HEIGHT ZETA FUNCTIONS OF EQUIVARIANT COMPACTIFICATIONS OF UNIPOTENT GROUPS

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Abstract. We prove Manin's conjecture for bi-equivariant compactifications of unipotent groups.

Contents

Introduction

Let F be a number field and X an algebraic variety over F; we write $X(F)$ for the set of its F-rational points. The *height* of an F-rational point $\mathbf{x} = (x_0 : ... : x_n) \in \mathbb{P}^n(F)$ of a projective space is given by

$$H(\mathbf{x}) := \prod_v \max_j |x_j|_v,$$

where the product is over the set of all valuations of F and $|\cdot|_v$ is the standard v-adic absolute value. Let G be a linear algebraic group over F and

$$\rho \,:\, \mathrm{G} \to \mathrm{PGL}_{n+1}$$

a projective rational representation of G. Assume that there exists a point $e \in \mathbb{P}^n$ with trivial stabilizer (under the action of $\rho(\mathrm{G})$). We are interested in the asymptotics of

$$N(B) := \{\gamma \in \mathrm{G}(F) \,|\, H(\rho(\gamma)\cdot e) \le B\}, \quad B \to \infty.$$

An alternative geometric description of the problem is as follows: Consider the Zariski closure $X \subset \mathbb{P}^n$ of the orbit

$$\{\rho(\gamma)\cdot e \,|\, \gamma \in \mathrm{G}(F)\}.$$

Then X is an *equivariant* compactification of G, embedded by a G-linearized (ample) line bundle L. Choosing a particular height in the ambient projective space amounts to choosing an adelic *metrization* $\mathcal{L} := (L, \|\cdot\|_v)$ of L (see Section 8 for the definitions). In this setup, the problem is to understand

$$N(\mathcal{L}, B) := \{\gamma \in \mathrm{G}(F) \,|\, H_{\mathcal{L}}(\gamma) \le B\}, \quad B \to \infty, \tag{0.1}$$

where $H_{\mathcal{L}}$ is the height defined by $\mathcal{L}$.

In this paper we consider smooth projective *bi-equivariant* compactifications of a unipotent group G over F. This means that G is contained in X as a Zariski open subset and that the natural left *and* right action of G on itself extend to a left and right action of G on X. Alternatively, one may think of X as an equivariant compactification of the homogeneous space $\mathrm{G} \times \mathrm{G}/\mathrm{G}$.

The main result is the determination of the asymptotic (0.1) for arbitrary bi-equivariant compactifications X as above and $\mathcal{L} = -\mathcal{K}_X$, the *anticanonical* line bundle equipped with a smooth adelic metrization, proving Manin's conjecture [12] and its refinement by Peyre [22] for this class of varieties.

It turns out that the geometric language is more adequate for the description of the asymptotic behavior. More precisely, denote by $\mathrm{Pic}(X)$ the Picard group of X, this is a free abelian group generated by the classes of the boundary components D_α, and we will generally identify divisors and their classes in $\mathrm{Pic}(X)$. Our main result is a proof of Manin's conjecture:

Theorem 1. *Let X be a smooth projective bi-equivariant compactification of* G*, with boundary*

$$X \setminus \mathrm{G} = \cup_{\alpha \in \mathcal{A}} D_\alpha$$

a normal crossings divisor consisting of geometrically irreducible components. Then

$$N(-\mathcal{K}_X, B) = \frac{\tau(-\mathcal{K}_X)}{(b-1)!} B \log(B)^{b-1}(1 + o(1)), \qquad \text{as } B \to \infty,$$

where $b = \mathrm{rk}\,\mathrm{Pic}(X) = \#\mathcal{A}$ is the number of boundary components and $\tau(-\mathcal{K}_X)$ is the Tamagawa number defined by Peyre [22].

We now give an outline of the proof. In Section 2 we recall some basic structural results concerning nilpotent algebras and unipotent groups. In Section 3 we discuss coadjoint orbits and their parametrization and in Section 4 integral structures. In Section 5 we collect facts regarding unitary representations of unipotent groups over the adeles. In Section 6 we study the action of the universal enveloping algebra in representation spaces. All of the above material is standard and can be found in the books [7], [10] and the papers [17], [21].

In Section 7 we turn to equivariant compactifications of unipotent groups and describe the relevant geometric invariants and constructions. In Section 8 we introduce the height pairing

$$H = \prod_v H_v \, : \, \mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{G}(\mathbb{A}) \to \mathbb{C},$$

generalizing the usual heights, and the height zeta function

$$\mathcal{Z}(\mathbf{s}; g) := \sum_{\gamma \in \mathrm{G}(F)} H(\mathbf{s}; \gamma g)^{-1}. \tag{1.1}$$

By the projectivity of X, the series converges to a function which is continuous and bounded in g and holomorphic in $\mathbf{s}$ for $\Re(\mathbf{s})$ contained in some cone $\Lambda \subset \mathrm{Pic}(X)_{\mathbb{R}}$. Our goal is establish analytic properties, and in particular, obtain a meromorphic continuation of the 1-parameter *height zeta functions*

$$\mathcal{Z}(sL) = \sum_{\gamma \in \mathrm{G}(F)} H_{\mathcal{L}}(sL, \gamma)^{-1},$$

obtained by restricting the multiparameter zeta function $\mathcal{Z}(\mathbf{s}; g)$ to the complex line through L and the identity $g = e \in G(\mathbb{A}_F)$.

To describe the polar set, we use the classes D_α as a basis of $\mathrm{Pic}(X)$. In this basis, the pseudo-effective cone $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ consists of classes $(l_\alpha) \in \mathrm{Pic}(X)_{\mathbb{R}}$ with $l_\alpha \geq 0$ for all α. Let

$$-K_X = \kappa = \sum_{\alpha \in \mathcal{A}} \kappa_\alpha D_\alpha \in \mathrm{Pic}(X)_{\mathbb{R}},$$

be the anticanonical class. We know (see Proposition 7.3) that $\kappa_\alpha \geq 2$, for all $\alpha \in \mathcal{A}$. Conjecturally, analytic properties of *height zeta functions* $\mathcal{Z}(sL)$ depend on the location of $L = (l_\alpha) \in \mathrm{Pic}(X)$ with respect to the anticanonical class and the cone $\Lambda_{\mathrm{eff}}(X)$ (see [12], [22] and [3]). Precisely, define

- $a(L) := \inf\{a \mid aL + K_X \in \Lambda_{\mathrm{eff}}(X)\} = \max_\alpha(\kappa_\alpha / l_\alpha)$;
- $b(L) := \#\{\alpha \,|\, \kappa_\alpha = a(L) l_\alpha\}$;

- $\mathcal{C}(L) := \{\alpha \mid \kappa_\alpha \neq a(L)l_\alpha\}$;
- $c(L) := \prod_{\alpha \notin \mathcal{C}(L)} l_\alpha^{-1}$.

Then, conjecturally,

$$\mathcal{Z}(sL) = \frac{c(L)\tau(\mathcal{L})}{(s-a(L))^{b(L)}} + \frac{h(s)}{(s-a(L))^{b(L)-1}}, \tag{1.2}$$

where $h(s)$ is a holomorphic function (for $\Re(s) > a(L) - \delta$, some $\delta > 0$) and $\tau(\mathcal{L})$ is a positive real number. Given this, Tauberian theorems imply

$$N(\mathcal{L}, B) = \frac{c(L)\tau(\mathcal{L})}{a(L)(b(L)-1)!} B^{a(L)} \log(B)^{b(L)-1}(1 + o(1)),$$

as $B \to \infty$, for certain constants $\tau(\mathcal{L})$ defined in [3]. Here we establish this for $L = -K_X$, via a spectral expansion of $\mathcal{Z}(\mathbf{s}; g)$ from Equation (1.1).

The bi-equivariance of X implies that H is invariant under the action *on both sides* of a compact open subgroup K of the finite adeles $\mathrm{G}(\mathbb{A}_{\mathrm{fin}})$. Furthermore, we assume that H_v is smooth for archimedean v. We observe that

$$\mathcal{Z}(\mathbf{s}; g) \in \mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))^{\mathrm{K}}$$

and we proceed to analyze its spectral decomposition. We get a formal identity

$$\mathcal{Z}(\mathbf{s}; g) = \sum_{\varrho} \mathcal{Z}_\varrho(\mathbf{s}; g), \tag{1.3}$$

where the summation is over all irreducible unitary representations $(\varrho, \mathcal{H}_\varrho)$ of $\mathrm{G}(\mathbb{A})$ occurring in the right regular representation of $\mathrm{G}(\mathbb{A})$ in $\mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))$. These are parametrized by F-rational orbits $\mathcal{O} = \mathcal{O}_\varrho$ under the coadjoint action of G on the dual of its Lie algebra $\mathfrak{g}^*$. The relevant orbits are *integral* - there exists a lattice in $\mathfrak{g}^*(F)$ such that $\mathcal{Z}_\varrho(\mathbf{s}; g) = 0$ unless the intersection of $\mathcal{O}$ with this lattice is nonempty. The pole of highest order is contributed by the trivial representation and integrality insures that this representation is "isolated".

Let ϱ be a representation as above. Then ϱ arises from some

$$\pi = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(\psi),$$

where $\mathrm{M} \subset \mathrm{G}$ is an F-rational subgroup and ψ is a certain character of $\mathrm{M}(\mathbb{A})$ (see Proposition 5.5). In particular, for the trivial representation,

$\mathrm{M} = \mathrm{G}$ and ψ is the trivial character. Further, there exists a finite set of places $S = S_\varrho$ such that $\dim\, \varrho_v = 1$ for $v \notin S$ and consequently

$$\mathcal{Z}_\varrho(\mathbf{s}; g') = \mathcal{Z}^S(\mathbf{s}; g') \cdot \mathcal{Z}_S(\mathbf{s}; g'). \tag{1.4}$$

We prove that

$$\mathcal{Z}^S(\mathbf{s}; g') := \prod_{v \notin S} \int_{\mathrm{M}(F_v)} H_v(\mathbf{s}; m_v g'_v)^{-1} \overline{\psi}(m_v g'_v) dm_v,$$

(with an appropriately normalized Haar measure dm_v on $\mathrm{M}(F_v)$). The function $\mathcal{Z}_S$ is the projection of $\mathcal{Z}$ to $\otimes_{v \in S} \varrho_v$.

The first key result is the explicit computation of *height integrals*:

$$\int_{\mathrm{M}(F_v)} H_v(\mathbf{s}; m_v g'_v)^{-1} \overline{\psi}(m_v g'_v) dm_v$$

for almost all v (see Section 9). This has been done in [5] for equivariant compactifications of additive groups $\mathbf{G}_a^n$; the same approach works here as well. We regard the height integrals as geometric versions of Igusa's integrals (see [6]).

For the trivial representation and $v \notin S$, we have
(1.5)

$$\int_{\mathrm{G}(F_v)} H(\mathbf{s}; g_v)^{-1} dg_v = q_v^{-\dim X} \left(\sum_{A \subseteq \mathcal{A}} D_A^0(\mathbf{k}_v) \prod_{\alpha \in A} \frac{q_v - 1}{q_v^{s_\alpha - \kappa_\alpha + 1} - 1} \right),$$

where

$$D_A := \cap_{\alpha \in A} D_\alpha, \quad D_A^0 := D_A \setminus \cup_{A' \supsetneq A} D_{A'}$$

and q_v is the cardinality of the residue field $\mathbf{k}_v$ at v. Restricting to the line through $-K_X$, we find that the resulting Euler product $\mathcal{Z}^S(-sK_X)$ is regularized by a product of (truncated) Dedekind zeta functions, thus is holomorphic for $\Re(s) > 1$, admits a meromorphic continuation to $\Re(s) > 1 - \delta$, for some $\delta > 0$, and has an isolated pole of order $\mathrm{rk}\ \mathrm{Pic}(X)$ at $s = 1$, with the expected leading coefficient $\tau(-\mathcal{K}_X)$. Similarly, we identify the poles of $\mathcal{Z}^S$ for nontrivial representations: again, they are regularized by products of (truncated) Dedekind zeta functions and thus admit a meromorphic continuation to the same domain, with at most an isolated pole at $s = 1$; but the order of the pole at $s = 1$ is strictly smaller than $\mathrm{rk}\ \mathrm{Pic}(X)$.

Next we need to estimate $\dim\, \varrho_v$ and the local integrals for nonarchimedean $v \in S$ (see Sections 5.7 and 9). Then we turn to archimedean places. Using integration by parts, we prove in Lemma 9.7 that for all

$\epsilon > 0$ and all (left or right) G-invariant differential operators ∂ there exist constants $\mathsf{c} = \mathsf{c}(\epsilon, \partial)$ and $N = N(\partial)$ such that

$$\int_{\mathrm{G}(F_v)} |\partial H_v(\mathbf{s}; g_v)^{-1} dg_v|_v \leq \mathsf{c} \cdot \|\mathbf{s}\|^N, \tag{1.6}$$

for all $\mathbf{s}$ with $\Re(s_\alpha) > \kappa_\alpha - 1 + \epsilon$, for all $\alpha \in \mathcal{A}$.

Let v be real. It is known that ϱ_v admits a standard model $(\pi_v, \mathsf{L}^2(\mathbb{R}^r))$, where $2r = \dim \mathcal{O}$. More precisely, there exists an isometry

$$j \ : \ (\pi_v, \mathsf{L}^2(\mathbb{R}^r)) \to (\varrho_v, \mathcal{H}_v),$$

an analog of the Θ-distribution. Moreover, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ surjects onto the Weyl algebra of differential operators with polynomial coefficients acting on the smooth vectors $\mathsf{C}^\infty(\mathbb{R}^r) \subset \mathsf{L}^2(\mathbb{R}^r)$. In particular, we can find an operator Δ acting as the r-dimensional harmonic oscillator

$$\prod_{j=1}^{r} \left(\frac{\partial^2}{\partial x_j^2} - a_j x_j^2\right),$$

with $a_j > 0$. We choose an orthonormal basis of $\mathsf{L}^2(\mathbb{R}^r)$ consisting of Δ-eigenfunctions $\{\tilde{\omega}_\lambda\}$ (which are well known) and analyze

$$\int_{\mathrm{G}(F_v)} H_v(\mathbf{s}; g_v)^{-1} \overline{\omega}_\lambda(g_v) dg_v,$$

where $\omega_\lambda = j^{-1}(\tilde{\omega}_\lambda)$. Using integration by parts and (1.6) we find that for all $n \in \mathbb{N}$ there exist constants $\mathsf{c} = \mathsf{c}(n, \Delta)$ and $N \in \mathbb{N}$ such that this integral is bounded by

$$\mathsf{c} \cdot \lambda^{-n} \cdot \|\mathbf{s}\|^N, \tag{1.7}$$

for $\mathbf{s}$ with $\Re(s_\alpha) > \kappa_\alpha - 1 + \epsilon$, for all α. This estimate suffices to conclude that for *each* ϱ the function $\mathcal{Z}_{S_\varrho}$ is holomorphic in a neighborhood of κ; indeed it will be majorized by

$$\sum_\lambda \lambda^{-n},$$

the spectral zeta function of a compact manifold, which converges for sufficiently large $n \geq 0$ (see Section 9 and the Appendix).

Now the issue is to prove the convergence of the sum in (1.3). Using any element $\partial \in \mathfrak{U}(\mathfrak{g})$ acting in $\mathcal{H}_\varrho$ by a scalar $\lambda(\partial) \neq 0$ (for example, any element in the center of $\mathfrak{U}(\mathfrak{g})$) we can improve the bound (1.7) to

$$\mathsf{c} \cdot \lambda^{-n_1} \cdot \lambda(\partial)^{-n_2} \cdot \|\mathbf{s}\|^N$$

(for any $n_1, n_2 \in \mathbb{N}$ and some constants $\mathsf{c} = \mathsf{c}(n_1, n_2, \Delta, \partial)$ and $N = N(\Delta, \partial)$. However, we have to insure the uniformity of such estimates over the set of all ϱ. This relies on a parametrization of coadjoint orbits. There is a finite set Σ of "packets" of coadjoint orbits, each parametrized by a locally closed subvariety $Z_\sigma \subset \mathfrak{g}^*$, and for each σ a finite set of F-rational polynomials $\{P_{\sigma,j}\}$ on $\mathfrak{g}^*$ such that each $P_{\sigma,j}$ is invariant under the coadjoint action and nonvanishing on the stratum Z_σ. Consequenty, the corresponding derivatives

$$\partial_{\sigma,j} \in \mathfrak{U}(\mathfrak{g})$$

act in $\mathcal{H}_\varrho$ by multiplication by the scalar

$$\lambda_{\varrho,j}(\ell) = P_{\sigma,j}(2\pi i \ell), \quad \ell \in \mathcal{O}.$$

Recall that ℓ varies over a lattice; applying several times $\partial_\sigma = \prod_j \partial_{\sigma,j}$ we obtain the uniform convergence of the right hand side in (1.3).

