \\ & & \downarrow & & \\ & & M_a(T) \times M_{v-a}(T) & & \end{array}$$

Hilbert-Chow contractions. The case (3.5) requires additional explanation; this is the 'Hilbert-Chow' case of [BM14b, §10], [BM14a, §5]: For concreteness, take $X = M_v(T)$ with Mukai vector $v = (1, 0, 1-n)$, where $\dim(X) = 2n$; set $a = (0, 0, -1)$ so that $v - a = (1, 0, 2-n)$. (Indeed, up to derived equivalence this is the only case cf. [BM14a, Lem. 8.5].) Thus

$$M_v(T) = T^{[n]}, \quad M_{v-a}(T) = T^{[n-1]},$$

and $M_a(T)$ parametrizes shifted point sheaves $\mathcal{O}_p[-1]$. Given distinct $p_1, \ldots, p_n \in T$, the natural inclusion of ideal sheaves gives an exact sequence

$$0 \to \mathcal{I}_{p_1,\ldots,p_n} \to \mathcal{I}_{p_1,\ldots,p_{n-1}} \to \mathcal{I}_{p_1,\ldots,p_{n-1}}|p_n \simeq \mathcal{O}_{p_n} \to 0$$

and thus an exact triangle

$$\mathcal{O}_{p_n}[-1] \to \mathcal{I}_{p_1,\ldots,p_n} \to \mathcal{I}_{p_1,\ldots,p_{n-1}}.$$

This reflects the fact that the vector space

$$\mathrm{Hom}(\mathcal{I}_{p_1,\ldots,p_{n-1}}, \mathcal{O}_{p_n}) = \mathrm{Ext}^1(\mathcal{I}_{p_1,\ldots,p_{n-1}}, \mathcal{O}_{p_n}[-1]) \simeq \mathbb{C}.$$

Now suppose that $p_{n-1} = p_n$; then

$$\mathrm{Hom}(\mathcal{I}_{p_1,\ldots,p_{n-1}}, \mathcal{O}_{p_{n-1}}) = T_{p_{n-1}} \simeq \mathbb{C}^2$$

which means that E is birationally a $\mathbb{P}^1$-bundle over

$$\Delta = \{(\Sigma, p) : p \in \Sigma\} \subset T^{[n-1]} \times T \simeq M_{v-a}(T) \times M_a(T).$$

In particular, the exceptional locus in the Hilbert-Chow case is irreducible.

Gieseker-Uhlenbeck contractions. Decompositions

$$v = a + (v - a), \quad (a, a) = 0, (v, a) =, 2$$

correspond to divisorial contractions (see (4.1)). These have been systematically studied in [BM14a, §8]. Typical extensions parametrized by the exceptional locus take the form

$$\mathcal{O}_p[-1] \to \mathcal{E} \to \mathcal{E}^{**},$$

where $\mathcal{E}$ has cotorsion supported at p and $\mathcal{E}^{**}$ is its reflexive hull, which is locally free.

Isotropic vectors and primitivity. Why do we allow only *primitive* isotropic vectors in basic decompositions? The analogy with K3 surfaces in Remark 4 suggests an answer: non-trivial semiample divisors with higher cohomology must be multiples of elliptic fibrations. Here we explain this from the moduli space perspective.

Suppose we have a decomposition of the form

$$v = \underbrace{a + \cdots + a}_{N \text{ times}} + b, \quad N \geq 1,$$

where a and b are primitive and a is isotropic. We continue to assume that $(v,a), (v,b) \geq 0$ and $(b,b) \geq -2$. We analyze strata in the exceptional locus associated with such decompositions.

Consider the moduli space $M_{Na}(T)$ for an appropriate generic stability condition [BM14a, Thm. 2.15]. Since a is isotropic and the generic point of $M_a(T)$ parametrizes simple objects, we have

$$M_{Na}(T) = (\underbrace{M_a(T) \times \cdots \times M_a(T)}_{N \text{ times}})/\mathfrak{S}_N,$$

the Nth symmetric product. Moreover, $M_b(T)$ has dimension $(b,b) + 2 \geq 0$ and the generic point is stable. Given $\mathcal{A} \in M_a(T)$ and $\mathcal{B} \in M_b(T)$ the non-split extensions

$$0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$$

are parametrized by a projective space of dimension

$$(a,b) - 1 = (a,v) - 1.$$

Now consider those of the form

$$0 \to \mathcal{A}^{\oplus N} \to \mathcal{E}' \to \mathcal{B} \to 0,$$

where we assume the restriction to each summand $\mathcal{A}$ is non-trivial. The isomorphism classes of $\mathcal{E}'$ that arise in this way—neglecting the extension data—are parametrized by

$$\underbrace{\mathbb{P}^{(a,v)-1} \times \cdots \times \mathbb{P}^{(a,v)-1}}_{N \text{ times}}.$$

Putting everything together, the expected codimension of the corresponding stratum is

$$N((v,a) - 1),$$

which is typically larger than the codimension (4.1) of the stratum associated with the decomposition

$$v = a + (v - a).$$

Thus decompositions involving isotropic vectors with multiplicities correspond to non-maximal strata.

5. Enumeration of rays

In this section, for each monodromy orbit of extremal rays we describe the geometry of the exceptional locus of the associated contraction. This completes the analysis started in [HT10b] by employing the recent work of Bayer and Macrì [BM14a, BM14b]. We organize the information first by dimension (or equivalently, by (v, v)) and then by the magnitude of (v, a). Such explicit descriptions have been used in connection with the following problems:

- constructing explicit Azumaya algebras realizing transcendental Brauer-Manin obstructions to weak approximation and the Hasse principle [HVAV11, HVA13];
- modular constructions of isogenies between K3 surfaces and interpretation of moduli spaces of K3 surface with level structure [MSTVA14];
- explicit descriptions of derived equivalences among K3 surfaces and perhaps varieties of K3 type;
- analysis of birational and biregular automorphisms of holomorphic symplectic varieties, see e.g. [HT10a, BCNWS14].