The last technical point is to prove that both expressions (1.1) and (1.3) for $\mathcal{Z}(-sK_X; g)$ define *continuous* functions on $\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})$. Then (1.3) gives the desired meromorphic continuation of $\mathcal{Z}(-sK_X; e)$.

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2. Nilpotent Lie algebras and unipotent groups

In this section we recall basic properties of nilpotent Lie algebras and unipotent groups. We work over a field F of characteristic zero.

2.1. **Nilpotent algebras.** Let $\mathfrak{g} = (\mathfrak{g}, [,])$ be an n-dimensional Lie algebra over F: an affine space over F of dimension n together with a bracket $[\cdot,\cdot]$ satisfying the Jacobi identity. Denote by $\mathfrak{z}_\mathfrak{g}$ the center of $\mathfrak{g}$. For a subset $\mathfrak{h} \subset \mathfrak{g}$ we denote by

$$\mathfrak{n}_\mathfrak{g}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subset \mathfrak{h}\}$$

its normalizer and by

$$\mathfrak{z}_\mathfrak{g}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [X, Y] = 0, \forall Y \in \mathfrak{h}\}$$

its centralizer. Let

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_k \subset \mathfrak{g}$$

be a sequence of subalgebras. A *weak* Malcev basis through this sequence is a basis $(X_1, \ldots, X_n)$ of $\mathfrak{g}$ such that

- for all $j \in 1, \ldots, k$ there exists an n_j such that $\mathfrak{g}_j = \langle X_1, \ldots, X_{n_j}\rangle$;

- for all $i = 1, ..., n$ the F-vector space $\langle X_1, ..., X_i\rangle$ is a Lie subalgebra.

Assume that all $\mathfrak{g}_j$ above are ideals. A *strong* Malcev basis through this sequence is a weak Malcev basis such that

- for all $i = 1, ..., n$ the F-vector space $\langle X_1, ..., X_i\rangle$ is an ideal.

The *ascending central series* of $\mathfrak{g}$ is defined as

$$\begin{array}{rcl} \mathfrak{g}_0 & := & 0; \\ \mathfrak{g}_j & := & \{x \in \mathfrak{g} \,|\, [x, \mathfrak{g}] \subseteq \mathfrak{g}_{j-1}\}. \end{array}$$

From now on we will assume that $\mathfrak{g}$ is *nilpotent*, that is, there exists an n such that $\mathfrak{g}_n = \mathfrak{g}$.

Example 2.2. Some common examples are:

- the Heisenberg algebra $\mathfrak{h}_3 := \langle X, Y, Z\rangle$, $[X, Y] = Z$;
- the upper-triangular algebra $\mathfrak{n}_n \subset \mathfrak{gl}_n$;
- the algebra $\mathfrak{k}_4 = \langle X_1, X_2, X_3, Y\rangle$: $[X_i, X_j] = 0, [Y, X_i] = X_{i-1}$.

Lemma 2.3. *If $\mathfrak{g}$ is nilpotent then for any ascending sequence of algebras (resp. ideals) there exists a weak (resp. strong) Malcev basis passing through it.*

Proof. Indeed,

$$\mathfrak{h} \subsetneq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}),$$

and for any $X \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$ the vector space $\mathfrak{h} \oplus FX$ is a subalgebra. Same argument works for ideals. □

There is no canonical choice of a Malcev basis through a given subalgebra.

Lemma 2.4. *(Kirillov's lemma) Let $\mathfrak{g}$ be a noncommutative nilpotent Lie algebra with 1-dimensional center $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) = \langle Z\rangle$. Then there exist $X, Y \in \mathfrak{g}$ such that*

- $[X, Y] = Z$;
- $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(Y) \oplus FX$.

Proof. Choose some $Y \in \mathfrak{g}_2 \setminus \mathfrak{g}_1$. Then $\mathfrak{g}_0 := \mathfrak{z}_{\mathfrak{g}}(Y)$ is a subalgebra of codimension one and there is an X in its complement as required. □

Notation 2.5. We refer to the quadruple $(Z, Y, X, \mathfrak{g}_0)$ in Lemma 2.4 as a reducing quadruple.

2.6. **Polarizations.** Denote by $\mathfrak{g}^*$ the dual Lie algebra. Each $\ell \in \mathfrak{g}^*$ determines a skew-symmetric bilinear form

$$\begin{array}{rcccc} \mathrm{B}_\ell & : & \mathfrak{g}\times\mathfrak{g} & \to & F \\ & & (X,Y) & \to & \ell([X,Y]). \end{array}$$

For any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ denote by

$$\mathfrak{r}_\ell(\mathfrak{h}) := \mathfrak{h} \cap \mathfrak{h}^{\perp_\ell} = \{h \in \mathfrak{h} \mid \ell([h,h']) = 0,\ \forall\, h' \in \mathfrak{h}\}$$

its *radical* with respect to B_ℓ.

Definition 2.7. *A subalgebra* $\mathfrak{m}_\ell \subset \mathfrak{g}$ *is called* polarizing *for* ℓ *if*

- $\mathfrak{m}_\ell$ *is isotropic for* B_ℓ*, that is* $\mathrm{B}_\ell(m,m') = 0$ *for all* $m, m' \in \mathfrak{m}_\ell$*;*
- $\mathfrak{m}_\ell$ *has maximal dimension among all subalgebras of* $\mathfrak{g}$ *isotropic for* B_ℓ*.*

Example 2.8. For the Heisenberg algebra $\mathfrak{h}_3$ and any ℓ with $\ell(Z) \neq 0$ a polarizing subalgebra is the ideal $\mathfrak{m}_\ell = \langle Z, Y\rangle$.

Remark 2.9. A polarizing algebra $\mathfrak{m}_\ell$ is not necessarily an ideal. An $\ell \in \mathfrak{g}^*$ can have many polarizing subalgebras. In general, there does not exist a finite set of subalgebras such that for each $\ell \in \mathfrak{g}^*$ one of the subalgebras in this set is polarizing for ℓ.

A *canonical* construction of a polarizing algebra (by Vergne [30]) goes as follows: fix a strong Malcev basis $(X_1, ..., X_n)$ for $\mathfrak{g}$. Put

$$\mathfrak{m}_\ell := \sum_{j=1}^{n} \mathfrak{r}_\ell(\mathfrak{g}_j),$$

where $\mathfrak{g}_j := \langle X_1, ..., X_j\rangle$ and $\mathfrak{r}_\ell(\mathfrak{g}_j)$ is the radical.

Alternatively, a polarizing subalgebra may be constructed inductively:

Case 1. If

$$\mathfrak{z}_\ell := \mathfrak{z}_\mathfrak{g} \cap \mathrm{Ker}(\ell) \neq 0$$

consider the projection

$$pr\,:\,\mathfrak{g} \to \mathfrak{g}_0 := \mathfrak{g}/\mathfrak{z}_\ell$$

and write ℓ_0 for the induced linear form on $\mathfrak{g}_0$. If $\mathfrak{m}_{\ell_0} \subset \mathfrak{g}_0$ is a polarizing algebra for ℓ_0 the preimage $pr^{-1}(\mathfrak{m}_{\ell_0})$ is a polarizing algebra for ℓ.

Case 2. Otherwise, $\mathfrak{z}(\mathfrak{g}) = \langle Z\rangle$ and $\ell(Z) \neq 0$. Then there exists a $Y \in \mathfrak{g}_2 \setminus \mathfrak{g}_1$ such that $\mathrm{codim}\, \mathfrak{z}_\mathfrak{g}(Y) = 1$ (by Lemma 2.4). Let ℓ_Y be the restriction of ℓ to $\mathfrak{z}_\mathfrak{g}(Y)$ and $\mathfrak{m}_Y$ a polarizing algebra for ℓ_Y in $\mathfrak{z}_\mathfrak{g}(Y)$. Then $\mathfrak{m}_\ell = \mathfrak{m}_Y$.

Proposition 2.10. *Let $Z \subset \mathfrak{g}^*$ be an algebraic variety, defined over F. There exists a Zariski open subset $Z^0 \subset Z$, a positive integer $k \leq \dim \mathfrak{g}$ and an F-morphism*

$$\mathrm{pol} \, : \, Z^0 \to \mathrm{Gr}(k, \mathfrak{g})$$

such that for every point ℓ in Z^0 the image $\mathrm{pol}(\ell)$ *in the Grassmannian of k-dimensional subspaces in $\mathfrak{g}$ corresponds to a polarizing subalgebra for ℓ.*

Proof. Consider $\mathfrak{g}^*$ over the function field of Z and apply Vergne's construction to the generic point.

Alternatively, consider the subvariety of all subalgebras $\mathfrak{m} \subset \mathfrak{g}$ over the function field $F(Z)$ of dimension k such that $\ell([m, m]) = 0$, with $\ell \in \mathfrak{g}^*(F(Z))$. Take the maximal k such that this variety has an $F(Z)$-rational point. This point defines an $F(Z)$-rational point in $\mathrm{Gr}(k, \mathfrak{g})$. Specializing, we get polarizations on some open subset $Z^0 \subset Z$. □

2.11. **Unipotent groups.** Let V be a finite dimensional vector space over F and $\mathrm{N} \subset \mathrm{GL}(V)$ the subgroup of all upper-triangular unipotent matrices. Denote by $\mathfrak{n}$ the F-vector space of all upper-triangular nilpotent matrices. The (standard) maps

$$\begin{array}{rccc} \exp : & \mathfrak{n} & \to & \mathrm{N} \\ \log : & \mathrm{N} & \to & \mathfrak{n} \end{array}$$

are biregular F-morphisms (polynomial maps) between algebraic varieties.

Let G be a (connected) unipotent linear algebraic group defined over F. Then there exists an F-rational representation

$$\rho_F \, : \, \mathrm{G} \to \mathrm{GL}(V),$$

for some V, realizing G as a closed subgroup of N. We fix this representation. Then

$$\mathfrak{g} := \log(\mathrm{G}) \subset \mathfrak{n}$$

is the Lie algebra of G. This coincides with the usual definition of $\mathfrak{g}$ as the F-algebra of left-invariant F-derivations of the algebra or rational functions $F[G]$.

3. Coadjoint orbits

3.1. **Orbits.** Both $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ are defined over F. For all fields E/F we can consider the E-rational points of $\mathfrak{g}$ and $\mathfrak{g}^*$, which we denote by $\mathfrak{g}(E)$, resp. $\mathfrak{g}^*(E)$.

Denote by Ad (resp. Ad^*) the adjoint (resp. coadjoint) action of G on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$), both are algebraic actions defined over F. Let $\mathcal{O}_\ell$ be the coadjoint orbit through $\ell \in \mathfrak{g}^*(F)$. It is a *symplectic* algebraic variety: the skew-symmetric bilinear form

$$\begin{array}{cccc} \mathrm{B}_\ell & : & \mathfrak{g} \times \mathfrak{g} & \to & F \\ & & (X,Y) & \mapsto & \ell([X,Y]) \end{array}$$

descends to a *nondegenerate* algebraic 2-form Ω_ℓ on the orbit $\mathcal{O}_\ell$.

Lemma 3.2. *The map*

$$\begin{array}{ccc} \mathrm{G} & \to & \mathcal{O}_\ell \\ g & \mapsto & \mathrm{Ad}^*(g)^{-1} \circ \ell \end{array}$$

induces an exact sequence

$$0 \to \mathfrak{r}_\ell \to \mathfrak{g} \xrightarrow{pr_\ell} \mathcal{T}_\ell(\mathcal{O}_\ell) \to 0,$$

where $\mathcal{T}_\ell$ is the tangent space at ℓ and $\mathfrak{r}_\ell$ is the radical of the skew-symmetric form B_ℓ.

Lemma 3.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with 1-dimensional center and reducing quadruple $(Z, Y, X, \mathfrak{g}_0)$. Let $\ell \in \mathfrak{g}^*$ be such that $\ell(Z) \neq 0$. Let*

$$pr : \mathfrak{g} \to \mathfrak{g}_0.$$

be the natural projection and ℓ_0 the restriction of ℓ to $\mathfrak{g}_0$. Then

$$pr(\mathcal{O}_\ell) = \sqcup_{t\in F} \mathrm{Ad}^* \exp(tX)(\mathcal{O}_{\ell_0})$$

and $pr^{-1}(\ell_0) = \ell$.

Proof. See [7], p. 69. □

3.4. **Basic invariant theory.** In this section we describe the geometric structure of the set of coadjoint orbits. The main result is the following

Proposition 3.5. *One has a decomposition*

$$\mathfrak{g}^* = \sqcup_{\sigma\in\Sigma} Z_\sigma$$

into a finite union of irreducible algebraic varieties $\{Z_\sigma\}_{\sigma\in\Sigma}$. For each Z_σ there exists a finite set of polynomials $\{P_{\sigma,j}\}_{j\in \mathsf{J}_\sigma}$ on $\mathfrak{g}^*$ separating the orbits: $P_{\sigma,j}$ are invariant on each orbit in Z_σ and for every pair $\ell, \ell' \in \mathfrak{g}^*$ contained in different orbits $\mathcal{O}, \mathcal{O}' \subset Z_\sigma$ there exist $j, j' \in \mathsf{J}_\sigma$ such that $P_{\sigma,j}(\ell) \neq P_{\sigma,j'}(\ell')$.

We follow the exposition in [25]. Let V be an irreducible algebraic variety over a field F. Denote by $F[V]$ the ring of regular functions on V and by $F(V)$ its function field. Let G be an algebraic group over F. A *regular* action of G on V is an F-homomorphism $\rho_{\mathrm{reg}} : \mathrm{G} \to \mathrm{Aut}(V)$ such that the induced map $\rho : \mathrm{G} \times V \to V$ is a morphism of algebraic varieties. A *rational* action is a homomorphism $\rho_{\mathrm{rat}} : \mathrm{G} \to \mathrm{Bir}(V)$ such that the induced map ρ is defined and coincides with some *rational* map ρ^0 on a dense Zariski open subset. An *orbit* $\mathcal{O}_\ell$ through a point $\ell \in V$ is the image in V of $\mathrm{G} \times \ell$ under ρ.

The action ρ_{rat} induces an action on $F(V)$. Regular functions $\phi \in F[V]$ satisfying $g \cdot \phi = \phi$ for all $g \in \mathrm{G}$ are called *integral* invariants for the action, rational functions $\phi \in F(V)$ with the same property are called *rational invariants.* Integral invariants form a subalgebra in $F[V]$, denoted by $F[V]^{\mathrm{G}}$, and rational invariants a field, $F(V)^{\mathrm{G}}$. A rational invariant ϕ separates the orbits $\mathcal{O}, \mathcal{O}'$ if ϕ is defined in the points of both orbits and if for all $\ell \in \mathcal{O}, \ell' \in \mathcal{O}'$ one has $\phi(\ell) \neq \phi(\ell')$. A set $\Phi = \{\phi\}$ of rational invariants separates generic orbits if there exists a Zariski open dense subset $V^0 \subset V$ with the property that for every pair of points $\ell, \ell' \in V^0$ contained in different orbits $\mathcal{O}, \mathcal{O}'$ there exists a rational invariant $\phi \in \Phi$ separating $\mathcal{O}, \mathcal{O}'$. In this case Φ generates the field of rational invariants $F(V)^{\mathrm{G}}$.

Theorem 3.6. *(Rosenlicht,* [27]*) For every (rational) action of an algebraic group* G *on an irreducible algebraic variety there exists a finite set* Φ *of rational invariants separating generic orbits.*

Theorem 3.7. *Assume that V is an affine algebraic variety and that* G *is a unipotent group acting on V. Then every rational invariant is representable as a quotient of integral invariants. In particular, there exists a finite set of integral invariants separating generic orbits.*

Proposition 3.5 follows: in $V = \mathfrak{g}^*$ we find a dense Zariski open subset V^0 and a set of integral invariants $\{P_{0,j}\}$ separating generic orbits. We can stratify the complement $V \setminus V^0$ into a finite disjoint union of irreducible affine algebraic varieties of smaller dimension and continue by induction.

3.8. **Parametrization.** In this section we make the parametrization of coadjoint orbits more explicit. This will be useful in Section 5.7, where we estimate certain multiplicities in terms of relative Pfaffians.

We follow the exposition in [7], Section 3. Consider the coadjoint action of G on the affine space $V = \mathfrak{g}^*$. Fix a strong Malcev basis

$(X_1, ..., X_n)$ for $\mathfrak{g}$, passing through the ideals $\mathfrak{g}_j$ of the ascending central series (see Section 2). The dual basis $(\ell_1, ..., \ell_n)$ is a Jordan-Hölder basis of V, for the coadjoint action of G.

Denote by V_j the F-span of $\{\ell_{j+1}, ..., \ell_n\}$. The canonical projection

$$pr_j : V \to V/V_j$$

is $\mathrm{Ad}^*(\mathrm{G})$-equivariant. For $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{N}^n$ consider the subset

$$Z_{\mathbf{d}} := \{v \in V \mid \dim \mathrm{Ad}^*(\mathrm{G})(pr_j(v)) = d_j,\ \forall j \in [1, ..., n]\}.$$

The set $\mathbf{D}$ of $\mathbf{d}$ with $Z_{\mathbf{d}} \neq \emptyset$ is finite and *partially* ordered: $\mathbf{d} \succeq \mathbf{d}'$ iff $d_j \geq d'_j$ for all $j \in [1, ..., n]$. It has a *unique* maximal element corresponding to $\mathbf{d}^{\max}$ with

$$d_j^{\max} = \max_{\mathbf{d} \in \mathbf{D}} \{d_j\}$$

for all $j \in [1, ..., n]$. Define

$$\mathbf{D}_1 := \{\mathbf{d}^{\max}\},\ \mathbf{D}_{k+1} := \{\mathbf{d} | \, \mathbf{d} \text{ maximal in } \mathbf{D} \setminus (\cup_{k' \leq k} \mathbf{D}_{k'})\}.$$

Fix an order $>_k$ in each $\mathbf{D}_k$. This gives an order $\succeq$ in $\mathbf{D}$:

$$\mathbf{D}_k \ni \mathbf{d} \succeq \mathbf{d}' \in \mathbf{D}_{k'}$$

if either $k < k'$ or (if $k = k'$) $\mathbf{d} >_k \mathbf{d}'$.

For $\mathbf{d} \in \mathbf{D}$ define

$$\begin{array}{llll} \mathsf{I}_{\mathbf{d}} := & \{i \,|\, d_i = d_{i-1} + 1\}, & V_{\mathbf{d}}^{\mathsf{I}} := & \langle \ell_i \mid i \in \mathsf{I}_{\mathbf{d}} \rangle_F \\ \mathsf{J}_{\mathbf{d}} := & \{j \,|\, d_j = d_{j-1}\}, & V_{\mathbf{d}}^{\mathsf{J}} := & \langle \ell_j \mid j \in \mathsf{J}_{\mathbf{d}} \rangle_F, \end{array}$$

(with $d_0 = 0$). We have a decomposition of the F-vector space

$$\mathfrak{g}^* = V_{\mathbf{d}}^{\mathsf{I}} \oplus V_{\mathbf{d}}^{\mathsf{J}}.$$

Remark 3.9. A posteriori, $\mathsf{I}_{\mathbf{d}}$ is always even.

Proposition 3.10. *The stratification of $V = \mathfrak{g}^*$ into strata $Z_{\mathbf{d}}$ satisfies the following properties:*

- *For all $\mathbf{d} \in \mathbf{D}$ the set $\cup_{\mathbf{d}' \succeq \mathbf{d}} Z_{\mathbf{d}}$ is an $\mathrm{Ad}^*(\mathrm{G})$-invariant subset of V.*
- *Each $\mathrm{Ad}^*(\mathrm{G})$-orbit in the stratum $Z_{\mathbf{d}}$ meets $V_{\mathbf{d}}^{\mathsf{J}}$ in exactly one point.*
- *$\Sigma_{\mathbf{d}} := Z_{\mathbf{d}} \cap V_{\mathbf{d}}^{\mathsf{J}}$ is algebraic (a locally closed subvariety of $\mathfrak{g}^*$).*
- *The union $\Sigma = \sqcup_{\mathbf{d}} \Sigma_{\mathbf{d}}$ parametrizes all $\mathrm{Ad}^*(\mathrm{G})$-orbits in $\mathfrak{g}^*$.*

Moreover, for each $\mathbf{d}$ with $\mathsf{I}_{\mathbf{d}} = \{i_1, ..., i_{2k}\}$ there exist rational functions $P_1, ..., P_n \in F(V \times V_{\mathbf{d}}^{\mathsf{I}})$ such that:

- *For each $r = 1, ..., n$ the restriction of P_r to $Z_{\mathbf{d}} \times V_{\mathbf{d}}^{\mathsf{I}}$ has no poles.*
- *For each $z \in Z_{\mathbf{d}}$ and each r the restriction of P_r to $z \times V_{\mathbf{d}}^{\mathsf{I}}$ is a polynomial and $P_r(z, v) = P_r(z', v)$ for all z' in the orbit $\mathrm{Ad}^*(\mathrm{G})(z)$ and all $v \in V_{\mathbf{d}}^{\mathsf{I}}$.*
- *The vector $w = \sum_{r=1}^{n} P_r(z; v)\ell_r$ is the unique vector in the $\mathrm{Ad}^*(\mathrm{G})$-orbit through z whose projection to $V_{\mathbf{d}}^{\mathsf{I}}$ is v.*

Proof. See [7], Section 3.1. □

3.11. **Pfaffians.** We fix a strong Malcev basis $(X_1, ..., X_n)$ for $\mathfrak{g}$. Denote by $(\ell_1, ..., \ell_n)$ the dual basis of $\mathfrak{g}^*$. Fix a stratum $Z = Z_{\mathbf{d}}$ with with $\mathsf{I}_{\mathbf{d}} = \{i_1, ..., i_{2k}\}$. Let $\ell \in Z$ and Ω_ℓ be the canonical symplectic 2-form on $\mathcal{O}_\ell$. Write

$$\Omega_\ell = 2 \cdot \sum_{i_r < i_{r'}} \mathrm{B}_\ell(X_{i_r}, X_{i_{r'}})\bar{\ell}_{i_r} \wedge \bar{\ell}_{i_{r'}}$$

and let $\mu(\ell) = \wedge^k \Omega_\ell$. Then

$$\mu(\ell) = 2^k k! \mathrm{Pf}(\ell) \cdot \bar{\ell}_{i_1} \wedge \cdots \wedge \bar{\ell}_{i_{2k}} \tag{3.1}$$

for some function $\mathrm{Pf}(\ell)$, called the *relative Pfaffian*. Clearly, $\mathrm{Pf}(\ell)$ is a sum of terms each of which is a product of factors of the form

$$\mathrm{B}_\ell(X_{i_r}, X_{i_{r'}})$$

and thus a *polynomial* function on $\mathfrak{g}^*$. Now we notice that the formula (3.1) is well defined for *any* $\ell \in \mathfrak{g}^*$.

For $\ell \in Z$, we have

$$\mu(\ell) = 2^k k! \det(M) \cdot \bar{\ell}_{i_1} \wedge \cdots \wedge \bar{\ell}_{i_{2k}}$$

which implies that on the stratum $\mathrm{Pf}(\ell) = \det(M)$ and

$$\mathrm{Pf}(\ell)^2 = \det(M)^2 = \det(\mathrm{B}_\ell(X_{i_r}, X_{i_{r'}})).$$

Since B_ℓ is non-degenerate on $\mathcal{O}_\ell$, $\mathrm{Pf}(\ell) \neq 0$. Further,

$$\mathsf{c}(\ell) = 2^k k! \mathrm{Pf}(\ell)^{-1}$$

and

$$\mu(\ell) = \mathsf{c}(\ell) \cdot \tilde{\mu}(\ell)$$

Thus, $\mathrm{Pf}(\ell)$ is $\mathrm{Ad}^*(\mathrm{G})$-invariant on $Z_{\mathbf{d}}$.

4. Integral structures

We will also need *integral* structures on all objects: $\mathfrak{g}$, strata defined in Sections 7.2 and 3.8, polarizing subalgebras etc. A precise choice of such structures is not essential for analytic considerations below; it suffices to observe that different choices of integral structures affect only finitely many places of F. In particular, they do not affect analytic properties of Euler products and height zeta functions.

Notation 4.1. Let F be a finite extension of $\mathbb{Q}$. Denote by F_v the completion of F with respect to a valuations v; for v nonarchimedean, denote by $\mathfrak{o}_v$ the ring of v-adic integers. Denote by $\mathbb{A} := \prod_v' F_v$ the ring of adeles of F.

4.2. **On $\mathfrak{g}$.** Let $\mathfrak{g} = \langle X_1, ..., X_n \rangle$ be an n-dimensional Lie algebra over F, with a fixed basis $\mathbf{X}$. Let $\mathfrak{g}_{\mathfrak{o}}'$ be the $\mathfrak{o}_F$-module

$$\mathfrak{o}_F X_1 + \cdots + \mathfrak{o}_F X_n.$$

There is an integer $a \in \mathbb{Z}$ such that $\mathfrak{g}_{\mathfrak{o}} = a \cdot \mathfrak{g}_{\mathfrak{o}}'$ is a *Lie order.* Indeed, write

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

with $c_{ij}^k \in F$. Then

$$[aX_i, aX_j] = \sum (ac_{ij}^k) aX_k$$

and we can choose $a \in \mathbb{Z}$ such that all $ac_{ij}^k \in \mathfrak{o}_F$.

Definition 4.3. *A Lie order $\mathfrak{g}_{\mathfrak{o}}$ is called* admissible *if $\exp(\mathfrak{g}_{\mathfrak{o}})$ is a subgroup of $\mathrm{G}(F)$.*

Assume that $\mathfrak{g}$ is nilpotent. By the Baker-Campbell-Hausdorff formula

$$X * Y := \log(\exp(X) \cdot \exp(Y)) = X + Y + \sum_{j=2}^{k} \mathrm{b}_j(X, Y)$$

where $X, Y \in \mathfrak{g}(F)$ and b_j is a sum of j-fold brackets with coefficients in F.

Definition 4.4. *A Lie order $\mathfrak{g}_{\mathfrak{o}}$ is called* universal *if $X, Y \in \mathfrak{g}_{\mathfrak{o}}$ implies that*

$$\mathrm{b}_j(X, Y) \in \mathfrak{g}_{\mathfrak{o}}$$

for all $j \geq 2$.

Clearly, a universal Lie order is admissible.

Lemma 4.5. *There exists an $a \in \mathbb{Z}$ such that $\mathfrak{g}_{\mathfrak{o}} = a \cdot \mathfrak{g}'_{\mathfrak{o}}$ is a universal order.*

Proof. For $X, Y \in \mathfrak{g}'_{\mathfrak{o}}$ we have

$$(aX) * (aY) = \sum \mathrm{b}_j(aX, aY) = aX + aY + \sum_{j \geq 2} \mathrm{b}_j(X, Y) a^j.$$

Now choose $a \in \mathbb{Z}$ such that for all $j \geq 2$ and a^{j-1} times every coefficient of b_j is in $\mathfrak{o}_F$. Then $a^{j-1}\mathrm{b}_j(X, Y) \in \mathfrak{g}'_{\mathfrak{o}}$ and

$$\mathrm{b}_j(aX, aY) \in a\mathfrak{g}'_{\mathfrak{o}}.$$

□

Example 4.6. For $\mathfrak{g} = \mathfrak{n}_4$ (from Example 2.2) we have

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]].$$

Then $6 \cdot \mathfrak{g}_{\mathbb{Z}}$ is universal.

Let v be a nonarchimedean valuation. Write

$$\mathfrak{g}(F_v) := \mathfrak{g} \otimes_F F_v, \;\; \mathfrak{g}(\mathfrak{o}_v) := \mathfrak{g}_{\mathfrak{o}} \otimes_{\mathbb{Z}} \mathfrak{o}_v.$$

If $\mathfrak{g}_{\mathfrak{o}}$ is universal then $\mathfrak{g}(\mathfrak{o}_v)$ is an admissible lattice in $\mathfrak{g}(F_v)$. In fact, if $o \in \mathfrak{o}_v$ then

$$\mathrm{b}_j(X \times o, Y \otimes o) = \mathrm{b}_j(X, Y) \otimes o^j \in \mathfrak{g}(\mathfrak{o}_v).$$

In particular, $\exp(\mathfrak{g}(\mathfrak{o}_v))$ is a subgroup of $\mathrm{G}(F)$.

We apply the preceding discussion as follows: Let $\mathfrak{g} = \mathfrak{g}_F$ be a nilpotent Lie algebra over a number field F. Fix a strong Malcev basis $(X_1, ..., X_n)$ of $\mathfrak{g}$. Choose an $a \in \mathbb{Z}$ such that $\mathfrak{g}_{\mathfrak{o}} = a \cdot \mathfrak{g}'_{\mathfrak{o}}$ is a universal order in $\mathfrak{g}_F$. Then $a\mathfrak{g}(\mathfrak{o}_v) := a\mathfrak{g}'_{\mathfrak{o}} \otimes_{\mathbb{Z}} \mathfrak{o}_v$ is an admissible lattice in $\mathfrak{g}_F$. Set $\mathrm{K}_v := \exp(a\mathfrak{g}(\mathfrak{o}_v)) \subset \mathrm{G}(F_v)$. This is a compact subgroup.

Remark 4.7. Once a representation $\rho_F : \mathrm{G} \to \mathrm{N}$ as in Section 2.11 is fixed, we have

$$\mathrm{K}_v = \mathrm{G}(F_v) \cap \mathrm{N}(\mathfrak{o}_v)$$

for almost all v (N is defined over $\mathbb{Z}$).

4.8. **Measures.** We fix Haar measures dx_v on $\mathbf{G}_a(F_v)$ for all v (normalized as in [29]). For all but finitely many v the volume of $\mathbf{G}_a(\mathfrak{o}_v)$ with respect to dx_v is equal to 1. Thus we have an induced Haar measure dx on $\mathbf{G}_a(\mathbb{A}) = \mathbb{A}$ and on $\mathbf{G}_a^n(\mathbb{A})$ for all n. Using the homeomorphism between $\mathrm{G}(\mathbb{A})$ and $\mathbf{G}_a^n(\mathbb{A})$ we may realize the Haar measure $dg = \prod_v dg_v$ on $\mathrm{G}(\mathbb{A})$ as the product measure

$$dg = dx_1 \dots dx_n.$$

Similarly, we have $\mathrm{vol}(\mathrm{G}(\mathfrak{o}_v) = 1$, for almost all v. Let $\mathfrak{m} \subset \mathfrak{g}$ be a subalgebra and $\mathrm{M} \subset \mathrm{G}$ the corresponding subgroup. The induced integral structure on $\mathfrak{m}$ allows us to obtain a normalized Haar measure on $\mathrm{M}(\mathbb{A})$, again we have $\mathrm{vol}(\mathrm{M}(\mathfrak{o}_v)) = 1$, for almost all v.

Notation 4.9. For each $\Sigma_{\mathbf{d}} \in \Sigma$ we fix a finite set of $\mathrm{Ad}^*(G)$-invariant polynomials $P_{\mathbf{d},j} \in F[\mathfrak{g}^*]$ separating the orbits, as in Proposition 3.10. Let $v \in S_\infty$, $\ell \in Z_{\mathbf{d}}(F)$ and $\mathcal{O}_\ell$ be the corresponding $\mathrm{Ad}^*(G)$-orbit in $\mathfrak{g}^*(F_v)$. Define the norm of the orbit

$$\|\mathcal{O}_\ell\|_\infty := \max_{v \in S_\infty} \max_{j \in \mathrm{J}_{\mathbf{d}}} |P_{\mathbf{d},j}(\ell)|_v. \tag{4.1}$$

A priori, the definition of this norm depends on the choice of $\mathrm{Ad}^*(G)$-invariant polynomials separating the orbits in Z_σ. However, for any $\mathrm{Ad}^*(G)$-invariant polynomial $P \in F[\mathfrak{g}^*]$ there exists an $N = N(P)$ such that

$$\max_{v \in S_\infty} |P(\ell')|_v \leq \|\mathcal{O}_\ell\|_\infty^N,$$

for all $\ell' \in \mathcal{O}_\ell$. In particular, a norm as in (4.1), defined via a different choice of polynomials separating the orbits in $Z_{\mathbf{d}}$, will be comparable, up to powers. We have a fundamental finiteness result: let $\mathfrak{l} \subset \mathfrak{o}_F$ be any lattice. Then there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{\ell \in \mathfrak{g}^*(\mathfrak{l})/\mathrm{Ad}^*(G)} \|\mathcal{O}_\ell\|_\infty^{-n},$$

is convergent for all $n \geq n_0$.