For example, when we have an exceptional divisor of the form

$$\begin{array}{ccc} \mathbb{P}^{r-1} & \rightarrow & E \\ & & \downarrow \\ & & S \times M \end{array}$$

where S is a K3 surface and M is holomorphic symplectic (perhaps a point!), we may interpret M as a parameter space of Brauer-Severi varieties over S. These naturally defined families can be quite useful for arithmetic applications.

We write $S^{[m]}$ as shorthand for the deformation equivalence class of the Hilbert scheme of a K3 surface. Note that the notation $S \times S^{[m]}$ just means a product of a K3 surface and such a manifold. 'Discriminant' refers to the lattice $\langle v, a\rangle$. We note cases where there are inclusions

$$\mathcal{H} = \langle v, a\rangle \supsetneq \langle v', a'\rangle = \mathcal{H}'$$

as then the exceptional locus associated to $\mathcal{H}$ may be reducible, as noted in Question 6. Here we analyze whether these arise from basic decompositions of v.

5.1. **Dimension four.** The case $(v, v) = 2$ has been explored in [HT09].

(a,a)	(a,v)	$(v-a,v-a)$	**Discriminant**	**Interpretation**
-2	0	0	-4	$\mathbb{P}^1$-bundle over S
-2	1	-2	-5	$\mathbb{P}^2$
0	1	0	-1	$\mathbb{P}^1$-bundle over S

The only inclusion of lattices takes the first case to the third:

$$\begin{array}{c|cc} & v & a \\ \hline v & 2 & 1 \\ a & 1 & 0 \end{array} \supsetneq \begin{array}{c|cc} & v & a' \\ \hline v & 2 & 2 \\ a' & 2 & 0 \end{array}$$

This is induced by $a' = 2a$ which does not correspond to a basic decomposition. We know the exceptional locus of the Hilbert-Chow contraction is irreducible, so Question 6 has a positive answer in this case.

5.2. **Dimension six.** We take $(v,v) = 4$. The case of Lagrangian $\mathbb{P}^3$'s, where $(a,a) = -2$ and $(a,v) = 2$, was examined in [HHT12].

(a,a)	(a,v)	$(v-a,v-a)$	**Discriminant**	**Interpretation**
-2	0	2	-8	$\mathbb{P}^1$-bundle over $S^{[2]}$
-2	1	0	-9	$\mathbb{P}^2$-bundle over S
-2	2	-2	-12	$\mathbb{P}^3$
0	1	2	-1	$\mathbb{P}^1$-bundle over $S \times S$
0	2	0	-4	$\mathbb{P}^1$-bundle over $S \times S'$, S and S' are isogenous

Again, the only possible inclusion involves the Hilbert-Chow case, which has irreducible exceptional locus.

The last entry was omitted in [HT10b, Table H3] but was included in the general conjecture proposed in that paper. We sketch the geometry in this case: Suppose

$$X = M_v(T), \quad v = (r, Nh, s), \quad N \neq 0$$

for some K3 surface (T,h); we assume that $a = (r', N'h, s')$. Express

$$H^2(X,\mathbb{Z}) = v^\perp \subset \widetilde{\Lambda} \simeq U \oplus H^2(T,\mathbb{Z})$$

so there is a saturated embedding of the primitive cohomology

$$H^2(T,\mathbb{Z})_\circ = h^\perp \hookrightarrow H^2(X,\mathbb{Z}).$$

The factors of the center of $X \to X'$ are $S = M_a(T)$ and $S' = M_{v-a}(T)$, which have cohomology groups

$$H^2(M_a(T),\mathbb{Z}) = a^\perp/\mathbb{Z}a, \quad H^2(M_{v-a}(T),\mathbb{Z}) = (v-a)^\perp/\mathbb{Z}(v-a).$$

We also have embeddings

$$H^2(T,\mathbb{Z})_\circ \hookrightarrow H^2(S,\mathbb{Z}), \quad H^2(T,\mathbb{Z})_\circ \hookrightarrow H^2(S',\mathbb{Z}),$$

that fail to be saturated in some cases.

5.3. **Dimension eight.** Here we have $(v, v) = 6$:

(a,a)	(a,v)	$(v-a,v-a)$	**Discriminant**	**Interpretation**
-2	0	4	-12	$\mathbb{P}^1$-bundle over $S^{[3]}$
-2	1	2	-13	$\mathbb{P}^2$-bundle over $S^{[2]}$
-2	2	0	-16	$\mathbb{P}^3$-bundle over S
-2	3	-2	-21	$\mathbb{P}^4$
0	1	4	-1	$\mathbb{P}^1$-bundle over $S \times S^{[2]}$
0	2	2	-4	$\mathbb{P}^1$-bundle over $S' \times S^{[2]}$ S, S' are isogenous
0	3	0	-9	$\mathbb{P}^2$-bundle over $S \times S'$ S, S' are isogenous

The last entry was also omitted in [HT10b, Table H4] but included in the general conjecture. The geometry is similar to example in Section 5.2.

Here we do have an inclusion of lattices not involving the Hilbert-Chow example. Write

$$\mathcal{H} = \begin{array}{c|cc} & v & a \\ \hline v & 6 & 2 \\ a & 2 & 0 \end{array} \supsetneq \mathcal{H}' = \begin{array}{c|cc} & v & a' \\ \hline v & 6 & 2 \\ a' & 2 & -2 \end{array}$$

with $a' = v - 2a$ and $v - a' = 2a$. However, this is not a basic decomposition so it does not arise from an additional component of the exceptional locus. Note that in the decomposition

$$v = a' + (v - a') = a' + 2a$$

we have an isotropic vector with multiplicity two.

5.4. **Dimension ten.** In this case $(v, v) = 8$.