5. Representations: basics

In this section we describe Kirillov's orbit method in the theory of unitary representations of nilpotent groups over local fields and its generalization to adeles by Moore (see [17] and [21]).

5.1. **The orbit method.**

Notation 5.2. Let F be a number field, v a valuation and F_v the v-adic completion of F. Denote by $\mathfrak{m}_v \subset \mathfrak{o}_v$ the maximal ideal in the ring of $\mathfrak{o}_v$ of v-adic integers (for nonarchimedean v). We write $\mathbf{k} = \mathbf{k}_v$ for the residue field of $\mathfrak{o}_v$ and $q = q_v$ for the cardinality of $\mathbf{k}_v$. We denote by $\mathrm{Val}(F) = \{|\cdot|_v\} = S_{\mathrm{fin}} \cup S_\infty$ the set of all valuations of F, here S_∞ is the set of archimedean and S_{fin} the set of nonarchimedean valuations. We normalize the valuations in such a way that for any Haar measure μ_v on F_v one has $\mu_v(a\mathcal{M}) = |a|_v \mu_v(\mathcal{M})$ for all measurable subsets $\mathcal{M} \subset F_v$ and all $a \in F_v^*$. We continue to denote by $\mathbb{A} = \mathbb{A}_F$ the adele ring of F.

For any finite set $S \subset \mathrm{Val}(F)$ we put $\mathbb{A}_S = \prod_{v\in S} F_v$, $\mathbb{A}^S = \prod'_{v\notin S} F_v$ (restricted product). We abbreviate $\mathbb{A}_{\mathrm{fin}} = \mathbb{A}^{S_\infty}$ and $\mathbb{A}_\infty = \mathbb{A}_{S_\infty}$.

First we recall basic facts concerning harmonic analysis on additive groups (cf., for example, [29]). For any prime number p, we have an embedding $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Using it we can define a (unitary) character ψ_p of the additive group $\mathbf{G}_a(\mathbb{Q}_p)$ by

$$\psi_p\,:\, x_p \mapsto \exp(2\pi i x_p).$$

At the infinite place of $\mathbb{Q}$ we put

$$\psi_\infty\colon x_\infty \mapsto \exp(-2\pi i x_\infty),$$

(here x_∞ is viewed as an element in $\mathbb{R}/\mathbb{Z}$). Taking the product we get a character ψ of $\mathbf{G}_a(\mathbb{A}_{\mathbb{Q}})$ and, by composition with the trace, a character $\psi = \psi_1 = \prod_v \psi_v$ of $\mathbf{G}_a(\mathbb{A})$. This defines a Pontryagin duality

$$\mathbf{G}_a(\mathbb{A}) \to \mathbf{G}_a(\mathbb{A})^*$$
$$(a_v) \mapsto ((x_v) \to \prod_v \psi_v(a_v x_v)).$$

The subgroup $\mathbf{G}_a(F) \subset \mathbf{G}_a(\mathbb{A})$ is discrete, cocompact and selfdual under the above duality.

Denote by $\mathfrak{g}^*$ the dual to the Lie algebra $\mathfrak{g}$ of G. It inherits the F-rational structure from $\mathfrak{g}$. For every F-rational linear form $\ell \in \mathfrak{g}^*(F)$ let $\mathfrak{m}_\ell$ be a maximal subordinate to ℓ subalgebra of $\mathfrak{g}$ (see Section 2). Then ℓ defines a character on the adelic points $\mathrm{M}_\ell(\mathbb{A})$ of the subgroup $\mathrm{M}_\ell = \exp(\mathfrak{m}_\ell) \subset \mathrm{G}$

$$\psi_\ell = \psi_1 \circ \ell \circ \log\,:\, \mathrm{M}_\ell(\mathbb{A}) \to \mathbb{S}^1 \subset \mathbb{C}^*.$$

Let

$$\pi_\ell = \mathrm{Ind}_{\mathrm{M}_\ell(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\psi_\ell)$$

be the induced representation. Then

- π_ℓ is irreducible;
- π_ℓ does not depend on the choice of $\mathfrak{m}_\ell$ (up to isomorphy);
- π_ℓ does not depend on the choice of ℓ in the Ad^*-orbit $\mathcal{O}_\ell$ (up to isomorphy).

This is the orbit picture proposed by Kirillov: irreducible unitary representations of a unipotent group G are parametrized by orbits of the coadjoint action of G on $\mathfrak{g}^*(F)$.

Notation 5.3. Let $\mathrm{G}_\infty := \prod_{v\in S_\infty} \mathrm{G}(F_v)$ and Γ be a discrete cocompact subgroup in G_∞ (e.g., the image of $\mathrm{G}(\mathfrak{l})$, where $\mathfrak{l} \subseteq \mathfrak{o}_F$ is a sublattice).

Notation 5.4. Denote by

$$\mathcal{H} := \mathsf{L}^2(\mathrm{G}(F)\backslash \mathrm{G}(\mathbb{A}))$$

the space of (left) $\mathrm{G}(F)$-invariant square-integrable (on the quotient) functions on $\mathrm{G}(\mathbb{A})$ (and similarly $\mathsf{L}^2(\Gamma\backslash \mathrm{G}_\infty)$). Denote by

$$\mathcal{H}^{\mathrm{K}} = \mathsf{L}^2(\mathrm{G}(F)\backslash \mathrm{G}(\mathbb{A}))^{\mathrm{K}}$$

the subspace of (right) K-fixed vectors.

Proposition 5.5. [21] *For each $\ell \in \mathfrak{g}^*$ there exists an isometry*

$$\begin{array}{cccl} j_\ell & : & \pi_\ell & \to \varrho_\ell \\ j_\ell & : & \mathrm{Ind}_{\mathrm{M}_\ell(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\psi_\ell) & \to \mathcal{H}_\ell \subset \mathcal{H} \\ & & \phi(x) & \mapsto \sum_{\gamma \in \mathrm{M}_\ell(F)\backslash \mathrm{G}(F)} \phi(\gamma \cdot x), \end{array}$$

where $(\varrho_\ell, \mathcal{H}_\ell)$ is a unitary irreducible representation occurring in the right regular representation of $\mathrm{G}(\mathbb{A})$ on $\mathcal{H}$. Moreover, j_ℓ induces an isometry on the subspaces of K*-fixed vectors and*

$$\mathcal{H}^{\mathrm{K}} = \oplus_{\ell \in \mathfrak{g}^*(F)/\mathrm{Ad}^*} \mathcal{H}_\ell^{\mathrm{K}},$$

as a direct sum of irreducible unitary representations of $\mathrm{G}(\mathbb{A})$, each occurring with multiplicity one.

Notation 5.6. We denote by $\mathcal{R}(\mathrm{K}) = \{\varrho\}$ the set of irreducible unitary representation of $\mathrm{G}(\mathbb{A})$ occcuring in $\mathcal{H}^{\mathrm{K}}$.

5.7. **Multiplicities.** Let F be a number field and $S_G \subset S_{\mathrm{fin}}$ a finite set of nonarchimedean valuations of F such that for all $v \notin S_G$ one has

$$\exp(\mathfrak{g}(\mathfrak{o}_v)) = \mathrm{G}(\mathfrak{o}_v) = \mathrm{K}_v.$$

For $v \in S$ the compacts K_v are defined as in Section 4.

Notation 5.8. Let π_v be a unitary representation of $\mathrm{G}(F_v)$. We denote by

$$m(\pi_v, \mathrm{K}_v, \mathbf{1})$$

the multiplicity of the trivial representation $\mathbf{1}$ occurring in the restriction of π_v to K_v.

Proposition 5.9. *Let $\ell \in \mathfrak{g}^*(F_v)$ and π_v be the unitary irreducible representation of $\mathrm{G}(F_v)$ corresponding to the orbit $\mathcal{O}_\ell$. Then there exist constants c_v, with $\mathsf{c}_v = 1$ for all $v \notin S_G$ such that*

$$m(\pi_v, \mathrm{K}_v, \mathbf{1}) \leq \mathsf{c}_v |\mathrm{Pf}(\ell)|_v^{-1},$$

where $\mathrm{Pf}(\ell)$ is the Pfaffian (defined in Section 3.11), evaluated at $\mathcal{O}_\ell$.

Notation 5.10. Let S_π be the set of nonarchimedean v such that either $v \in S_G$ or $m(\pi_v, \mathrm{K}_v, \mathbf{1}) \neq 1$.

Remark 5.11. In [15], [26], [11] one can find bounds for $m(\pi_v, \mathrm{K}_v, \mathbf{1})$ in terms of the the number of $\mathrm{Ad}^*(\mathrm{K}_v)$-orbits on $\mathcal{O}_\ell(\mathrm{K}_v)$. An estimate in terms of Pfaffians has been derived in [8].

For completeness, we include a proof of this proposition.

We choose the Haar measure dg_v on $\mathrm{G}(F_v)$ so that

$$\int_{\mathrm{K}_v} dg_v = 1.$$

We normalize the measure dX_v on $\mathfrak{g}(F_v)$ such that

$$\int_{\mathrm{G}(F_v)} \phi(g_v) dg_v = \int_{\mathfrak{g}(F_v)} \phi(\exp(X_v)) dX_v$$

for all smooth compactly supported functions $\phi \in \mathsf{C}_c^\infty(\mathrm{G}(F_v))$.

For $\ell = \ell_v \in \mathfrak{g}^*(F_v)$ denote by $d\mu(\ell) = d\mu_v(\ell)$ the (canonical) v-adic measure on the ($2k$-dimensional) orbit $\mathcal{O}_\ell(F_v) \subset \mathfrak{g}^*(F_v)$ associated with the top degree form $\mu(\ell) := \wedge^k \Omega_\ell$, where Ω_ℓ is the *canonical* invariant algebraic 2-form on $\mathcal{O}_\ell$.

For appropriate functions ϕ on $\mathrm{G}(F_v)$ define the Fourier transform

$$\hat{\phi}(\ell) := \int_{\mathfrak{g}(F_v)} \phi(\exp(X_v)) \psi(\langle \ell, X_v \rangle) dX_v.$$

Let π_v be an irreducible unitary representation of $\mathrm{G}(F_v)$. The function ϕ defines the operator

$$\pi_v(\phi) := \int_{\mathrm{G}(F_v)} \phi(g_v) \pi_v(g_v) dg_v.$$

It is of trace class.

Lemma 5.12. *Let v be a nonarchimedean valuation and π_v an irreducible unitary representation of $\mathrm{G}(F_v)$ corresponding to $\ell \in \mathfrak{g}^*(F_v)$. Let $\phi \in \mathsf{C}_c^\infty(\mathrm{G}(F_v))$. Then*

$$\operatorname{tr} \pi_v(\phi) = \frac{1}{2^k k!} \int_{\mathcal{O}_\ell} \hat{\phi}(\ell) d\mu_v(\ell)$$

Proof. See [7], p. 145. □

Let $\chi = \chi_v$ be the characteristic function of K_v. Then the convolution $\chi * \chi = \chi$. Thus, $\pi_v(\chi)$ is a self-adjoint projection on $\mathcal{H}_{\pi_v}$. Moreover,

$$\operatorname{tr}(\pi_v(\chi)) = m(\pi_v, \mathrm{K}_v, \mathbf{1})$$

and

$$\hat{\chi}(\ell_v) = \int \chi_0(X_v)\psi(\langle \ell_v, X_v\rangle)dX_v$$

where χ_0 is the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$. It follows that $\hat{\chi}$ is the characteristic function of the dual lattice $\mathfrak{g}^*(\mathcal{O}_v) \subset \mathfrak{g}^*(F_v)$. Therefore,

$$m(\pi_v, \mathrm{K}_v, \mathbf{1}) = \frac{1}{2^k k!}\int_{\mathcal{O}_\ell} \hat{\chi}(\ell)d\mu_v(\ell), \tag{5.1}$$

where $d\mu_v(\ell)$ is the canonical measure on the orbit $\mathcal{O}(\ell)$.

The Lie algebra $\mathfrak{g}$ is equipped with a fixed strong Malcev basis $\langle X_1, ..., X_n\rangle$. The dual basis $\langle \ell_1, ..., \ell_n\rangle$ in $V := \mathfrak{g}^*$ is a Jordan-Hölder basis. Recall the stratification of representations explained in Section 3.8. Assume that the representation π_v belongs to the stratum $\Sigma_{\mathbf{d}}$ (as in Section 3.8). Denote by

$$V_{\mathsf{I}}^{\mathbf{d}} := \langle \ell_i; i \in \mathsf{I}_{\mathbf{d}}\rangle_{F_v}, \;\; V_{\mathsf{J}}^{\mathbf{d}} := \langle \ell_j; j \in \mathsf{J}_{\mathbf{d}}\rangle_{F_v}$$

the affine subspaces defined in 3.10. Recall that $V_{\mathsf{I}}^{\mathbf{d}} = \langle \ell_{i_1}, ..., \ell_{i_{2k}}\rangle$ is even dimensional. Regarding $X_{i_1}, ..., X_{i_{2k}}$ as (independent) linear forms on $V_{\mathsf{I}}^{\mathbf{d}}$ define a Haar measure on $V_{\mathsf{I}}^{\mathbf{d}}$ by

$$|d\mu| := |X_{i_1} \wedge \cdots \wedge X_{i_{2k}}|.$$

Let

$$\begin{array}{cccc} f_\ell & : & V_{\mathsf{I}}^{\mathbf{d}} & \rightarrow & \mathcal{O}_\ell \\ & & u & \mapsto & (u, P_\ell(u)) \end{array}$$

be the map parametrizing the orbit $\mathcal{O}_\ell$ (here P_ℓ is a polynomial on $V_{\mathsf{I}}^{\mathbf{d}}$). Then

$$\tilde{\mu}(\ell) := (df_\ell^*)^{-1}(X_{i_1} \wedge \cdots \wedge X_{i_{2k}})$$

is a G-invariant volume form on the orbit $\mathcal{O}_\ell$. Therefore,

$$\mu(\ell) = \mathsf{c}(\ell)\tilde{\mu}(\ell)$$

(and $\mathsf{c}(\ell)$ depends only on the orbit $\mathcal{O}_\ell$). Denote by $\mathfrak{o}_{v,\mathsf{I}}^*$ the image of the projection

$$pr_{v,\mathsf{I}} : \mathfrak{g}^*(\mathfrak{o}_v) \rightarrow V_{v,\mathsf{I}}$$

and by $\chi_{v,\mathsf{I}}^*$ the characteristic function of this set. Continuing from (5.1), we obtain

$$m(\pi_v, \mathrm{K}_v, \mathbf{1}) = \frac{|\mathsf{c}(\ell)|}{2^k k!}\int_{\mathcal{O}_\ell} \hat{\phi}(\ell)|d\tilde{\mu}_v(\ell)| \tag{5.2}$$

$$= \frac{|\mathsf{c}(\ell)|}{2^k k!}\int_{V_{v,\mathsf{I}}} \hat{\phi}(u, P_\ell(u))|d\mu_v(\ell)| \tag{5.3}$$

$$\leq \frac{|\mathsf{c}(\ell)|}{2^k k!}\int_{V_{v,\mathsf{I}}} \chi_{v,\mathsf{I}}^*(u)|d\mu_v(\ell)| \tag{5.4}$$

We fix an integer $a \in \mathbb{Z}$ so that $\mathfrak{g}_{\mathfrak{o}}$ is equal to the $\mathfrak{o}_F$-span of $\{aX_1, ..., aX_n\}$ (replacing X_j by $a^{-1}X_j$ we may assume from the beginning that

$$\mathfrak{g}_{\mathfrak{o}} = \mathfrak{o}_F X_1 \oplus ... \oplus \mathfrak{o}_F X_n.$$

Then

$$\begin{array}{rcl} \mathfrak{g}^*(\mathfrak{o}_v) & = & \mathfrak{o}_v \ell_1 \oplus ... \oplus \mathfrak{o}_v \ell_n \\ \mathfrak{o}^*_{v,\mathsf{I}} & = & \mathfrak{o}_v \ell_{i_1} \oplus ... \oplus \ell_{i_{2k}}. \end{array}$$

The Haar measure is normalized such that $\mathrm{vol}(\mathfrak{o}_v) = 1$. Then

$$\int_{V_{v,\mathsf{I}}} \chi^*_{v,\mathsf{I}}(u)|d\mu_v(\ell)| = 1.$$

It remains to observe that

$$\mathsf{c}(\ell) = \mathrm{Pf}(\ell)^{-1}.$$

5.13. **Spherical functions.**

Proposition 5.14. *For $\ell \in \mathfrak{g}^*(F)$ let $\mathfrak{m}_\ell \subset \mathfrak{g}$ be a polarizing subalgebra, $\mathrm{M} = \mathrm{M}_\ell = \exp(\mathfrak{m}_\ell)$, and $\psi = \psi_\ell = \psi_1 \circ \ell$ the corresponding adelic character of $\mathrm{M}(\mathbb{A})$. Let $\mathcal{H}_\ell$ be the associated irreducible unitary representation of $\mathrm{G}(\mathbb{A})$. Let K be as in Section 8 and assume that $\mathcal{H}_\ell^{\mathrm{K}} \neq 0$. Then for all $\omega \in \mathcal{H}_\ell^{\mathrm{K}}$, with $\|\omega\|_{\mathsf{L}^2} = 1$, all $v \notin S_{\varrho_\ell}$ and all (integrable) functions H_v on $\mathrm{G}(F_v)$ such that*

$$H_v(k_v g_v) = H_v(g_v),$$

for all $k_v \in \mathrm{K}_v, g_v \in \mathrm{G}(F_v)$, one has

$$\int_{\mathrm{G}(F_v)} H_v(g_v)\omega_v(g_v)dg_v = \int_{\mathrm{M}(F_v)} H_v(h_v)\psi_v(h_v)dh_v,$$

(where dh_v is normalized as in Section 4.8).