(a,a)	(a,v)	$(v-a,v-a)$	**Discriminant**	**Interpretation**
-2	0	6	-16	$\mathbb{P}^1$-bundle over $S^{[4]}$
-2	1	4	-17	$\mathbb{P}^2$-bundle over $S^{[3]}$
-2	2	2	-20	$\mathbb{P}^3$-bundle over $S^{[2]}$
-2	3	0	-25	$\mathbb{P}^4$-bundle over S
-2	4	-2	-32	$\mathbb{P}^5$
0	1	6	-1	$\mathbb{P}^1$-bundle over $S \times S^{[3]}$
0	2	4	-4	$\mathbb{P}^1$-bundle over $S' \times S^{[3]}$ S, S' are isogenous
0	3	2	-9	$\mathbb{P}^2$-bundle over $S' \times S^{[2]}$ S, S' are isogenous
0	4	0	-16	$\mathbb{P}^3$-bundle over $S \times S'$ S, S' are isogeneous

The sublattices

$$\begin{array}{c|cc} & v & a \\ \hline v & 8 & 1 \\ a & 1 & 0 \end{array} \supsetneq \begin{array}{c|cc} & v & Na \\ \hline v & 8 & N \\ Na & N & 0 \end{array}$$

explain all embeddings among the lattices in the table above, preserving v. None of these arise from basic decompositions.

5.5. **Higher dimensional data.** For later applications, we enumerate possible discriminants of the lattice $\langle v, a\rangle$. Note that this lattice need not be saturated in the Mukai lattice, so each row below may correspond to multiple cases. When $(v,v) = 10$ we have

(a,a)	(a,v)	**Discriminant**
-2	$b = 0,1,2,3,4,5$	$-20 - b^2$
0	$b = 0,1,2,3,4,5$	$-b^2$
2	$b = 5$	$20 - b^2$

The case $(v,v) = 12$ yields

(a,a)	(a,v)	**Discriminant**
-2	$b = 0,1,2,3,4,5,6$	$-24 - b^2$
0	$b = 0,1,2,3,4,5,6$	$-b^2$
2	$b = 5,6$	$24 - b^2$

5.6. **Characterizing Lagrangian $\mathbb{P}^n$'s.** Smoothly embedded rational curves in a K3 surface

$$\ell := \mathbb{P}^1 \subset S$$

are characterized as (-2)-curves $(\ell,\ell) = -2$. Suppose that X is deformation equivalent to $S^{[n]}$ and we have a smoothly embedded

$$\mathbb{P}^n \subset X$$

with $\ell \subset \mathbb{P}^n$ a line. For $n = 2, 3$ we showed in [HT09, HHT12] that these are unique up to monodromy and satisfy

$$(\ell, \ell) = -\frac{n+3}{2}.$$

For $n = 4$ Bakker and Jorza [BJ14] computed

$$(\ell, \ell) = -\frac{7}{2}.$$

Furthermore, Bakker [Bak13, Cor. 23] has offered sufficient conditions to guarantee that Lagrangian planes form a single monodromy orbit. Previously [HT10b, Thesis 1.1], we suggested that the intersection theoretic properties of these classes should govern the cone of effective curves. Markman, Bayer and Macrì offered counterexamples to our original formulation [HT10b, Conj. 1.2] in [BM14b, §10].

Our purpose here is to illustrate that there may be multiple orbits of Lagrangian projective spaces under the monodromy action. For each n, the lattice

$$\mathcal{G} = \begin{array}{c|cc} & v & a \\ \hline v & 2(n-1) & n-1 \\ a & n-1 & -2 \end{array}$$

gives rise to a Lagrangian projective space with

$$R = [\ell] = \pm\theta^\vee(a).$$

We expect a second orbit in cases when there exists an embedding $\mathcal{G} \hookrightarrow \mathcal{H}$ as a finite index sublattice (cf. [BM14a, §14] and Question 6). For these to exist, the discriminant

$$\operatorname{disc}(\mathcal{G}) = -(n-1)(n+3)$$

should be divisible by a square. It is divisible by 9 when $n \equiv 6 \pmod 9$; we consider when $n = 15$.

Let $\mathcal{H}$ denote the lattice

$$\mathcal{H} = \begin{array}{c|cc} & v & a \\ \hline v & 28 & 14 \\ a & 14 & 6 \end{array}$$

which has discriminant -28. The lattice

$$\mathcal{G} = \begin{array}{c|cc} & v & a' \\ \hline v & 28 & 14 \\ a' & 14 & -2 \end{array}$$

can be realized as an index three sublattice of $\mathcal{H}$ via $a' = 3a - v$. Thus we obtain *two* basic decompositions of v. The associated contraction

$X \to X'$ thus should have reducible exceptional locus, with one component isomorphic to $\mathbb{P}^{15}$ and the other of codimension $(a, v-a) - 1 = 7$.

5.7. **Another example with interesting exceptional locus.** Consider the lattice

$$\mathcal{H} = \begin{array}{c|cc} & v & a \\ \hline v & 10 & 5 \\ a & 5 & 2 \end{array}$$

associated with a 12-dimensional holomorphic symplectic manifold X.

Note that $v^\perp$ is generated by a spherical class $s_1 = v - 2a$, which we take to be effective. Thus we have the decomposition

$$v = 2a + (v - 2a)$$

which yields a codimension-one stratum in X isomorphic to a $\mathbb{P}^1$-bundle over M_{2a}. The formalism of Bayer-Macrì [BM14a, Thm. 5.7a and Lem. 7.5] implies there is one exceptional divisor arising from the extremal ray associated with $\mathcal{H}$.

The lattice $\mathcal{H}$ represents (-2) infinitely many times. Here are the vectors b with $(b, b) = \pm 2$ and $(v, b) \geq 0$:

b	(b,b)	(v,b)
$2a - v$	-2	0
$s_3 = 3a - v$	-2	5
$7a - 2v$	-2	15
$\vdots$	$\vdots$	$\vdots$
$4a - v$	2	10
a	2	5
$v - a$	2	5
$3v - 4a$	2	10
$\vdots$	$\vdots$	$\vdots$
$5v - 7a$	-2	15
$s_2 = 2v - 3a$	-2	5
$s_1 = v - 2a$	-2	0

While we have

$$\begin{array}{c|cc} & s_2 & s_3 \\ \hline s_2 & -2 & 7 \\ s_3 & 7 & -2 \end{array}.$$

the decomposition

$$v = s_2 + s_3$$

is not basic as $(s_1, s_2) < 0$. We do not expect this decomposition to correspond to a Lagrangian $\mathbb{P}^6$.

Example 7. Here is a concrete example: Let T denote a K3 surface with $\mathrm{Pic}(T) = \mathbb{Z}f$ with $f^2 = 22$. Let $v = (1, f, 6)$ and $a = (-1, 0, 1)$. Elements of $M_v(T)$ generically take the form $\mathcal{I}_\Sigma(f)$ where $\Sigma \subset T$ is of length six. The distinguished spherical class is $v - 2a = (3, f, 4)$, which arises in the uniform construction of T in [Muk06, §3].

6. Orbits and extremal rays

We fix a primitive vector $v \in \widetilde{\Lambda}$ such that $v^\perp = H^2(X, \mathbb{Z})$, as in Section 2. Write

$$(v, v)\, a - (a, v)\, v = M\varrho \tag{6.1}$$

where $M > 0$ and $\varrho \in v^\perp$ is primitive. The *divisibility* $\mathrm{dv}(\varrho)$ is defined as the positive integer such that

$$\big(\varrho, H^2(X, \mathbb{Z})\big) = \mathrm{dv}(\varrho)\mathbb{Z},$$

so that $R = \varrho/\mathrm{dv}(\varrho)$ represents (via duality) a class in $H_2(X, \mathbb{Z})$ and an element of the discriminant group $d(H^2(X, \mathbb{Z})) = H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z})$.

Note that a projects to a negative class in $v^\perp$ if and only if

$$(a, a)\,(v, v) < (a, v)^2\,,$$

i.e., the lattice $\langle a, v\rangle$ has signature $(1, 1)$. The autoduality of the positive cone and the fact that nef divisors have non-negative Beauville-Bogomolov squares imply that the Mori cone contains the positive cone. Thus we may restrict our attention to ϱ with $(\varrho, \varrho) < 0$.

We exhibit representatives of these orbits in the special case where $X = S^{[n]}$; we use Example 2 and write $\delta^\vee = \frac{1}{2n-2}\delta$ so that

$$d(H^2(S^{[n]}, \mathbb{Z})) = (\mathbb{Z}/2(n-1)\mathbb{Z}) \cdot \delta^\vee.$$

Our objective is to write down *explicit* examples where they arise from extremal rational curves:

Theorem 8. *Retain the notation introduced above and assume that $R^2 < 0$. Then there exists a K3 surface S with $\mathrm{Pic}(S) \simeq \mathbb{Z}f$ that admits an extremal rational curve $\mathbb{P}^1 \subset S^{[n]}$ such that $\mathbb{R}_{\geq 0}[\mathbb{P}^1]$ is equivalent to $\mathbb{R}_{\geq 0}R$ under the action of the monodromy group.*

In particular, the cone of effective curves of $S^{[n]}$ is generated by $\delta^\vee$ and $[\mathbb{P}^1]$.

We will develop several lemmas to prove this theorem. A direct computation (cf. [BM14a, Prop. 12.6]) gives:

Lemma 9. *Retain the notation introduced above and assume that*

$$(v, v) = 2(n-1), \quad |\,(a, v)\,| \leq v^2/2, \quad \textit{and} \;\; a^2 \geq -2.$$

Then we have

$$(\rho, \rho) > -2(n-1)^2(n+3), \quad (R, R) > -(n+3)/2.$$

Markman [Mar11, Lemma 9.2] shows that the image G_n of the monodromy representation consists of the orientation-preserving automorphisms of the lattice $H^2(X, \mathbb{Z})$ acting via ± 1 on $d(H^2(X, \mathbb{Z}))$. In particular, $d(H^2(X, \mathbb{Z}))$ has a distinguished generator $\pm\delta^\vee$, determined up to sign. We consider orbits of primitive vectors $\varrho \in H^2(X, \mathbb{Z}) = v^\perp$ under the action of automorphisms of $H^2(X, \mathbb{Z})$ acting trivially on $d(H^2(X, \mathbb{Z}))$. A classical result of Eichler [Eic74] (see also [GHS10, Lemma 3.5]) shows that there is a unique orbit of primitive elements $\varrho' \in H^2(X, \mathbb{Z})$ such that

$$(\varrho', \varrho') = (\varrho, \varrho), \quad \varrho'/\mathrm{dv}(\varrho') = \varrho/\mathrm{dv}(\varrho) \in d(H^2(X, \mathbb{Z})). \tag{6.2}$$

The same holds true even if we restrict to the subgroup preserving orientations. Let $G_n^+ \subset G_n$ denotes the orientation preserving elements; for this group, the second part of (6.2) may be relaxed to

$$\varrho'/\mathrm{dv}(\varrho') = \pm\varrho/\mathrm{dv}(\varrho) \in d(H^2(X, \mathbb{Z})).$$

Lemma 10. *Each G_n^+-orbit of primitive vectors in $H^2(X, \mathbb{Z})$ has a representative of the form*

$$\varrho = sf - t\delta, \quad \gcd(s, t) = 1, \; s, t > 0, \; s | 2(n-1), \tag{6.3}$$

where $f \in H^2(S, \mathbb{Z})$ is primitive with $f^2 = 2d > 0$. Here $\mathrm{dv}(\varrho) = s$, $R = \varrho/s$, and $[R] = -2t(n-1)/s \in d(H^2(X, \mathbb{Z}))$.