Proof. Define the function

$$\tilde{\psi} = \prod_v \tilde{\psi}_v \in \mathrm{Ind}_{\mathrm{M}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\psi) =: \pi$$

as follows:

$$\begin{array}{rcll} \tilde{\psi}_v(g_v) & = & 0 & \text{if } g_v \notin \mathrm{M}(F_v)\mathrm{K}_v \\ \tilde{\psi}_v(h_v k_v) & = & \psi_v(h_v) & \text{otherwise} \end{array}$$

For all $v \notin S_\pi$ we have

$$\psi_v|_{\mathrm{M}(F_v)\cap \mathrm{K}_v} = 1.$$

By definition, for $v \notin S_\pi$ the representation π_v has a unique K_v-fixed vector (of norm 1). A direct computation shows that

$$\|\tilde{\psi}\|_{\mathsf{L}^2(\mathrm{M}(\mathbb{A})\backslash \mathrm{G}(\mathbb{A}))} = 1.$$

Therefore, the (local) spherical function φ_v (normalized by $\varphi_v(e)=1$) is given by

$$\varphi_v(g_v) = \langle \pi_v(g_v)\tilde{\psi}_v, \tilde{\psi}_v\rangle.$$

Now we compute:

$$\begin{aligned}
\int_{\mathrm{G}(F_v)} H_v(g_v)\varphi_v(g_v)dg_v &= \int_{\mathrm{G}(F_v)}\int_{\mathrm{K}_v\cap \mathrm{M}(F_v)\backslash \mathrm{K}_v} H_v(k_vg_v)\tilde{\psi}_v(k_vg_v)dk_v'dg_v \\
&= \int_{\mathrm{K}_v\cap \mathrm{M}(F_v)\backslash \mathrm{K}_v} dk_v' \int_{\mathrm{G}(F_v)} H_v(k_vg_v)\tilde{\psi}_v(k_vg_v)dg_v \\
&= \mathrm{vol}\cdot \int_{\mathrm{K}_v\backslash \mathrm{G}(F_v)} H_v(g_v)\tilde{\psi}_v(g_v)dg_v \\
&= \mathrm{vol}\cdot \int_{\mathrm{M}(F_v)\cap \mathrm{K}_v\backslash \mathrm{H}(F_v)} H_v(h_v)\tilde{\psi}_v(h_v)dh_v \\
&= \mathrm{vol}\cdot \int_{\mathrm{M}(F_v)} H_v(h_v)\psi_v(h_v)dh_v.
\end{aligned}$$

Here dk_v' is the induced measure and $\mathrm{vol} = \mathrm{vol}(\mathrm{K}_v\cap \mathrm{M}(F_v)\backslash \mathrm{K}_v)$. □

6. Universal enveloping algebra

Now we turn to archimedean places.

6.1. **Basics.** Let

$$\mathfrak{T}(\mathfrak{g}) := \oplus_{j\ge 0}\mathfrak{g}^{\otimes j}$$

be the tensor algebra,

$$\mathfrak{S}(\mathfrak{g}) := \mathfrak{T}(\mathfrak{g})/\langle X\otimes Y - Y\otimes X\rangle$$

the symmetric algebra and

$$\mathfrak{U}(\mathfrak{g}) := \mathfrak{T}(\mathfrak{g})/\langle X\otimes Y - Y\otimes X - [X,Y]\rangle$$

the universal enveloping algebra of $\mathfrak{g}$. There is an injective map

$$\mathfrak{g}\to \mathfrak{T}(\mathfrak{g})\to \mathfrak{U}(\mathfrak{g})$$

and a $\mathfrak{g}$-module isomorphism (symmetrization)

$$\mathrm{sym}\,:\,\mathfrak{S}(\mathfrak{g})\to \mathfrak{U}(\mathfrak{g}),$$

which is defined on monomials by

$$Y_1\cdots Y_r \mapsto \frac{1}{r!}\sum_{\sigma\in \mathbb{S}_r} Y_{\sigma(1)}\cdots Y_{\sigma(r)}$$

(where $\mathbb{S}_n$ is the symmetric group). Each $Y\in \mathfrak{g}$ defines a differential operator

$$\partial_Y\,:\,f(g)\mapsto \frac{\mathrm{d}}{\mathrm{d}t}f(g\cdot \exp(tY))|_{t=0}$$

on smooth functions on $\mathrm{G}(F_v)$, for any archimedean v. This gives a surjective algebra homomorphism from $\mathfrak{U}(\mathfrak{g})$ onto the algebra of left-invariant differential operators on $\mathsf{C}^\infty(\mathrm{G}(F_v))$. In particular, $\mathfrak{U}(\mathfrak{g})$ acts in the space of smooth vectors of every irreducible unitary representation $(\varrho, \mathcal{H})$ of $\mathrm{G}(F_v)$. For $\partial \in \mathfrak{U}(\mathfrak{g})$ we will denote by $\varrho(\partial)$ the corresponding operator.

We will use the canonical identification $\mathfrak{S}(\mathfrak{g}) = F[\mathfrak{g}^*]$ (by duality):

$$Y \to f_Y \in F[\mathfrak{g}^*], \;\; f_Y(\ell) = \ell(Y); \;\; Y \in \mathfrak{g}, \ell \in \mathfrak{g}^*.$$

The adjoint action of G on $\mathfrak{g}$ extends to $\mathrm{Aut}_F(\mathfrak{U}(\mathfrak{g}))$ and $\mathrm{Aut}_F(\mathfrak{S}(\mathfrak{g}))$.

Lemma 6.2. *The symmetrization* sym *is equivariant with respect to the adjoint action of* G *and maps the space of* Ad^**-invariant polynomials on* $\mathfrak{g}^*$ *bijectively onto the center* $\mathfrak{Z}\mathfrak{U}(\mathfrak{g})$ *of* $\mathfrak{U}(\mathfrak{g})$.

6.3. **Scalar operators.**

Proposition 6.4. *Let v be an archimedean valuation, $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit and*

$$(\varrho_\ell, \mathcal{H}_\ell) \sim \mathrm{Ind}_{\mathrm{M}_\ell(F_v)}^{\mathrm{G}(F_v)}(\psi_\ell)$$

for some $\ell \in \mathcal{O}$ and some polarizing M_ℓ. *For $P \in F[\mathfrak{g}^*]$ let $\partial_P \in \mathfrak{U}(\mathfrak{g})$ be the corresponding differential operator. Assume that the restriction of P to $\mathcal{O}$ is identically constant*

$$P(2\pi i \ell) = P(2\pi i \ell')$$

for all $\ell, \ell' \in \mathcal{O}$. Then

$$\varrho_\ell(\partial_P) f = P(2\pi i \ell) \cdot f$$

for all smooth vectors $f \in \mathcal{H}_\ell$ and all $\ell \in \mathcal{O}$.

In particular, let $\partial_z \in \mathfrak{Z}\mathfrak{U}(\mathfrak{g})$ and P_z be the corresponding Ad^**-invariant polynomial (see Lemma 6.2). Then, for all orbits $\mathcal{O}$ and all $\ell \in \mathcal{O}$, the operator $\varrho_\ell(\partial_z)$ acts in $\mathcal{H}_\ell$ by multiplication by*

$$P_z(2\pi i \ell).$$

Proof. We follow closely the exposition in [7], p. 186. The proof proceeds by induction on the dimension of $\mathfrak{g}$. We explain the case when v is real, the complex places being similar.

Assume that there is a nontrivial ideal $\mathfrak{z}_0 \subset \mathfrak{z}_{\mathfrak{g}}$ such that ϱ_ℓ restricted to $\exp(\mathfrak{z}_0)$ is trivial. Consider the projections

$$\begin{array}{lccl} pr: & \mathfrak{g} & \to & \mathfrak{g}_0 := \mathfrak{g}/\mathfrak{z}_0, \\ pr: & \mathrm{G} & \to & \mathrm{G}_0. \end{array}$$

The induced injection

$$in\,:\,\mathfrak{g}_0^* \hookrightarrow \mathfrak{z}_0^\perp \subset \mathfrak{g}^*$$

maps $\mathcal{O}$ isomorphically onto $\mathcal{O}_0$. The maps are equivariant with respect to the (co)adjoint actions of G and G_0. They extend naturally to symmetric algebras, universal enveloping algebras and polynomial functions. In particular,

$$pr\,:\,F[\mathfrak{g}^*] \to F[\mathfrak{g}_0^*]$$

is simply the restriction of the polynomial $P \in F[\mathfrak{g}^*]$ to $\mathfrak{z}_0^\perp = \mathfrak{g}_0^*$. The representations ϱ_ℓ and $\varrho_{\ell_0} = \varrho_\ell \circ pr$ correspond to the same orbit

$$\mathcal{O} = \mathcal{O}_0 \subset \mathfrak{g}_0^* \subset \mathfrak{g}^*.$$

We have an equivariant commutative diagram

$$\begin{array}{ccccc} F[\mathfrak{g}^*] & \simeq & \mathfrak{S}(\mathfrak{g}) & \xrightarrow{\text{sym}} & \mathfrak{U}(\mathfrak{g}) \\ \downarrow & & \downarrow & & \downarrow \\ F[\mathfrak{g}_0^*] & \simeq & \mathfrak{S}(\mathfrak{g}_0) & \xrightarrow{\text{sym}} & \mathfrak{U}(\mathfrak{g}_0). \end{array}$$

Let $\ell \in \mathfrak{g}^*$, let M_ℓ be a polarizing subgroup for ℓ, and ϱ_ℓ the corresponding irreductible unitary representation of $\mathrm{G}(F_v)$ as in Section 5. We have

$$\varrho_\ell(\partial_P) = \varrho_{\ell_0}(\partial pr(P)) = pr(P)(2\pi i \ell_0) \cdot \mathrm{Id}_0 = P(2\pi i \ell) \cdot \mathrm{Id},$$

as claimed.

Now assume that $\dim \mathfrak{z}_{\mathfrak{g}} = 1$ and that ϱ_ℓ is nontrivial on $\mathfrak{g}$ ($\ell(Z) \neq 0$). Choose a reducing quadruple as in Kirillov's lemma:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus FX$$

(2.5). This time we have an injection

$$in\,:\,\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$$

and an induced projection

$$pr\,:\,\mathfrak{g}^* \to \mathfrak{g}_0^*.$$

We have

$$\ell_0 := pr(\ell) = \ell|_{\mathfrak{g}_0}.$$

By Lemma 3.3, we have

$$pr(\mathcal{O}_\ell) = \sqcup_{t \in \mathbb{R}} \mathcal{O}_{\ell_t},$$

where

$$\mathcal{O}_{\ell_t} := \{\mathrm{G}_0 \cdot \ell_t\}\mathrm{Ad}^*(\exp(tX))(\ell_0).$$

The Ad^*-invariance of P at $\ell \in \mathcal{O}$ implies that the restriction of P to $\mathcal{O}$ does not depend on X. In particular, the restriction of P to each $\mathcal{O}_{\ell_t}$ is invariant under the adjoint action of G_0.

We have a direct integral decomposition

$$\varrho = \int_{\mathbb{R}}^{\oplus} \varrho_t dt,$$

where ϱ_t is the unitary irreducible representation of $\mathrm{G}_0(F_v)$ associated to the orbit $\mathcal{O}_{\ell_t}$. This decomposition passes to smooth vectors (see [7], p. 188). By the induction hypothesis,

$$\begin{array}{rcl} \varrho_t(pr(P))f & = & pr(P)(2\pi i\ell_t)f \\ & = & pr(P)(2\pi i(\mathrm{Ad}^*(\exp(tX))\ell_0))f \\ & = & P(2\pi i(\mathrm{Ad}^*(\exp(tX))\ell))f \\ & = & P(2\pi i\ell)f \end{array}$$

for all smooth vectors f in the representation space of ϱ_t, since the projection $pr : \mathfrak{g}^* \to \mathfrak{g}_0^*$ is equivariant for the coadjoing action of G, resp. G_0, and P is invariant. Further, since $\varrho_0(pr(P)) = \varrho(P)$, as elements in $\mathfrak{U}(\mathfrak{g})$, $\varrho(P)$ is determined by the restriction of ϱ to G_0 and

$$\varrho(P) = \int_{\mathbb{R}}^{\oplus} \varrho_t(pr(P))dt = P(2\pi i\ell)\mathrm{Id}.$$

□

7. Geometry

Here we work over an arbitrary field F of characteristic zero.

Notation 7.1. For a smooth projective algebraic variety X over F we denote by $\mathrm{Pic}(X)$ its Picard group and by $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ the (closed) cone of pseudo-effective divisors on X. We will often identify line bundles, the corresponding divisors and their classes in $\mathrm{Pic}(X)$. We write $\mathcal{L} = (L, \|\cdot\|)$ when we want to emphasize that the line bundle L is adelically metrized. If X has an action by a group G we write $\mathrm{Pic}^{\mathrm{G}}(X)$ for the group of isomorphism classes of G-linearized line bundles on X.

7.2. **Main invariants.** Let X be a smooth projective variety with $-K_X$ contained in the interior of the effective cone. Then $\Lambda_{\mathrm{eff}}(X)$ is a rational finitely generated cone, by [4]. In this case, given a line bundle L on X, we let

$$a(L) := \inf\{a \,|\, a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)$$

and $b(L)$ be the codimension of the face of $\Lambda_{\rm eff}(X)$ containing $a(L)[L]+[K_X]$. When X is *singular* and $\rho : \tilde{X} \to X$ a desingularization satisfying the conditions above, we can define

$$a(L) := a(\rho^*(L)),\ \ b(L) := b(\rho^*(L)).$$

These invariants are well-defined (see [13] for more details).

Proposition 7.3. *Let X be a smooth projective equivariant compactification of a unipotent algebraic group* G. *Let $D := X \setminus \mathrm{G}$ be the boundary and $(D_\alpha)_{\alpha\in\mathcal{A}}$ the set of its irreducible components. Then*

- $\mathrm{Pic}^{\mathrm{G}}(X) = \mathrm{Pic}(X)$*;*
- $\mathrm{Pic}(X)$ *is freely generated by the classes D_α;*
- $\Lambda_{\rm eff}(X) = \oplus_\alpha \mathbb{R}_{\geq 0} D_\alpha$*;*
- $-K_X = \sum_\alpha \kappa_\alpha D_\alpha$ *with $\kappa_\alpha \geq 2$ for all $\alpha \in \mathcal{A}$.*

Proof. Analogous to the proofs in Section 2 of [14]. In particular, it suffices to assume that X carries only a one-sided action of G. Notice that every line bundle admits a unique G-linearization. □

Corollary 7.4. *The divisor of every irreducible polynomial*

$$f \in F[\mathrm{G}] = F[x_1, ..., x_n]$$

can be written as

$$\mathrm{div}(f) = E(f) - \sum_\alpha d_\alpha(f) D_\alpha$$

where $E(f)$ is the unique irreducible component of $\{f = 0\}$ in G *and $d_\alpha(f) \geq 0$ for all α.*

Proposition 7.5. *Let X be a smooth equivariant compactification of a unipotent group* G *(with Lie algebra $\mathfrak{g}$). Let $\mathfrak{m} \subset \mathfrak{g}$ be a subalgebra and $Y := Y_{\mathfrak{m}} \subset X$ the compactification of* $\mathrm{M} = \exp(\mathfrak{m}) \subset \mathrm{G}$. *Then*

$$(a(-K_X|_Y), b(-K_X|_Y)) < (a(-K_X), b(-K_X)),$$

in the lexicographic ordering. In particular, the set of pairs

$$(a(-K_X|_Y), (b(-K_X|_Y))$$

is finite, as $\mathfrak{m}$ ranges over the set of all subalgebras.