This is quite standard—see the first paragraph of the proof of [GHS10, Prop. 3.6] for the argument via Eichler's criterion.

Lemma 11. *Fix a constant C and the orbit of a primitive vector $\varrho \in H^2(S^{[n]}, \mathbb{Z})$ with $C \le \varrho^2 < 0$ and $\mathrm{dv}(\varrho) = s$. Then there exists an even integer $2d > 0$ and a representation (6.3) such that for every*

$$\varrho_0 = \sigma f - \tau\delta, \; \sigma, \tau > 0$$

with $C \le \varrho_0^2 < 0$ we have $t/s > \tau/\sigma$.

Proof. First, let $\mu < s$ be a positive integer such that $t + \mu$ is divisible by s. If we express t/s as a continued fraction

$$t/s = [a_0, a_1, \ldots, a_r]$$

then $[a_1, \ldots, a_r]$ depends only on μ/s and $a_0 = \lfloor t/s \rfloor$. We regard $a_1, \ldots, a_r$ as fixed and a_0 as varying.

If the representation (6.3) is to hold we must have

$$\varrho^2 = 2ds^2 - 2(n-1)t^2$$

which implies

$$\begin{aligned} d &= (n-1)\left(\frac{t}{s}\right)^2 + \frac{\varrho^2}{2s^2} \\ &= \frac{2(n-1)t^2+\varrho^2}{2s^2}. \end{aligned}$$

If the fraction is an integer for some t it is an integer for an arithmetic sequence of t's. Thus there are solutions for $t \gg 0$, and we may assume d is large.

Now suppose that $\tau_j/\sigma_j = [a_0, \dots, a_j]$ for some $j < r$. We estimate

$$\frac{d}{n-1} - \left(\frac{\tau_j}{\sigma_j}\right)^2 = \left(\frac{t}{s}\right)^2 + \frac{\varrho^2}{2(n-1)s^2} - \left(\frac{\tau_j}{\sigma_j}\right)^2$$

using the continued fraction expressions. Substituting yields

$$(a_0 + \frac{1}{[a_1,\dots,a_r]})^2 + \frac{\varrho^2}{2(n-1)s^2} - (a_0 + \frac{1}{[a_1,\dots,a_j]})^2$$

and cancelling the a_0^2 terms gives

$$2a_0(\frac{1}{[a_1,\dots,a_r]} - \frac{1}{[a_1,\dots,a_j]}) + (\frac{1}{[a_1,\dots,a_r]} - \frac{1}{[a_1,\dots,a_j]})^2 + \frac{\varrho^2}{2(n-1)s^2}.$$

This can be made arbitrarily large in absolute value if $a_0 \gg 0$. Therefore, for $j < r$ we conclude

$$2d\sigma_j^2 - 2(n-1)\tau_j^2 \notin [C, 0).$$

Suppose we have σ and τ as specified above in the assumption of the Lemma. It follows that

$$\frac{C}{2(n-1)\sigma^2} < \frac{d}{n-1} - \left(\frac{\tau}{\sigma}\right)^2 < 0;$$

dividing both sides by $\sqrt{d/(n-1)} + \frac{\tau}{\sigma}$, which we may assume is larger than $\frac{|C|}{(n-1)}$, we obtain

$$\frac{1}{2\sigma^2} > |\sqrt{d/(n-1)} - \frac{\tau}{\sigma}|.$$

It follows (see [HW60, Thm. 184], for example) that τ/σ is necessarily a continued fraction approximation for $\sqrt{d/(n-1)}$, say, $\tau_{r'}/\sigma_{r'}$. Given a representation (6.3) we may assume that t/s is a continued fraction approximation as well.

Let $\tau_j/\sigma_j = [a_0, \dots, a_j]$ denote the sequence of continued fraction approximations of $\sqrt{d/(n-1)}$, starting from

$$\tau_0 = \lfloor \sqrt{d/(n-1)} \rfloor, \sigma_0 = 1.$$

Note that

$$\tau_{2w-2}/\sigma_{2w-2} < \tau_{2w}/\sigma_{2w} < \sqrt{d/(n-1)} < \tau_{2w+1}/\sigma_{2w+1} < \tau_{2w-1}/\sigma_{2w-1}$$

for each $w \in \mathbb{N}$, thus

$$2d\sigma_j^2 - 2(n-1)\tau_j^2 < 0$$

precisely when j is odd. Our estimate above shows that $r' > r$ whence

$$\tau/\sigma = \tau_{r'}/\sigma_{r'} < \tau_r/\sigma_r = t/s,$$

which is what we seek to prove. □

Proof. We complete the proof of Theorem 8.

Lemma 9 shows that each $\varrho_0 \in H^2(S^{[n]}, \mathbb{Z})$ associated with a negative extremal rays satisfies

$$C = -2(n+3)(n-1)^2 \leq \varrho_0^2 < 0.$$

Lemma 10 allows us to assume ϱ_0 is equivalent under the monodromy action to one of the lattice vectors satisfying the hypotheses of Lemma 11.

Take S to be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}f$ and $f^2 = 2d$; thus we have $\mathrm{Pic}(S^{[n]}) = \mathbb{Z}f \oplus \mathbb{Z}\delta$. The cone of effective curves of $S^{[n]}$ has two generators, one necessarily $\delta^\vee$. We choose d via Lemma 11. We know from [BM14a, Thm. 12.2] that the generator of $\mathbb{Q}_{\geq 0}\varrho \cap H_2(S^{[n]}, \mathbb{Z})$, is effective with some multiple generated by a rational curve $\mathbb{P}^1 \subset S^{[n]}$. However, Lemma 11 ensures that all the other $\varrho_0 \in \mathrm{Pic}(S^{[n]})$ satisfying

$$C \leq \varrho_0^2 < 0, \quad (\varrho_0, f) > 0,$$

are contained in the cone spanned by $\delta^\vee$ and ϱ. Thus our rational curve is necessarily extremal. □

7. Automorphisms on Hilbert schemes not coming from K3 surfaces

Oguiso-Sarti asked whether $S^{[n]}, n \geq 3$ can admit automorphisms not arising from automorphisms of S. Beauville gave examples for $n = 2$, e.g., the secant line involution for generic quartic surfaces; recently, a systematic analysis has been offered in [BCNWS14]. A related question of Oguiso is to exhibit automorphisms of $S^{[n]}$ not arising from automorphisms of any K3 surface T with $T^{[n]} \simeq S^{[n]}$ (see Question 6.7 in his ICM talk [Ogu14]).

Proposition 12. *There exists a polarized K3 surface (S, h) such that $S^{[3]}$ admits an automorphism α not arising from S. Moreover, there exists no K3 surface T with $T^{[3]} \simeq S^{[3]}$ explaining α.*

For simplicity we will restrict to those with $\mathrm{Pic}(S) = \mathbb{Z}h$ with $h^2 = d$. We have

$$\mathrm{Pic}(S^{[3]}) = \mathbb{Z}h \oplus \mathbb{Z}\delta,$$

with 2δ the class of the non-reduced subschemes. Recall that

$$(h, h) = d, \quad (h, \delta) = 0, \quad (\delta, \delta) = -4.$$

Lemma 13. *Suppose there exists an element $g \in \mathrm{Pic}(S^{[3]})$ such that*

$$(g, g) = 2 \quad \text{and} \quad g.R > 0$$

for each generator R of the cone of effective curves of $S^{[3]}$. Then $S^{[3]}$ admits an involution associated with reflection in g:

$$D \mapsto -D + (D, g)g.$$

Proof. This follows from the Torelli Theorem. The reflection is a monodromy operator in the sense of Markman and the positivity of the intersections guarantees it preserves the nef cone. □

Example 14. Let $d = 6$. Given three points on a degree six K3 surface, the plane they span meets the K3 surface in three additional points, yielding an involution $S^{[3]} \dashrightarrow S^{[3]}$.

However, this breaks down along triples of collinear points, which are generally parametrized by maximal isotropic subspaces of the (unique smooth) quadric hypersurface containing X. These are parametrized by a $\mathbb{P}^3 \subset S^{[3]}$. Here we have $g = h - \delta$ and the offending R is Poincaré dual to a multiple of $2h - 3\delta$. The class of the line in $\mathbb{P}^3$ is $h - (3/2)\delta$, interpreting $H_2(S^{[3]}, \mathbb{Z})$ as a finite extension of $H^2(S^{[3]}, \mathbb{Z})$.

Returning to arbitrary d, we apply the ampleness criterion to find the extremal curves. One is proportional to δ. The second generator is given by $R = ah - b\delta$, with (a, b) non-negative relatively prime integers satisfying one of the following:

(1) $da^2 - 4b^2 = -2$
(2) $da^2 - 4b^2 = -4$, with a divisible by 4
(3) $da^2 - 4b^2 = -4$, with a divisible by 2 but not 4
(4) $da^2 - 4b^2 = -12$, with a divisible by 2 but not 4
(5) $da^2 - 4b^2 = -36$, with a divisible by 4

The smallest example is $d = 114$ and $g = 3h - 16\delta$. To check the ampleness criterion, the first step is to write down all the (a, b) where $114a^2 - 4b^2$ is 'small' using the continued fraction expansion

$$\sqrt{114}/2 = [5; 2, 1, 20, 1, 2, 10],$$

which gives the following

a	b	$114a^2 - 4b^2$
1	5	14
2	11	-28
3	16	2
62	331	-28
65	347	14
127	678	-30
192	1025	-4

The class $R = 192h - 1025\delta$ is the second extremal generator; note it satisfies $R.g = 64 > 0$ which means that g is ample on $S^{[3]}$.

Now -36 is not 'small' for $114a^2 - 4b^2$ so we need to analyze this case separately. However, the equation

$$114a^2 - 4b^2 = -36$$

only has solutions when a and b are both divisible by three.

8. Ambiguity in the ample cone

The following addresses a question raised by Huybrechts:

Theorem 15. *There exist polarized manifolds of K3 type (X, g) and (Y, h) admitting an isomorphism of Hodge structures*

$$\phi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}), \quad \phi(g) = h$$

not preserving ample cones.

This contradicts our speculation that the Hodge structure determines the ample cone of a polarized holomorphic symplectic manifold; we also need to keep track of the Markman extension data. In particular, ϕ is not a parallel transport operator [Mar11, §7.1 and 9.1].

We first explain the idea: Let Λ_n denote the lattice isomorphic to $H^2(X, \mathbb{Z})$ where X is deformation equivalent to $S^{[n]}$ where S is a K3 surface. Given an isomorphism $X \simeq S^{[n]}$ we have a natural embedding $\Lambda_n \hookrightarrow \widetilde{\Lambda}$. Let $d(\Lambda_n)$ denote the discriminant group with the associated $(\mathbb{Q}/2\mathbb{Z})$-valued quadratic form. There is a natural homomorphism

$$\mathrm{Aut}(\Lambda_n) \to \mathrm{Aut}(d(\Lambda_n))$$

which is surjective by Nikulin's theory of lattices. The automorphisms of Λ_n extending to automorphisms of $\widetilde{\Lambda}$ are those acting via ± 1 on $d(\Lambda_n)$ [Mar11, §9].

We choose n such that $\mathrm{Aut}(d(\Lambda_n)) \supsetneq \{\pm 1\}$, exhibit an $\alpha \in \mathrm{Aut}(\Lambda_n)$ not mapping to ± 1, and show that α fails to preserve the ample cone by verifying that its dual

$$\alpha^* : \Lambda_n^* \to \Lambda_n^*$$

fails to preserve the extremal rays identified by Bayer-Macrì.