Proof. In the additive case when $\mathrm{G} = \mathbb{G}_a^n$, this is the content of Lemma 7.3 in [5]. The case of general linear groups is covered in [13]. □

7.6. **Uniformity.** Let X be a smooth projective equivariant compactification of G and $\mathfrak{m} \subset \mathfrak{g}$ a subalgebra. Denote by $Y = Y_{\mathfrak{m}}$ the Zariski closure of $\exp(\mathfrak{m})$ in X. It is a compactification of $\mathrm{M} = \exp(\mathfrak{m})$, not necessarily smooth. Denote by $D = D_{\mathfrak{m}}$ the boundary $Y \setminus \mathrm{M}$.

Proposition 7.7. *There exist constants* $\mathsf{d}, \mathsf{d}', \mathsf{n} > 0$ *such that for every subalgebra* $\mathfrak{m} \subset \mathfrak{g}$ *there exists am equivariant blow-up* $\tilde{Y}_{\mathfrak{m}}$ *with support in the boundary* $Y_{\mathfrak{m}} \setminus \exp(\mathfrak{m})$ *such that*

- $\tilde{Y}_{\mathfrak{m}}$ *is smooth and projective;*
- *the boundary of* $\tilde{Y}_{\mathfrak{m}}$ *is a strict normal crossings divisor;*
- *the number of boundary components is bounded by* n*;*
- *the degree of every boundary component of* $\tilde{Y}_{\mathfrak{m}}$ *is bounded by* d*;*
- *for every linear subspace* $\mathfrak{e} \subset \mathfrak{m}$ *the degree of the intersection of the Zariski closure in* $\tilde{Y}_{\mathfrak{m}}$ *of* $\exp(\mathfrak{e})$ *with every boundary component is bounded by* d'.

Proof. Noetherian induction. For each $k \le \dim \mathfrak{g}$ consider the Grassmannian Gr of k-planes in $\mathfrak{g}$. This induces an algebraic family $\mathcal{Y} \to \mathrm{Gr}$ of subvarieties $Y_{\mathfrak{m}} \subset X$ and a family $\mathcal{D} \subset \mathcal{Y}$ of boundary divisors. Taking a (finite) flattening stratification of the base we reduce to the case when $\mathcal{Y} \to B, \mathcal{D} \to B$ are flat over B. Now we use embedded resolution of singularities over the function field of B. Next we complete (projectively) to a family $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ over B and restrict to a Zariski open subset B^0 where both $\tilde{\mathcal{Y}}^0, \tilde{\mathcal{D}}^0$ are flat over B^0, $\tilde{\mathcal{Y}}^0$ is smooth and $\mathcal{D}^0$ is strict normal crossings. We repeat the process for each irreducible component of the complement to B^0.

Each $\mathfrak{m} \in \mathfrak{g}$ will belong to one of the finitely many families constructed above. For each family we can find uniform bounds as claimed. □

8. Height zeta function

8.1. **Heights.** Let X be a smooth projective algebraic variety over a number field F. A smooth adelic metrization of a line bundle L on X is a family of v-adic norms $\|\cdot\|_v$ on $L \otimes_{F_v} F$ for all $v \in \mathrm{Val}(F)$ such that

- for $v \in S_\infty$ the norm $\|\cdot\|_v$ is C^∞;
- for $v \in S_{\mathrm{fin}}$ the norm of every local section of L is locally constant in the v-adic topology;
- there exist a finite set $S \subset \mathrm{Val}(F)$, a flat projective scheme (an integral model) $\mathcal{X}$ over $\mathrm{Spec}(\mathfrak{o}_S)$ with generic fiber X together with a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that for all $v \notin S$ the v-adic metric is given by the integral model.

If $\mathcal{X}$ carries an action of an integral model $\mathcal{G}$ of G over an open dense subset of $\mathrm{Spec}(\mathfrak{o}_F)$ extending the action of G on X and the line bundle $\mathcal{L}$ has a $\mathcal{G}$-linearization extending the G-linearization of L then we call the smooth adelic metrization equivariant.

Proposition 8.2. *Let* G *be a unipotent algebraic group defined over a number field F and X a smooth projective bi-equivariant compactification of* G*. Then there exist a compact open subgroup*

$$\mathrm{K} = \prod_v \mathrm{K}_v \subset \mathrm{G}(\mathbb{A}_{\mathrm{fin}})$$

and a height pairing

$$H = \prod_{v \in \mathrm{Val}(F)} H_v \,:\, \mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{G}(\mathbb{A}) \to \mathbb{C}$$

such that

- *for all $L \in \mathrm{Pic}(X)$ the restriction of H to $L \times \mathrm{G}(F)$ is a height corresponding to some smooth adelic metrization of L;*
- *the pairing is exponential in the* $\mathrm{Pic}(X)$ *component:*

$$H_v(\mathbf{s}+\mathbf{s}'; g) = H_v(\mathbf{s}; g) H_v(\mathbf{s}'; g)$$

  for all $\mathbf{s}, \mathbf{s}' \in \mathrm{Pic}(X)_{\mathbb{C}}$, all $g \in \mathrm{G}(\mathbb{A})$ and all $v \in \mathrm{Val}(F)$;
- *for all $v \in S_{\mathrm{fin}}$ one has $H_v(\mathbf{s}; kgk') = H_v(\mathbf{s}; g)$ for all $\mathbf{s} \in \mathrm{Pic}(S)_{\mathbb{C}}$ and $k, k' \in \mathrm{K}_v$.*

Proof. We follow closely the proof of Lemma 3.2 in [5]. First we observe that in our situation (with an action of $\mathrm{G} \times \mathrm{G}$)

$$\mathrm{Pic}^{\mathrm{G}\times\mathrm{G}}(X) = \mathrm{Pic}(X).$$

Let $\mathcal{L} = (L, \|\cdot\|_v)$ be a very ampe line bundle equipped with a locally constant v-adic norm (where $v \notin S_\infty$). The space of F-rational global sections $H^0(X, L)$ contains a unique (upto multiplication by F^*) $\mathrm{G}\times\mathrm{G}$-invariant section f. Consider the morphism of (left) multiplication

$$m \,:\, \mathrm{G} \times X \to X$$

and the projection

$$pr \,:\, X \to X.$$

The trivial line bundle $m^*(\mathcal{L}) \otimes pr^*(L)$ carries the tensor product metric. Restricting to $\mathrm{G}(F_v) \times \mathrm{G}(F_v)$, we find that the norm of the canonical section 1 is given by

$$(g, x) \mapsto \|f(gx)\|_v \|f(x)\|_v^{-1}.$$

It extends to a locally constant function on $\mathrm{G}(F_v) \times X(F_v)$. Since it is locally constant and equal to 1 on $\{1\} \times X(F_v)$ there exists a compact open subgroup $\mathrm{K}_v \subset \mathrm{G}(\mathfrak{o}_v)$ such that the above function equals 1 on $\mathrm{K}_v \times X(F_v)$. Moreover, for almost all v the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$ is equal to $\mathrm{G}(\mathfrak{o}_v)$.

We can use the same section f for the right action of G. If $\mathcal{L}$ is not ample, we can represent it as $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ with very ample $\mathcal{L}_1$, $\mathcal{L}_2$ and apply the same argument.

Now choose a basis for $\mathrm{Pic}(X)$ consisting of very ample line bundles, fix smooth adelic bi-equivariant metrizations on these generators as described above and extend these adelic metrizations to arbitrary $L \in \mathrm{Pic}(X)$ (by linearity). Then there exists a compact open subgroup

$$\mathrm{K} = \prod_{v \in S_{\mathrm{fin}}} \mathrm{K}_v$$

stabilizing *all* line bundles metrized in this way. □

8.3. **Spectral analysis.** In this section we begin to analyze the spectral decomposition of the height zeta function

$$\mathcal{Z}(\mathbf{s}; g) := \sum_{\gamma \in \mathrm{G}(F)} H(\mathbf{s}; \gamma g)^{-1}. \tag{8.1}$$

Proposition 8.4. *There exists an $N > 0$ such that the series (8.1) converges absolutely and uniformly to a holomorphic in $\mathbf{s}$ and continuous in g function, for g and $\mathbf{s}$ contained in compacts in $\mathrm{G}(\mathbb{A})$ and, respectively, in the domain in $\mathrm{Pic}(X)_{\mathbb{R}}$ defined by $\Re(s_\alpha) > N$, for all α.*

Proof. Analogous to the proof of Proposition 4.4 in [5] (follows from the projectivity of X). □

Proposition 8.5. *One has a formal identity*

$$\mathcal{Z}(\mathbf{s}; g) = \sum_{\varrho \in \mathcal{R}(\mathrm{K})} \mathcal{Z}_\varrho(\mathbf{s}; g). \tag{8.2}$$

Here the sum is over all irreducible unitary representations occurring $\mathsf{L}^2(\mathrm{G}(F)\backslash \mathrm{G}(\mathbb{A}))^{\mathrm{K}}$, see 5.6.

Proof. We use the right-K-invariance of the height function. □

Notation 8.6. Denote by S_X the set of all places $v \notin S_\infty$ such that either:

- residual characteristic of v is 2 or 3;
- $\mathrm{K}_v \neq \mathrm{G}(\mathfrak{o}_v)$;
- $\mathrm{vol}(\mathrm{G}(\mathfrak{o}_v))$ with respect to dg_v is not equal to 1;
- the coadjoint action of G on $\mathfrak{g}^*$ is not defined over $\mathfrak{o}_v$;

- over $\mathfrak{o}_v$, the union $\cup_\alpha D_\alpha$ is not a union of smooth relative divisors with strict normal crossings.

Remark 8.7. For all $v \notin (S_X \cup S_\infty)$ the height H_v is invariant with respect to the right and left $\mathrm{G}(\mathfrak{o}_v)$-action.

Notation 8.8. Denote by S_ϱ the set of all nonarchimedean places v such that either

- $v \in S_X$;
- $|\mathrm{Pf}(\ell)|_v \neq 1$, (where ℓ is contained in the orbit corresponding to ϱ);
- $v \in S_{\pi_\ell}$.

By the results in Section 5.7, for all $\varrho \in \mathcal{R}(\mathrm{K})$ and all nonarchimedean $v \notin S_\varrho$ one has $\dim \mathcal{H}_{\varrho_v} = 1$. For all other $v \notin S_\infty$, $\dim \mathcal{H}_{\varrho_v} < \infty$, controlled by Proposition 5.9.

Choose a norm 1 generator

$$\omega_\varrho = \otimes_{v \notin (S_\varrho \cup \infty)} \omega_{\varrho,v}$$

of the 1-dimensional space

$$\otimes_{v \notin (S \cup \infty)} \mathcal{H}_{\varrho_v}^{\mathrm{K}_v}$$

and an orthonormal basis $\mathcal{B}_{S_\varrho} = \{\omega_{\varrho,\lambda'}\}$ for the finite dimensional space

$$\otimes_{v \in S_\varrho} \mathcal{H}_{\varrho_v}^{\mathrm{K}_v}.$$

Fix an elliptic operator Δ acting on $\Gamma\backslash\mathrm{G}_\infty$ as in Section 10 (where G_∞ and Γ are defined in Notation 5.3). Finally, choose an orthonormal basis $\mathcal{B}_{\varrho,\infty} := \{\omega_{\varrho,\lambda}\}$ of

$$\otimes_{v \in S_\infty} \mathcal{H}_{\varrho_v} \subset \mathsf{L}^2(\Gamma\backslash\mathrm{G}_\infty)$$

consisting of Δ-eigenfunctions.

Lemma 8.9. *For all $\varrho \in \mathcal{R}(\mathrm{K})$, the set*

$$\mathcal{B}_\varrho = \omega_\varrho \otimes \mathcal{B}_{S_\varrho} \otimes \mathcal{B}_{\varrho,\infty}$$

is a complete orthonormal basis of $\mathcal{H}_\varrho$.

From now on we fix a basis $\mathcal{B}_\varrho$ as in Lemma 8.9. Considerations in Section 5.1, in particular, Proposition 5.14, imply:

Proposition 8.10. *For every $\varrho \in \mathcal{R}(\mathrm{K})$ one has (formally)*

$$\mathcal{Z}_\varrho(\mathbf{s}; g) = \mathcal{Z}^\varrho(\mathbf{s}; g) \cdot \mathcal{Z}_{S_\varrho}(\mathbf{s}; g) \cdot \mathcal{Z}_{\varrho,\infty}(\mathbf{s}; g),$$

where

$$\mathcal{Z}^{\varrho}(\mathbf{s};g) := \prod_{v\notin(S_\varrho\cup\infty)} \int_{\mathrm{G}(F_v)} H_v(\mathbf{s};g'_vg_v)^{-1}\bar{\omega}_{\varrho,v}(g'_vg_v)dg'_v,$$

$$\mathcal{Z}_{S_\varrho}(\mathbf{s};g) = \sum_{\lambda'} \int_{\mathrm{G}(\mathbb{A}_{S_\varrho})} H_{\mathbb{A}_{S_\varrho}}(\mathbf{s};g'_{S_\varrho}g_{S_\varrho})^{-1}\overline{\omega}_{\lambda'}(g'_{S_\varrho}g_{S_\varrho})dg'_{S_\varrho},$$

and

$$\mathcal{Z}_{\varrho,\infty}(\mathbf{s};g) = \sum_{\lambda} \int_{\mathrm{G}_\infty} H_\infty(\mathbf{s};g'_\infty g_\infty)^{-1}\overline{\omega}_{\lambda}(g'_\infty g_\infty)dg'_\infty.$$

The following sections justify this formal expansion, by

- analyzing the contributions from all places $v \notin (S_\varrho \cup \infty)$,
- estimating contributions for places $v \in S_\varrho$, and
- deriving upper bounds at places $v \in S_\infty$, via integration by parts for suitable differential operators.

9. Analytic properties of the height zeta function

We have a general integrability result (see, e.g., Section 8 in [5]):

Lemma 9.1. *For all places v, the integral*

$$I_v(\mathbf{s}) := \int_{\mathrm{G}(F_v)} H_v(\mathbf{s};g_v)^{-1}dg_v$$

is absolutely convergent to a holomorphic function in $\mathbf{s}$ in the domain $\Re(s_\alpha) > \kappa_\alpha - 1$, for all α. For every $\epsilon > 0$ there exists a constant $\mathsf{c}_v(\epsilon)$ such that

$$|I_v(\mathbf{s})| < \mathsf{c}_v(\epsilon),$$

for all $\mathbf{s}$ with $\Re(s_\alpha) > \kappa_\alpha - 1 + \epsilon$, for all α.

We start by considering nonarchimedian places of good reduction and the contribution of the trivial representation to the height zeta function. We have a stratification of X by locally closed subvarieties

$$D_A^0 := D_A \setminus \cup_{A'\supsetneq A} D_{A'}$$

where

$$D_A := \cap_{\alpha\in A} D_\alpha$$

and $A \subset \mathcal{A}$. In particular, $D_\emptyset = \mathrm{G}$. A key result is the following computation:

Proposition 9.2. *For all $v \notin S_X \cup S_\infty$ and all $\mathbf{s}$ with $\Re(s_\alpha) > \kappa_\alpha - 1$, for all α, one has*

$$\int_{\mathrm{G}(F_v)} H(\mathbf{s}; g_v)^{-1} dg_v = q_v^{-n} \left(\sum_{A \subseteq \mathcal{A}} D_A^0(\mathbf{k}_v) \prod_{\alpha \in A} \frac{q_v - 1}{q_v^{s_\alpha - \kappa_\alpha + 1} - 1} \right).$$

Proof. This is Theorem 7.1 in [5]. The proof proceeds as follows: for $v \notin S_X \cup S_\infty$ there is a *good* model $\mathcal{X}$ of X over $\mathfrak{o}_v$: all boundary components D_α (and G) are defined over $\mathfrak{o}_v$ and form a strict normal crossing divisor. We can consider the reduction map

$$\mathrm{red} \ : \ X(F_v) = X(\mathfrak{o}_v) \to X(\mathbf{k}_v) = \sqcup_{A \subset \mathcal{A}} D_A^0(\mathbf{k}_v).$$

The main observation is that in a neighborhood of the preimage in $X(F_v)$ of the point $\tilde{x}_v \subset D_A^0(\mathbf{k}_v)$ one can introduce local v-adic analytic coordinates $\{x_\alpha\}_{\alpha=1,\dots,n}$ such that

$$H_v(\mathbf{s}; g)^{-1} = \prod_{\alpha \in A} |x_\alpha|_v^{s_\alpha}.$$

Now it suffices to keep track of the change of the measure dg_v:

$$dg_v = \prod_{\alpha \notin A} dx_\alpha \cdot \prod_{\alpha \in A} |x_\alpha|_v^{-\kappa_\alpha} dx_\alpha,$$

where dx_α are standard Haar measures on F_v. The obtained integrals over the maximal ideal $\mathfrak{m}_v \subset \mathfrak{o}_v$ are elementary

$$\int_{\mathrm{red}^{-1}(\tilde{x}_v)} H_v(\mathbf{s}; g_v)^{-1} dg_v = \prod_{\alpha \notin A} \int_{\mathfrak{m}_v} dx_\alpha \cdot \prod_{\alpha \in A} \int_{\mathfrak{m}_v} q_v^{-(s_\alpha - \kappa_\alpha) v(x_\alpha)} dx_\alpha$$

(see Theorem 9.1 in [5]). Summing over all $\tilde{x}_v \in X(\mathbf{k}_v)$ we obtain the claim. □

Corollary 9.3. *Let*

$$\mathcal{Z}_0(\mathbf{s}, g) := \int_{\mathrm{G}(\mathbb{A})} H(\mathbf{s}; g' g)^{-1} dg'$$

be the contribution of the trivial representation to the spectral expansion (1.3)*. Then*

$$\mathbf{s} \mapsto \mathcal{Z}_0(\mathbf{s}; g)$$

is holomorphic for $\Re(s_\alpha) > \kappa_\alpha$, for all α, and continuous in g. Furthermore, the function

$$\mathsf{s} \mapsto \prod_{\alpha \in \mathcal{A}} (s_\alpha - \kappa_\alpha) \cdot \mathcal{Z}_0(\mathbf{s}; g)$$

is holomorphic for $\Re(s_\alpha) > \kappa_\alpha - 1/2$ *and continuous in* g*, with*

$$\lim_{\mathbf{s}\to\kappa} \prod_{\alpha\in\mathcal{A}} (s_\alpha - \kappa_\alpha) \cdot \mathcal{Z}_0(\mathbf{s}; e) = \tau(\mathcal{K}_X) \neq 0,$$

where $\tau(-\mathcal{K}_X)$ *is the Tamagawa number defined in* [22].

Proof. Apply Corollary 7.6 in [5]. □

The next step is to analyze contributions from nontrivial representations. We start by considering automorphic characters of G, i.e., 1-dimensional representations of $\mathrm{G}(\mathbb{A})$, trivial on $\mathrm{G}(F)$. In the framework of the orbit method, these correspond to linear forms $\ell \in \mathfrak{g}^*$ which are trivial on $[\mathfrak{g}, \mathfrak{g}]$. The treatment of these is analogous to the one presented in Section 10 of [5]. Let

$$\hat{H}(\mathbf{s}; \psi_\ell, g) := \int_{\mathrm{G}(\mathbb{A})} H(\mathbf{s}; g'g)^{-1}\overline{\psi}_\ell(g'g)dg'$$

be the Fourier transform of the height function with respect to ψ_ℓ. By Proposition 9.2 and Corollary 9.3, this integral converges to a holomorpic function for $\Re(s_\alpha) > \kappa_\alpha$, for all α. Let

$$\mathcal{Z}_1(\mathbf{s}; g) := \sum_{\ell\neq 0} \hat{H}(\mathbf{s}; \psi_\ell, g),$$

be the contribution of the set of all nontrivial automorphic characters to the spectral expansion (1.3).

Proposition 9.4. *The function*

$$\mathbf{s} \mapsto \mathcal{Z}_1(\mathbf{s}; g)$$

is holomorphic for $\Re(s_\alpha) > \kappa_\alpha$*, for all* α*, and continuos in* g*. Furthermore, the function*

$$\mathbf{s} \mapsto \prod_{\alpha\in\mathcal{A}} (s_\alpha - \kappa_\alpha) \cdot \mathcal{Z}_1(\mathbf{s}; g)$$

is holomorphic for $\Re(s_\alpha) > \kappa_\alpha - 1/2$*, with*

$$\lim_{\mathbf{s}\to\kappa} \prod_{\alpha\in\mathcal{A}} (s_\alpha - \kappa_\alpha) \cdot \mathcal{Z}_1(\mathbf{s}; e) = 0.$$

Proof. The proof is identical to the proof of Theorem 6.3 in [5]. We repeat the argument since it will be essential in the subsequent analysis of infinite-dimensional representations $\varrho \in \mathcal{R}(\mathrm{K})$.

The ℓ occurring in $\mathcal{Z}_1$ are parametrized by a lattice $\mathfrak{d}$, minus 0. For each ℓ we have a finite set of nonarchimedean places S_ℓ, of "bad" reduction, defined as in Notation 8.8. Put $S := S_\ell \cup S_\infty$. The main steps of the proof are:

(1) In the domain $\Re(s_\alpha) > \kappa_\alpha - 1/2$, provide an upper bound of the form
$$|\prod_{v\in S_\ell} \hat{H}_v(\mathbf{s};\psi_\ell, g)| \leq \|\mathcal{O}_\ell\|_\infty^\epsilon,$$
for some $\epsilon > 0$; here the norm $\|\mathcal{O}_\ell\|_\infty$, defined in Notation 4.9, is equivalent to the euclidean norm of the linear form ℓ as the element of the lattice $\mathfrak{d}$.

(2) Establish meromorphic continuation of the Euler product
$$\hat{H}^S(\mathbf{s};\psi_\ell, g) := \prod_{v\notin S} \hat{H}_v(\mathbf{s};\psi_\ell, g),$$
of the form
$$\hat{H}^S(\mathbf{s};\psi_\ell, g) = \prod_{\alpha\in\mathcal{A}(\ell)} \zeta_F(s_\alpha - \kappa_\alpha + 1)\cdot \phi(\mathbf{s};\psi_\ell, g),$$
where $\mathcal{A}(\ell) \subsetneq \mathcal{A}$, and ϕ is a holomorphic function in the domain $\Re(s_\alpha) > \kappa_\alpha - 1/2 + \epsilon$, for all α, satisfying a uniform bound
$$|\phi(\mathbf{s};\psi_\ell, g)| \leq \mathsf{c}'\cdot(1 + \|\Im(\mathbf{s})\|)^{N'}\cdot\|\mathcal{O}_\ell\|_\infty^\epsilon,$$
for some constants ϵ, c', and N'.

(3) In the domain $\Re(s_\alpha) > \kappa_\alpha - 1/2$, for all α, and for any $N \in \mathbb{N}$, obtain upper bounds of the shape
$$|\hat{H}_\infty(\mathbf{s};\psi_\ell, g)| \leq \mathsf{c}''\cdot(1 + \|\Im(\mathbf{s}\|)^{N'}\cdot(1 + \|\mathcal{O}_\ell\|_\infty)^{-N},$$
for some constants c'' and N', to insure convergence of the sum
$$\sum_\ell \hat{H}^S(\mathbf{s};\psi_\ell, g)\cdot \hat{H}_{S_\ell}(\mathbf{s};\psi_\ell, g)\cdot \hat{H}_{S_\infty}(\mathbf{s};\psi_\ell, g).$$

This is the content of Section 10 of [5]. We proceed to explain these steps in more detail.

Let f be a polynomial function on $\mathfrak{g}$ and let
$$\mathrm{div}(f) = E(f) - \sum_\alpha d_\alpha(f) D_\alpha,$$
$$\mathcal{A}_0(f) := \{\alpha \,|\, d_\alpha(f) = 0\}.$$
Every nontrivial linear form $\ell \in \mathfrak{g}^*$ defines a nontrivial rational function $f = f_\ell \in F(X)$, by
$$x \mapsto \langle \ell, \log(x)\rangle.$$
We write $d_\alpha = d_\alpha(f)$ for the multiplicities of f along the corresponding boundary strata D_α.

For $v \in S_\ell$, we replace ψ_ℓ by 1 and refer to Lemma 8.2 of [5] (see also Lemma 4.1.1 in [6] for a general integrability result of this type). This gives (1).

For $v \notin S_\ell \cup S_\infty$, we compute the integral defining $\hat{H}_v(\mathbf{s}; \psi_\ell, g)$ on residue classes, as in the proof of Proposition 9.2. Let $\tilde{x} \in X(\mathbf{k}_v)$ and $A = \{\alpha\,;\, \tilde{x} \in D_\alpha\}$. There are three cases:

Case 1. $A = \emptyset$. — Since ψ_ℓ is trivial on $\mathrm{G}(\mathfrak{o}_v)$, for $v \notin S_\ell \cup S_\infty$,

$$\int_{\mathrm{G}(\mathfrak{o}_v)} H_v(\mathbf{s}; g_v)^{-1}\overline{\psi}_\ell(g_v) dg_v = 1$$

Case 2. $A = \{\alpha\}$ and $\tilde{x} \notin E$. — We introduce v-adic analytic coordinates x_α and y_β around $\tilde{x}$ such that locally

$$f_\ell(x) = \langle \ell, \log(x) \rangle = u x_\alpha^{-d_\alpha}.$$

Then

$$\begin{aligned}\int_{\mathrm{red}^{-1}(\tilde{x})} &= \int_{\mathfrak{m}_v \times \mathfrak{m}_v^{n-1}} q^{-(s_\alpha - \kappa_\alpha) v(x_\alpha)} \overline{\psi}(u x_\alpha^{-d_\alpha})\, dx_\alpha d\mathbf{y} \\ &= \frac{1}{q^{n-1}} \sum_{n_\alpha \geq 1} q^{-(1+s_\alpha-\kappa_\alpha)n_\alpha} \int_{\mathfrak{o}_v^*} \psi(u \pi^{-n_\alpha d_\alpha} u_\alpha^{-d_\alpha})\, du_\alpha,\end{aligned}$$

where the last integral is elementary (see Lemma 10.3 of [5]).

Case 3. $\#A \geq 2$ or $\#A = 1$ and $\tilde{x} \in E$. — Replacing ψ by 1 we find that, for real $\mathbf{s}$, the contribution of these $\tilde{x}$ is bounded by

$$\sum_{\#A \geq 2} \frac{\#D_A^\circ(\mathbf{k}_v)}{q^n} \prod_{\alpha \in A} \frac{q-1}{q^{1+s_\alpha-\kappa_\alpha}-1} + \sum_{A=\{\alpha\}} \frac{\#(D_\alpha \cap E)(\mathbf{k}_v)}{q^n} \frac{q-1}{q^{1+s_\alpha-\kappa_\alpha}-1}.$$

Combining the calculations of the cases above, we obtain

$$\hat{H}_v(\mathbf{s}; \psi_\ell, g) = 1 + \sum_{\alpha \in \mathcal{A}_0(f)} \frac{\#D_\alpha^\circ(\mathbf{k}_v)}{q^n} \frac{q-1}{q^{1+s_\alpha-\kappa_\alpha}-1} + ET$$

with "an error term" ET on the order of $\mathrm{O}(q^{-(1+\delta)})$, for some $\delta = \delta(\epsilon)$, when $\Re(s_\alpha) > \kappa_\alpha - 1/2 + \epsilon$, for all α. This implies (2).

For $v \in S_\infty$ we use integration by parts with respect to suitable vector fields following the proof of Proposition 8.4 of [5]; this proves (3).

As in the proof of Proposition 10.2 of [5], we deduce from these estimates that $\hat{H}(\mathbf{s}; \psi_\ell, g)$ has a meromorphic continuation to the domain

$\Re(s_\alpha) > \kappa_\alpha - 1/2$, for all α:

$$\hat{H}(\mathbf{s};\psi_\ell,g) = \phi(\mathbf{s};\psi_\ell,g) \prod_{\alpha\in\mathcal{A}_0(f_\ell)} \zeta_F(1+s_\alpha-\kappa_\alpha), \tag{9.1}$$

where ϕ is a holomorpic function in this domain and ζ_F is the Dedekind zeta function. Moreover, for any $N>0$ there exist constants $N'>0$ and $\mathsf{c}(\epsilon,N)$ such that for any $\mathbf{s}\in\mathsf{T}_{-1/2+\epsilon}$, one has the estimate

$$|\phi(\mathbf{s};\psi_\ell,g)| \le \mathsf{c}(\epsilon,N)(1+\|\Im(\mathbf{s})|)^{N'}(1+\|\mathcal{O}_\ell\|)^{-N}.$$

This, in turn, implies the claimed properties of $\mathcal{Z}_1(\mathbf{s},g)$. $\square$

The proof of Proposition 9.4 (see Equation 9.1) and Corollary 9.3 imply:

Corollary 9.5. *Let $L=\sum_{\alpha\in\mathcal{A}} l_\alpha D_\alpha$, $l_\alpha>0$ for all α. Then the function*

$$s\mapsto \hat{H}(sL;\psi_\ell,g)$$

has the following properties:

- *is holomorphic for $\Re(s)>a(L)$,*
- *admits a meromorphic continuation to $\Re(s)<a(L)-\delta$, for some $\delta>0$,*
- *is holomorphic in this domain, except possibly at $s=a(L)$, where it could have a pole of multiplicity at most $b(L)$,*
- *the multiplicity of the pole is strictly smaller when $\ell\neq 0$.*

We generalize the argument above to the infinite-dimensional representations occurring in the expansion (1.3) as follows:

Consider a stratification of the set of $\mathrm{Ad}^*(G)$-orbits in $\mathfrak{g}^*$ into finitely many affine strata $Z_\sigma\subset\mathfrak{g}^*$ as in Sections 3.4 and 3.8; we may assume that for each $\sigma\in\Sigma$ we have a finite set of $\mathrm{Ad}^*(G)$-invariant polynomials $\{P_{\sigma,j}\}\in F[\mathfrak{g}^*]$ separating the orbits (see Theorem 3.7 and Proposition 3.10).

Passing to a finer stratification, if necessary, we may assume that for each σ, we have a finite collection of polynomial functions $Q_{\sigma,i}\in F[\mathfrak{g}^*]$ defining the F-morphism

$$\mathrm{pol}_\sigma : Z_\sigma \to \mathrm{Gr}(k_\sigma,\mathfrak{g})$$

from Proposition 2.10, i.e., for each $\ell\in Z_\sigma$ the image

$$\mathfrak{m}_\ell := \mathrm{pol}_\sigma(\ell)\subset \mathrm{Gr}(k_\sigma,\mathfrak{g})$$

is a polarizing subalgebra for ℓ. The corresponding family of subgroups $\{M_\ell\}_{\ell\in Z_\sigma}$ defines an equidimensional family of equivariant compactifications

$$M_\ell \subset Y_\ell \subset X,$$

with boundaries

$$D_\ell := Y_\ell \setminus M_\ell.$$

Considerations in Section 5.1 imply

$$\mathcal{R}(\mathrm{K}) = \sqcup_{\sigma\in\Sigma}\mathcal{R}(\mathrm{K})_\sigma$$

We restrict the height zeta function to a 1-parameter function

$$\mathcal{Z}(s;g) := \mathcal{Z}(-sK_X;g)$$

and decompose

$$\mathcal{Z}(s;g) = \mathcal{Z}_0(s;g) + \mathcal{Z}_1(s;g) + \sum_{\sigma\in\Sigma'}\mathcal{Z}_\sigma(s;g), \tag{9.2}$$

where the sum is over packets of infinite-dimensional automorphic representations. We will establish the meromorphic properties of each term in this sum. The pole or highest order at $s=1$ will be provided only by the trivial representation, i.e., $\mathcal{Z}_0(s;g)$.

For each stratum $\sigma\in\Sigma'$, fix integral models over $\mathfrak{o}_{S_\sigma}$, for some finite set of nonarchimedean places S_σ, i.e., we assume that the polynomials $P_{\sigma,j}$ and $Q_{\sigma,i}$ have coefficients in $\mathfrak{o}_{S_\sigma}$. We may assume that S_σ contains S_X. Then there is a sublattice $\mathfrak{d}_\sigma\subset\mathfrak{g}^*(F)$ of $\mathfrak{o}_{S_\sigma}$-integral points in $\mathfrak{g}^*(F)$ such that if $\varrho_\ell\in\mathcal{R}(\mathrm{K})_\sigma$ then $\ell\in Z_\sigma(\mathfrak{d}_\sigma)=Z_\sigma(F)\cap\mathfrak{d}_\sigma$. We can refine the expansion (9.2):

$$\mathcal{Z}(s;g) = \mathcal{Z}_0(s;g) + \mathcal{Z}_1(s;g) + \sum_{\sigma\in\Sigma'}\sum_{\ell\in Z_\sigma(\mathfrak{d}_\sigma)}\mathcal{Z}_{\varrho_\ell}(s;g), \tag{9.3}$$

with $\mathcal{Z}_{\varrho_\ell}\in\mathcal{H}_{ell}$, as in Section 8.3.

By Proposition 8.10,

$$\mathcal{Z}_{\varrho_\ell}(s;g) = \mathcal{Z}^{\varrho_\ell}(s;g)\cdot\mathcal{Z}_{S_{\varrho_\ell}}(s;g)\cdot\mathcal{Z}_{\rho_\ell,\infty}(s;g),$$

where, by Proposition 5.14

$$\mathcal{Z}^{\varrho_\ell}(s;g) = \prod_{v\notin(S\cup\infty)}\int_{\mathrm{M}_\ell(F_v)} H_v(-K_X;h_vg_v)^{-s}\overline{\psi}_\ell(h_vg_v)dh_v, \tag{9.4}$$

and the other factors are contributions from places in S_{ϱ_ℓ} (defined in Notation 8.8) and the places at infinity. The computation of the local intergrals in (9.4) is analogous to the one explained in the proof of Proposition 9.4, except that we cannot guarantee that Y_ℓ, the Zariski closure of M_ℓ in X, is smooth, with normal crossing boundary. However, the height integral can be computed on a desingularization $\tilde{Y}_\ell$ of Y_ℓ, constructed in Proposition 7.7. By Corollary 9.5, the analytic properties of the function

$$s \mapsto \mathcal{Z}^S_{\varrho\ell}(s;g)$$

are governed by the invariants

$$(a(-K_X|_{Y_\ell}), b(-K_X|_{Y_\ell})),$$

computed on the resolution $\tilde{Y}_\ell$ as in Section 7.2. Proposition 7.5 insures that

$$\mathcal{Z}^S_{\varrho\ell}(s;g) = \frac{1}{(s-1)^{b-1}} \cdot \phi_\ell(s;g), \tag{9.5}$$

where ϕ_ℓ is holomorphic for $\Re(s) > 1-\delta$, for some $\delta > 0$, i.e., $\mathcal{Z}^S_{\varrho\ell}(s;g)$ admits a meromorphic continuation to this domainm, with a possible pole at $s = 1$ of order *strictly smaller* than $b = b(-K_X)$, the rank of the Picard group of X.

Propositions 7.5 and 7.7 provide uniform control on the geometry of the occurring desingularizations Y_ℓ. In particular, only finitely many pairs $(a(-K_X|_{Y_\ell}), b(-K_X|_{Y_\ell}))$ arise, and the number and degrees of the corresponding boundary components are also uniformly bounded. The set of places of bad reduction of integral models of Y_ℓ is controlled by values of polynomials parametrizing orbits and defining the corresponding polarizing subalgebras in the stratum σ, i.e., by polynomial expressions in $P_{\sigma,j}(\ell)$. As in the proof of Proposition 9.4, we obtain the bound

$$|\phi_\ell(s;g)| \le \mathsf{c} \cdot |s|^N \cdot \|\mathcal{O}_\ell\|^\epsilon_\infty, \tag{9.6}$$

for some constants $\mathsf{c}, N, \epsilon > 0$, independent of ℓ and $g \in \mathrm{G}(F)\backslash \mathrm{G}(\mathbb{A}_F)$.

By Proposition 5.9, the sets S_{ϱ_ℓ} and the dimensions of $\mathcal{B}_{S_{\varrho_\ell}}$ are controlled in terms of an $\mathrm{Ad}^*(\mathrm{G})$-invariant polynomial $\mathrm{Pf} \in \mathfrak{o}_{S_\sigma}[\mathfrak{g}^*]$ (the Pfaffian). For $v \in S_{\varrho\ell}$ we can use the trivial estimate, replacing $\omega_{\lambda'}$ by 1, and using the integrability of height functions as in Lemma 8.2 of [5], to obtain:

Lemma 9.6. *There exist $\delta, \epsilon, \mathsf{c} > 0$ such that for all $\varrho_\ell \in \mathcal{R}(\mathrm{K})$ the function $\mathcal{Z}_{S_{\varrho_\ell}}$ is holomorphic in the domain $\Re(s) > 1-\delta$ and satisfies*

$$|\mathcal{Z}_{S_{\varrho_\ell}}(s;g)| \le \mathsf{c} \cdot (1 + \|\mathcal{O}_\ell\|_\infty)^\epsilon. \tag{9.7}$$

We now address contributions from archimedean places.

Lemma 9.7. *Let X be an equivariant compactification of a unipotent group* G *and* $\mathfrak{g}$ *the Lie algebra of* G*. For all $v \in S_\infty$, all $\epsilon > 0$, and all $\partial_v \in \mathfrak{g}(F_v)$ there exist constants $\mathsf{c}_v = \mathsf{c}_v(\epsilon, \partial_v)$ and $N = N(\partial_v) \in \mathbb{N}$ such that*

$$\int_{\mathrm{G}(F_v)} |\partial_v H_v(\mathbf{s}; g_v)^{-1}|_v dg_v < \mathsf{c}_v \cdot \|\mathbf{s}\|^N$$

for all $\mathbf{s}$ *with $\Re(s_\alpha) > \kappa_\alpha - 1 + \epsilon$, for all α.*

Proof. We use integration by parts with respect to ∂_v as in the proof of Proposition 6.4 in [5]. Assume that $F_v = \mathbb{R}$. Let $x \in X(\mathbb{R})$ and $A \subset \mathcal{A}$ be the set of all α such that $x \in D_\alpha(\mathbb{R})$. Let $f_\alpha = 0$ be a local equation for D_α in a neighborhood U of x. Then there exist functions $\varphi_\alpha \in \mathsf{C}^\infty(U)$ such that for all $g \in U \cap \mathrm{G}(\mathbb{R})$ we have

$$H_v(\mathbf{s}; g)^{-1} = \prod_{\alpha \in \mathcal{A}} \exp(-s_\alpha h_\alpha(g))$$

where

$$h_\alpha(g) = \log|f_\alpha(g)| + \varphi_\alpha(g).$$

Moreover, for any $\partial_v \in \mathfrak{g}(F_v)$ the derivative $\partial_v h_\alpha(g)$ extends to a C^∞-function on the compactification $X(\mathbb{R})$ (compare Proposition 2.2 in [5]). In particular, it is bounded on $\mathrm{G}(\mathbb{R}) \subset X(\mathbb{R})$ and the claim follows. Complex places are treated in the same way. □

Fix a stratum σ and consider an $\ell \in \sigma$. By Proposition 6.4, differential operators corresponding to the $\mathrm{Ad}(G)^*$-invariant polynomials $P_{\sigma,j}$, act in the representation space $\mathcal{H}_\ell$ by multiplication by $P_{\sigma,j}(2\pi i \ell)$. We apply Lemma 9.7:

Corollary 9.8. *For all $\epsilon > 0$ and all $n \in \mathbb{N}$ there exists a constant $\mathsf{c} = \mathsf{c}(\epsilon, N)$ such that for all* $\mathbf{s}$ *with $\Re(s_\alpha) > \kappa_\alpha - 1 + \epsilon$, for all α, for all $\varrho_\ell \in \mathcal{R}(\mathrm{K})$, and all eigenfunctions $\omega_{\varrho_\ell,\lambda}$ as in Lemma 8.9 one has*

$$|\int_{\mathrm{G}_\infty} H_\infty(\mathbf{s}; g_\infty)^{-1} \overline{\omega}_{\varrho_\ell,\lambda}(g_\infty) dg_\infty| \le \mathsf{c} \cdot \|\mathbf{s}\|^N \cdot \lambda^{-n} \cdot \|\mathcal{O}_\ell\|_\infty^{-n}$$

Proof. Proceed by intergration by parts, using the Laplacian $\Delta = \Delta_\ell$ as in Section 10. For any $N \in \mathbb{N}$ and any Δ-eigenfunction $\omega_{\varrho_\ell,\lambda} \in \mathcal{B}_{\varrho_\ell,\infty}$ with eigenvalue λ we have

$$\begin{aligned}\lambda^N \cdot \int_{\mathrm{G}_\infty} H_\infty(\mathbf{s}; g_\infty)^{-1} \bar{\omega}_{\varrho_\ell,\lambda}(g_\infty) dg_\infty &= \int_{\mathrm{G}_\infty} H_\infty(\mathbf{s}; g_\infty)^{-1} \Delta^N \bar{\omega}_{\varrho_\ell,\lambda}(g_\infty) dg_\infty \\ &= \int_{\mathrm{G}_\infty} \Delta^N H_\infty(\mathbf{s}; g_\infty)^{-1} \bar{\omega}_{\varrho_\ell,\lambda}(g_\infty) dg_\infty,\end{aligned}$$

which is majorized by

$$\|\omega\|_{\mathrm{L}^\infty(\Gamma\backslash \mathrm{G}_\infty)} \cdot \int_{\mathrm{G}_\infty} |\Delta^N H_\infty(\mathbf{s}; g_\infty)^{-1}| dg_\infty.$$

The third property in 10.2 bounds the norm of $\omega_{\varrho_\ell,\lambda}$ and Lemma 9.7 the integral. Similarly, applying differential operators corresponding to $P_{\sigma,j}$ which act by $P_{\sigma,j}(2\pi i\ell)$ on the eigenfunctions, and using the definition of $\|\mathcal{O}_\ell\|_\infty$ we obtain the claim. □

To establish analytic properties of the height zeta function we return to Equation 9.2 and consider

$$\sum_{\sigma\in\Sigma'} \mathcal{Z}_\sigma(s; g).$$

We have, formally,

$$\mathcal{Z}_\sigma(s; g) = \sum_{\ell\in\sigma} \mathcal{Z}_{\varrho_\ell}(s; g). \tag{9.8}$$

By Proposition 8.10

$$\mathcal{Z}_{\varrho_\ell}(s; g) = \mathcal{Z}^{\varrho_\ell}(s; g) \cdot \mathcal{Z}_{S_{\varrho_\ell}}(s; g) \cdot \mathcal{Z}_{\varrho_\ell,\infty}(s; g).$$

We combine Equations (9.5), (9.6), (9.7) and Corollary 9.8 to derive that there exists a $\delta > 0$ such that for all $\ell \in \sigma$ we have

$$\mathcal{Z}_{\varrho_\ell}(s; g) = \frac{1}{(s-1)^{b-1}} \cdot \Phi_\ell(s; g),$$

where Φ_ℓ is holomorphic in s, for $\Re(s) > 1 - \delta$ and continuous in g. Moreover, for all $n \in \mathbb{N}$ there exist constants $\mathsf{c} = \mathsf{c}(\delta, N)$ and N' such that

$$|\Phi_\ell(s; g)| \le \mathsf{c} \cdot |s|^{N'} \cdot \|\mathcal{O}_\ell\|_\infty^{-n},$$

in this domain.

10. Appendix: Elliptic operators

Let $U \subset \mathbb{R}^n$ be an open subset and

$$\Delta := \sum_{|\mathrm{J}|\le m} f_{\mathrm{J}}(x) \left(-i\frac{\partial}{\partial x}\right)^{\mathrm{J}}$$

a partial differential operator (we use the standard multi-index notations $\mathrm{J} = (j_1, ..., j_n)$ etc). Assume that $f_{\mathrm{J}} \in \mathsf{C}^\infty(\mathbb{R}^n)$ for all J. The principal symbol P_Δ of Δ is defined as

$$\mathrm{P}_\Delta(x, \xi) = \sum_{|\mathrm{J}|=m} f_{\mathrm{J}}(x)\xi^{\mathrm{J}}$$

(here $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$). The operator Δ is called *elliptic* in U if for all $x \in U$ the equality $\mathrm{P}_\Delta(x, \xi) = 0$ implies $\xi = 0$.

Let M be a C^∞-manifold equipped with a Riemannian metric and $\mathcal{T}(M)$ the tangent bundle of M. Consider the maps

$$\begin{array}{ccc} \mathcal{T}(M) & \rightarrow & \mathrm{End}_{\mathbb{C}}(\mathsf{C}^\infty(M)) \\ \partial & \mapsto & (f \mapsto \partial f) \\ \mathsf{C}^\infty(M) & \rightarrow & \mathrm{End}_{\mathbb{C}}(\mathsf{C}^\infty(M)) \\ g & \mapsto & (f \mapsto g \cdot f). \end{array}$$

The subalgebra $\mathcal{D}(M)$ of $\mathrm{End}_{\mathbb{C}}(\mathsf{C}^\infty(M))$ generated by the above endomorphisms is called the algebra of (finite order) differential operators of M.

Lemma 10.1. *Let $\Delta \in \mathcal{D}(M)$ be an operator of the form*

$$\Delta = \sum_j \partial_j^2,$$

where $\partial_j \in \mathsf{C}^\infty(\mathcal{T}(M))$ (and j runs over a finite set).

The operator Δ is elliptic iff there exists a constant $\mathsf{c} > 0$ such that for all $x \in M$ and all $\xi_x \in \mathcal{T}_x^*(M)$ one has

$$\sum_j (\partial_j(x), \xi_x)^2 \geq \mathsf{c} \cdot \|\xi_x\|^2,$$

(where $\|\cdot\|$ is the Riemannian metric on M).

A crucial ingredient in the proof of analytic properties of the height zeta function is the following basic fact about elliptic operators on compact manifolds.

Proposition 10.2. *Let M be a compact manifold and Δ an elliptic operator on $\mathsf{C}^\infty(M)$. Then*

- *the set $\mathrm{Spect}(\Delta)$ of eigenvalues of Δ is a discrete subset of $\mathbb{R}_{\geq 0}$;*
- *there exists a constant $\mathsf{c}_1 > 0$ such that the spectral zeta function*

  $$\sum_{\lambda \in \mathrm{Spect}(\Delta) \setminus 0} n_\lambda \lambda^{-s}$$

  converges absolutely and uniformly in compacts in the domain $\Re(s) > \mathsf{c}_1$ (here n_λ is the dimension of the λ-eigenspace);
- *there exist constants $\mathsf{c}_2, n > 0$ such that for all $\lambda \in \mathrm{Spect}(\Delta)$ and all λ-eigenvectors ω one has the estimate*

  $$\|\omega\|_{\mathsf{L}^2(M)} \leq \mathsf{c}_2(1 + \lambda^n)\|\omega\|_{\mathsf{L}^\infty(M)}.$$

We are interested in the case when $M = \Gamma\backslash \mathrm{G}_\infty$, where G_∞ is a C^∞-Lie group and Γ is a discrete cocompact subgroup. Denote by $\mathfrak{g}_\infty$ the Lie algebra of G_∞ and by

$$\exp\,:\,\mathfrak{g}_\infty \to \mathrm{G}_\infty$$

the exponential map. Choose a basis $\overline{\partial}_1, ..., \overline{\partial}_r$ of $\mathfrak{g}_\infty$. Each $\overline{\partial}_j$ can be regarded as a left-invariant vector field on G_∞.

Lemma 10.3. *The operator*

$$\overline{\Delta} := \sum_{j=1}^{r} \overline{\partial}_j^2$$

is an elliptic operator on G_∞.

Proof. We may assume that the metric on G_∞ is left invariant under the G_∞-action. Thus it suffices to check the estimate from Lemma 10.1 at the identity $e \in \mathrm{G}_\infty$. Choose a basis $\{\overline{\partial}_j^*\}$ of $\mathfrak{g}_\infty^* = \mathcal{T}_e^*(\mathrm{G}_\infty)$ dual to $\{\overline{\partial}_j\}$ and write

$$\xi_e = \sum_{j=1}^{r} \xi_j \overline{\partial}_j^*.$$

Then

$$\sum_{j=1}^{r} (\overline{\partial}_j(e), \xi_e)^2 = \sum_{j=1}^{r} \xi_j^2 = \|\xi_e\|^2$$

and we can take $\mathsf{c} = 1$. ☐

Consider the map

$$\begin{array}{ccc} \mathfrak{g}_\infty & \to & \mathcal{T}(M) \\ \overline{\partial} & \mapsto & \partial = (f \mapsto \frac{d}{dt}|_{t=0} f(x \cdot \exp(t\partial))). \end{array}$$

Lemma 10.4. *The operator*

$$\Delta := \sum_{j=1}^{r} \partial_j^2$$

is an elliptic operator on M.

Proof. The manifold $M = \Gamma\backslash \mathrm{G}_\infty$ is locally isomorphic to G_∞. ☐

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