We start by fixing notation: Consider

$$\Lambda_n \subset \widetilde{\Lambda} \simeq U \oplus H^2(S, \mathbb{Z}) \simeq U^4 \oplus (-E_8)^2$$

realized as the orthogonal complement of a vector $v \in U \simeq H^2(S, \mathbb{Z})^\perp$. Let e_1, f_1 denote a basis for this U satisfying

$$(e_1, e_1) = (f_1, f_1) = 0, (e_1, f_1) = 1;$$

let e_2, f_2 denote a basis for one of the hyperbolic summands $U \subset H^2(S, \mathbb{Z})$. We may assume $v = e_1 + nf_1$ and write $\delta = e_1 - nf_1$. Since $\Lambda_n^* \simeq H_2(X, \mathbb{Z})$ the classification of extremal rays is expressed via monodromy orbits of vectors $R \in H_2(X, \mathbb{Z})$. The pre-image of $\mathbb{Z}R$ in $\widetilde{\Lambda}$ is a rank-two lattice

$$\mathcal{H} \subset \widetilde{\Lambda}, \quad v, a \in \widetilde{\Lambda},$$

where a is as described in Theorem 3.

The first step is to give an n such that the group

$$(\mathbb{Z}/2(n-1)\mathbb{Z})^*$$

admits an element $\bar{\alpha} \neq \pm 1$ such that

$$\bar{\alpha}^2 \equiv 1 \pmod{4(n-1)}.$$

We choose $n = 7$ and $\bar{\alpha} = 5$.

Next, we exhibit an $\alpha \in \mathrm{Aut}(\Lambda_7)$ mapping to $\bar{\alpha}$. These exist by Nikulin's general theory, but we offer a concrete example of such an automorphism. Then we may take

$$\alpha(\delta) = 5\delta + 12(e_2 + f_2), \ \alpha(e_2) = \delta + 2e_2 + 3f_2, \ \alpha(f_2) = \delta + 3e_2 + 2f_2$$

and acting as the identity on the other summands.

The third step is to find an extremal ray that fails to be sent to an extremal ray under α^*. We are free to pick any representative in the orbit under the monodromy. Consider then the lattice

$$\mathcal{H}_1 := \begin{array}{c|cc} & v & a \\ \hline v & 12 & 5 \\ a & 5 & -2 \end{array}$$

with

$$a = 5f_1 + e_2 - f_2.$$

Consider the element

$$a' = v - 5a \in \Lambda_7;$$

the relevant ray R is a generator of

$$\mathbb{Q}a' \cap \Lambda_7^* \subset \Lambda_7 \otimes \mathbb{Q}.$$

Explicitly

$$a' = 5\delta - 12e_2 + 12f_2$$

and

$$\alpha(a') = 25\delta + 72e_2 + 48f_2 = 25e_1 - 150f_1 + 72e_2 + 48f_2.$$

Let $\mathcal{H}_2$ denote the saturated lattice containing $\alpha(a')$ and v. Note that $\alpha(a') - v$ is divisible by 12; write

$$b = \frac{\alpha(a') - v}{12} = 2e_1 - 13f_1 + 6e_2 + 4f_2.$$

In particular, $\langle v, b\rangle \subset \widetilde{\Lambda}$ is saturated. Thus we find:

$$\mathcal{H}_2 = \begin{array}{c|cc} & v & b \\ \hline v & 12 & -1 \\ b & -1 & -4 \end{array}$$

We put $\mathcal{H}_1$ and $\mathcal{H}_2$ in reduced form:

$$\mathcal{H}_1 \simeq \begin{pmatrix} 0 & 7 \\ 7 & -2 \end{pmatrix}, \quad \mathcal{H}_2 \simeq \begin{pmatrix} 0 & 7 \\ 7 & -4 \end{pmatrix}$$

which are *inequivalent* lattices of discriminant -49. We refer the reader to the Section 5.5: there is a *unique* lattice that appears of discriminant -49, i.e., the one associated with $\mathcal{H}_1$. Thus $\mathcal{H}_2$ is not associated with an extremal ray R'.

To recapitulate: Suppose we started with an X such that the vector a yields an extremal ray R. We apply the automorphism α to $H^2(X, \mathbb{Z})$ to get a new Hodge structure, equipped with an embedding into $\widetilde{\Lambda}$; surjectivity of Torelli [Huy99] guarantees the existence of another hyperkähler manifold Y with this Hodge structure and a compatible embedding $H^2(Y, \mathbb{Z}) \subset \widetilde{\Lambda}$. However, the class $R' \in H^2(Y, \mathbb{Z})$ corresponding to R is *not* in the monodromy orbit of any extremal ray.

To exhibit a concrete projective example of this type, we could carry out an analysis along the lines of Theorem 8 in Section 6. There we showed that each monodromy orbit of extremal rays R arises as from an extremal rational curve

$$\mathbb{P}^1 \subset S^{[n]} \simeq X$$

where (S, A) is a polarized K3 surface, perhaps of very large degree. The approach was to show that the *only* vectors in $\mathrm{Pic}(X)$ with 'small' norm are δ and ϱ, a positive integer multiple of R.

What happens when we apply the construction above to such an $X \simeq S^{[n]}$? The isomorphism

$$\alpha : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$$

implies $\mathrm{Pic}(Y) \simeq \mathrm{Pic}(X)$ as lattices, so their small vectors coincide. Furthermore, we may choose $Y \simeq T^{[n]}$ where (T, B) is a polarized K3 surface isogenous to (S, A), i.e., we have isomorphisms of polarized integral Hodge structures

$$H^2(S, \mathbb{Z}) \supset A^{\perp} \simeq B^{\perp} \subset H^2(T, \mathbb{Z}).$$

Moreover, we may assume that δ_X is taken to δ_Y, i.e., the extremal curve class $\delta^{\vee}_{S^{[n]}}$ maps to $\delta^{\vee}_{T^{[n]}}$. Consequently, there exists an ample divisor on $S^{[n]}$—for instance, $g := NA - \delta_{S^{[n]}}$ for $N \gg 0$—that goes to an ample divisor $h = \alpha(g)$ on $T^{[n]}$.

Let $\varrho \in \mathrm{Pic}(X)$ denote the class arising as a positive multiple of the extremal ray; note that $\varrho = \pm a'$ in the notation above. Now $\alpha(\varrho)$ does not correspond to an effective class, so the second extremal ray on Y corresponds to a subsequent vector of 'small' norm, i.e.,

$$\alpha^*(\text{cone of effective curves on } X) \subsetneq \text{cone of effective curves on } Y.$$

Remark 16. Markman has independently obtained an example along these lines; it is also of K3 type, deformation equivalent to $S^{[7]}$.

Remark 17. Explicit descriptions of cones of divisors on generalized Kummer manifolds have been found by Yoshioka [Yos12]. Qualitative descriptions of these cones, with applications to the Kawamata-Morrison conjecture, have been established by Markman-Yoshioka [MY14] and Amerik-Verbitsky [AV14].

References

[AV14] Ekaterina Amerik and Misha Verbitsky. Morrison-Kawamata cone conjecture for hyperkahler manifolds, 2014. arXiv:1408.3892.

[Bak13] Benjamin Bakker. A classification of extremal Langrangian planes in holomorphic symplectic varieties, 2013. arXiv:1310.6341.

[BCNWS14] Samuel Boissière, Andrea Cattaneo, Marc Nieper-Wisskirchen, and Alessandra Sarti. The automorphism group of the Hilbert scheme of two points on a generic projective $K3$ surface, 2014. arXiv:1410.8387.

[BHT13] Arend Bayer, Brendan Hassett, and Yuri Tschinkel. Mori cones of holomorphic symplectic varieties of $K3$ type. *Annales scientifiques de l'École normale supérieure*, to appear, 2013. arXiv:1307:2291.

[BJ14] Benjamin Bakker and Andrei Jorza. Lagrangian 4-planes in holomorphic symplectic varieties of $K3^{[4]}$-type. *Cent. Eur. J. Math.*, 12(7):952–975, 2014.

[BM14a] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.

[BM14b] Arend Bayer and Emanuele Macrì. Projectivity and birational geometry of Bridgeland moduli spaces. *J. Amer. Math. Soc.*, 27(3):707–752, 2014.

[Eic74] Martin Eichler. *Quadratische Formen und orthogonale Gruppen*. Springer-Verlag, Berlin, 1974. Zweite Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 63.

[GHS10] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli spaces of irreducible symplectic manifolds. *Compos. Math.*, 146(2):404–434, 2010.

[HHT12] David Harvey, Brendan Hassett, and Yuri Tschinkel. Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces. *Comm. Pure Appl. Math.*, 65(2):264–286, 2012.

[HT09] Brendan Hassett and Yuri Tschinkel. Moving and ample cones of holomorphic symplectic fourfolds. *Geom. Funct. Anal.*, 19(4):1065–1080, 2009.

[HT10a] Brendan Hassett and Yuri Tschinkel. Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces. *J. Inst. Math. Jussieu*, 9(1):125–153, 2010.

[HT10b] Brendan Hassett and Yuri Tschinkel. Intersection numbers of extremal rays on holomorphic symplectic varieties. *Asian J. Math.*, 14(3):303–322, 2010.

[Huy99] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[Huy11] Daniel Huybrechts. A global Torelli theorem for hyperkähler manifolds (after M. Verbitsky), 2011. Séminaire Bourbaki. Vol. 2010/2011, Exp. No. 1040.

[HVA13] Brendan Hassett and Anthony Várilly-Alvarado. Failure of the Hasse principle on general $K3$ surfaces. *J. Inst. Math. Jussieu*, 12(4):853–877, 2013.

[HVAV11] Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly. Transcendental obstructions to weak approximation on general K3 surfaces. *Adv. Math.*, 228(3):1377–1404, 2011.

[HW60] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. The Clarendon Press Oxford University Press, London, fourth edition, 1960.

[Mar08] Eyal Markman. On the monodromy of moduli spaces of sheaves on $K3$ surfaces. *J. Algebraic Geom.*, 17(1):29–99, 2008.

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.

[MSTVA14] Kelly McKinnie, Justin Sawon, Sho Tanimoto, and Anthony Várilly-Alvarado. Brauer groups on $K3$ surfaces and arithmetic applications, 2014. arXiv:1404.5460.

[Muk06] Shigeru Mukai. Polarized $K3$ surfaces of genus thirteen. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 315–326. Math. Soc. Japan, Tokyo, 2006.

[MY14] Eyal Markman and Kota Yoshioka. A proof of the Kawamata-Morrison conjecture for holomorphic symplectic varieties of $K3^{[n]}$ or generalized Kummer deformation type, 2014. arXiv:1402.2049.

[O'G99] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.

[Ogu14] K. Oguiso. Some aspects of explicit birational geometry inspired by complex dynamics, 2014. arXiv:1404.2982.

[SD74] B. Saint-Donat. Projective models of $K-3$ surfaces. *Amer. J. Math.*, 96:602–639, 1974.

[Ver13] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkahler manifolds. *Duke Mathematical Journal*, 162(15):2929–2986, 2013. arXiv:0908.4121.

[Yos12] Kota Yoshioka. Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface, 2012. arXiv:1206.4838.

Department of Mathematics, Rice University, MS 136, Houston, Texas 77251-1892, USA

Current address: Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, Rhode Island 02912, USA

E-mail address: bhassett@math.brown.edu

Courant Institute, New York University, New York, NY 10012, USA

E-mail address: tschinkel@cims.nyu.edu

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA