RATIONAL POINTS ON K3 SURFACES AND DERIVED EQUIVALENCE

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The geometry of vector bundles and derived categories on complex K3 surfaces has developed rapidly since Mukai's seminal work [Muk87]. Many foundational questions have been answered:

- the existence of vector bundles and twisted sheaves with prescribed invariants;
- geometric interpretations of isogenies between K3 surfaces [Orl97, Căl00];
- the global Torelli theorem for holomorphic symplectic manifolds [Ver13, Huy12b];
- the analysis of stability conditions and its implications for birational geometry of moduli spaces of vector bundles and more general objects in the derived category [BMT14, BM14, Bri07].

Given the precision and power of these results, it is natural to seek arithmetic applications of this circle of ideas. Questions about zero cycles on K3 surfaces have attracted the attention of Beauville-Voisin [BV04], Huybrechts [Huy12a], and other authors.

Our focus in this note is on *rational points* over non-closed fields of arithmetic interest. We seek to relate the notion of derived equivalence to arithmetic problems over various fields. Our guiding questions are:

Question 1. Let X and Y be K3 surfaces, derived equivalent over a field F. Does the existence/density of rational points of X imply the same for Y?

Given $\alpha \in \mathrm{Br}(X)$, let (X, α) denote the twisted K3 surface associated with α, i.e., if $\mathcal{P} \to X$ is an étale projective bundle representing α, of relative dimension $r-1$ then $(X, \alpha) = [\mathcal{P}/\,\mathrm{SL}_r]$.

Question 2. Suppose that (X, α) and (Y, β) are derived equivalent over F. Does the existence of a rational point on the former imply the same for the latter?

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Note that an F-rational point of (X, α) corresponds to an $x \in X(F)$ such that $\alpha|x = 0 \in \mathrm{Br}(F)$. After this paper was released, Ascher, Dasaratha, Perry, and Zhou [ADPZ15] found that Question 2 has a negative answer, even over local fields.

We shall consider these questions for F finite, p-adic, real, and local with algebraically closed residue field. These will serve as a foundation for studying how the geometry of K3 surfaces interacts with Diophantine questions over local and global fields. For instance, is the Hasse/Brauer-Manin formalism over global fields compatible with (twisted) derived equivalence? See [HVAV11, HVA13, MSTVA14] for concrete applications to rational points problems.

In this paper, we first review general properties of derived equivalence over arbitrary base fields. We then offer examples which illuminate some of the challenges in applying derived category techniques. The case of finite and real fields is presented first—here the picture is well developed. Local fields of equicharacteristic zero are also fairly well understood, at least for K3 surfaces with semistable or other mild reduction. The analogous questions in mixed characteristic remain largely open, but comparison with the geometric case suggests a number of avenues for future investigation.

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1. Generalities on derived equivalence for K3 surfaces

1.1. **Definitions.** Let X and Y denote K3 surfaces over a field F. Let p and q be the projections from $X \times Y$ to X and Y respectively.

Let $\mathcal{E} \in D^b(X \times Y)$ be an object of the bounded derived category of coherent sheaves, which may be represented by a perfect complex of locally free sheaves. The *Fourier-Mukai transform* is defined

$$\begin{aligned} \Phi_{\mathcal{E}} : D^b(X) &\to D^b(Y) \\ \mathcal{F} &\mapsto q_*(\mathcal{E} \otimes p^*\mathcal{F}), \end{aligned}$$

where push forward and tensor product are the derived operations.

We say X and Y are *derived equivalent* if there exists an equivalence of triangulated categories over F

$$\Phi : D^b(X) \xrightarrow{\sim} D^b(Y).$$

A fundamental theorem of Orlov [Orl97] implies that Φ arises as the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ for some perfect complex $\mathcal{E} \in D^b(X \times Y)$. When we refer to Fourier-Mukai transforms below they will induce derived equivalences.

1.2. **Mukai lattices.** Suppose X is a K3 surface defined over $\mathbb{C}$ and consider its Mukai lattice

$$\tilde{H}(X, \mathbb{Z}) = \tilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z})(-1) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})(1),$$

where we apply Tate twists to get a Hodge structure of weight two. Mukai vectors refer to type $(1,1)$ vectors in $\tilde{H}(X, \mathbb{Z})$. Let $(,)$ denote the natural nondegenerate pairing on $\tilde{H}(X, \mathbb{Z})$; this coincides with the intersection pairing on $H^2(X, \mathbb{Z})$ and the negative of the intersection pairing on the other summands. There is an induced homomorphism on the level of cohomology:

$$\begin{aligned} \phi_{\mathcal{E}} : \tilde{H}(X, \mathbb{Z}) &\to \tilde{H}(Y, \mathbb{Z}) \\ \eta &\mapsto q_*(\sqrt{\mathrm{Td}_{X \times Y}} \cdot \mathrm{ch}(\mathcal{E}) \cup p^*\eta). \end{aligned}$$

Note that

$$\phi_{\mathcal{E}} \, \mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\Phi_{\mathcal{E}}(\mathcal{F})).$$

For X defined over a non-closed field F, there are analogous constructions in ℓ-adic and other flavors of cohomology [LO11, § 2]. When working ℓ-adically, we interpret $\tilde{H}(X, \mathbb{Z}_\ell)$ as a Galois representation rather than a Hodge structure. Observe that $\phi_{\mathcal{E}}$ is defined on Hodge structures, de Rham cohomologies, and ℓ-adic cohomologies—and these are all compatible.

1.3. **Characterizations over the complex numbers.**

Theorem 3. [Orl97, §3] *Let X and Y be K3 surfaces over $\mathbb{C}$, with transcendental cohomology groups*

$$T(X) := \mathrm{Pic}(X)^{\perp} \subset H^2(X, \mathbb{Z}), \quad T(Y) := \mathrm{Pic}(Y)^{\perp} \subset H^2(Y, \mathbb{Z}).$$

The following are equivalent

- *there exists an isometry of Hodge structures* $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$*;*
- *there exists an isometry of Hodge structures* $T(X) \simeq T(Y)$*;*
- X *and* Y *are derived equivalent;*
- Y *is isomorphic to a moduli space of stable vector bundles over* X*, admitting a universal family* $\mathcal{E} \to X \times Y$*, i.e.,* $Y = M_v(X)$ *for a Mukai vector* v *such that there exists a Mukai vector* w *with* $(v,w) = 1$*.*

See [Orl97, §3.8] and [Huy08, §3] for discussion of the fourth condition; for our purposes, we need not distinguish among various notions of stability. This has been extended to arbitrary fields as follows:

Theorem 4. [LO11, Th. 1.1] *Let* X *and* Y *be K3 surfaces over an algebraically closed field* F *of characteristic* $\neq 2$*. Then the third and the fourth statements are equivalent.*

Morover, a derived equivalence between X and Y induces Galois-compatible isomorphisms between their ℓ-adic Mukai lattices. See [Huy15, §16.4] for more discussion.

1.4. **Descending derived equivalence.** Let F be a field of characteristic zero with algebraic closure $\bar{F}$. Given Y smooth and projective over F, let $\bar{Y}$ denote the corresponding variety over $\bar{F}$. We say Y is *of K3 type* if $\bar{Y}$ is deformation equivalent to the Hilbert scheme of a K3 surface.

Lemma 5. *Suppose* Y_1 *and* Y_2 *are of K3 type over* F *and there exists an isomorphism* $\iota : \bar{Y}_1 \xrightarrow{\sim} \bar{Y}_2$ *defined over* $\bar{F}$ *inducing*

$$\iota^* : H^2(\bar{Y}_2, \mathbb{Z}_\ell) \xrightarrow{\sim} H^2(\bar{Y}_1, \mathbb{Z}_\ell)$$

compatible with the action of $\mathrm{Gal}(\bar{F}/F)$*. Then* $Y_1 \simeq Y_2$ *over* F*.*

Proof. In characteristic zero, Galois fixed points of $\mathrm{Isom}(\bar{Y}_1, \bar{Y}_2)$ correspond to isomorphisms between the varieties defined over F. Automorphisms of $\bar{Y}_1$ (resp. $\bar{Y}_2$) act transitively and faithfully on this set by pre-composition (resp. post-composition).

The Torelli theorem implies that the automorphism group of a K3 surface has a faithful representation in its second cohomology. This holds true for manifolds of K3 type as well—see [Mar10, Prop. 1.9] as well as previous work of Beauville and Kaledin-Verbitsky. The Galois invariance of ι^* implies that ι is Galois-fixed, hence defined over F. □

Suppose that X and Y are K3 surfaces over F. Given a derived equivalence

$$\Phi : D(X) \xrightarrow{\sim} D(Y)$$

over F the induced

$$\phi : \tilde{H}(\bar{X}, \mathbb{Z}_\ell) \xrightarrow{\sim} \tilde{H}(\bar{Y}, \mathbb{Z}_\ell)$$

is compatible $\mathrm{Gal}(\bar{F}/F)$ actions. We consider whether the converse holds.

We use the notation $\mathcal{M}_v(X)$ for the moduli *stack* of vector bundles/complexes over X; this is typically a $\mathbb{G}_m$-gerbe over $M_v(X)$ due to homotheties.

Proposition 6. *Suppose that X and Y are K3 surfaces over F. Let $\Phi : D(\bar{X}) \xrightarrow{\sim} D(\bar{Y})$ be a derived equivalence over $\bar{F}$ such that the induced*

$$\phi : \tilde{H}(\bar{X}, \mathbb{Z}_\ell) \xrightarrow{\sim} \tilde{H}(\bar{Y}, \mathbb{Z}_\ell)$$

is compatible with Galois actions. Write $v = \phi^{-1}(1, 0, 0)$ and $w = \phi(1, 0, 0)$.

Then we have isomorphisms $Y \simeq M_v(X)$ and $X \simeq M_w(Y)$ inducing the derived equivlance over $\bar{F}$. Moreover, there exist $\mathbb{G}_m$-gerbes $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$ and isomorphisms $\mathcal{Y} \simeq \mathcal{M}_v(X)$ and $\mathcal{X} \simeq \mathcal{M}_w(Y)$ such that $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$ admit trivializations over $\bar{F}$.

Proof. Choosing an appropriate polarization on X, the moduli space $M_v(\bar{X})$ is a K3 surface. This follows from [Muk87, Prop. 4.2]; Mukai's argument shows that the requisite polarization can be found over a non-closed field, as we just have to avoid walls orthogonal to certain Mukai vectors.

There exists a Mukai vector $v' \in \tilde{H}(\bar{X}, \mathbb{Z}_\ell)$ that is Galois invariant and satisfies $(v, v') = 1$, e.g., $v' = \phi^{-1}(0, 0, -1)$. Thus the universal sheaf

$$\mathcal{E} \to X \times \mathcal{M}_v(X)$$

has the following property: on basechange to $\bar{F}$ it may be obtained as the pull-back of a universal sheaf on the coarse space

$$\bar{\mathcal{E}} \to \bar{X} \times M_v(\bar{X}).$$

Indeed, the universal sheaf exists wherever $M_{v'}(X)$ admits a rational point (see [LO11, Th. 3.16]).

We apply Lemma 5 to Y and $M_v(X)$. The Torelli Theorem implies $\bar{Y} \simeq M_v(\bar{X})$ with the induced isomorphism compatible with the Galois actions on ℓ-adic cohomology. Thus it descends to give $Y \simeq M_v(X)$ defined over F. □

Remark 7. Can X and Y fail to be derived equivalent over F? How do the pull-back homomorphisms

$$\mathrm{Br}(F) \to \mathrm{Br}(X), \quad \mathrm{Br}(F) \to \mathrm{Br}(Y)$$

compare? They must have the same kernel if X and Y are derived equivalent over F.

Remark 8. Sosna [Sos10] has given examples of complex K3 surfaces that are derived equivalent but not isomorphic to their complex conjugates.

1.5. Cycle-theoretic invariants of derived equivalence.

Proposition 9. *Let X and Y be derived equivalent K3 surfaces over a field F of characteristic $\neq 2$. Then* $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(Y)$ *are stably isomorphic as* $\mathrm{Gal}(\bar{F}/F)$*-modules, and* $\mathrm{Br}(X)[n] \simeq \mathrm{Br}(Y)[n]$ *provided n is not divisible by the characteristic.*

Even over $\mathbb{C}$, this result does not extend to higher dimensional varieties [Add13].

Proof. The statement on the Picard groups follows from the Chow realization of the Fourier-Mukai transform—see [LO11, §2.7] for discussion. We write

$$\tilde{H}_{\text{ét}}(X, \mu_n) = H^0_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) \oplus H^2_{\text{ét}}(X, \mu_n) \oplus H^4_{\text{ét}}(X, \mu_n^{\otimes 2}).$$

The Chow-theoretic interpretation of the Fourier-Mukai kernel gives the realization

$$\phi : \tilde{H}_{\text{ét}}(X, \mu_n) \to \tilde{H}_{\text{ét}}(Y, \mu_n),$$

compatible with cycle class maps [LO11, Prop. 2.10]. Modding out by the images of the cycle class maps we get the desired equality of Brauer groups. □

Recall that the *index* $\mathrm{ind}(X)$ of a smooth projective variety X over a field F is the greatest common divisor of the degrees of field extensions F'/F over which $X(F') \neq \emptyset$, or equivalently, the lengths of zero-dimensional subschemes $Z \subset X$.

Given a bounded complex of locally free sheaves on X

$$E = \{E_{-M} \to E_{-M+1} \to \cdots \to E_N\}$$

we may define the Chern character

$$\mathrm{ch}(E) = \sum_j (-1)^j \, \mathrm{ch}(E_j)$$

in Chow groups with $\mathbb{Q}$ coefficients. The degree zero and one pieces yield the rank and first Chern class of E, expressed as alternating sums of the ranks and determinants of the terms, respectively. Similarly, we may define

$$c_2(E) = -\operatorname{ch}_2(E) + \operatorname{ch}_1(E)^2/2,$$

a quadratic expression in the Chern classes of the E_j with *integer* coefficients. Modulo the $\mathbb{Z}$ algebra generated by the first Chern classes of the E_j, we may write

$$c_2(E) \equiv \sum_j (-1)^j c_2(E_j).$$

Lemma 10. *If (S, h) is a smooth projective surface over F then*

$$\begin{aligned} \operatorname{ind}(S) &= \gcd\{c_2(E) : E \text{ vector bundle on } S\} \\ &= \gcd\{c_2(E) : E \in D^b(S)\}. \end{aligned}$$

Proof. Consider the 'decomposible index'

$$\operatorname{inddec}(S) := \gcd\{D_1 \cdot D_2 : D_1, D_2 \text{ very ample divisors on } S\}$$

which is equal to

$$\gcd\{D_1 \cdot D_2 : D_1, D_2 \text{ divisors on } S\},$$

because for any divisor D the divisor $D + Nh$ is very ample for $N \gg 0$. All three quantities in the assertion divide $\operatorname{inddec}(S)$, so we work modulo this quantity.

By the analysis of Chern classes above, the second and third quantities agree. Given a reduced zero-dimensional subscheme $Z \subset S$ we have a resolution

$$0 \to E_{-2} \to E_{-1} \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$$

with E_{-2} and E_{-1} vector bundles. This implies that

$$\gcd\{c_2(E) : E \text{ vector bundle on } S\} | \operatorname{ind}(S).$$

Conversely, given a vector bundle E there exists a twist $E \otimes \mathcal{O}_S(Nh)$ that is globally generated and

$$c_2(E \otimes \mathcal{O}_S(Nh)) \equiv c_2(E) \pmod{\operatorname{inddec}(S)}.$$

Thus there exists a zero-cycle Z with degree $c_2(E \otimes \mathcal{O}_S(Nh))$ and

$$\operatorname{ind}(S) | \gcd\{c_2(E) : E \text{ vector bundle on } S\}.$$

□

Proposition 11. *If X is a K3 surface over a field F then*

$$\mathrm{ind}(X) \mid \gcd\{24, D_1 \cdot D_2 \text{ where } D_1, D_2 \text{ are divisors on } X\}.$$

This follows from Lemma 10 and the fact that $c_2(T_X) = 24$. Beauville-Voisin [BV04] and Huybrechts [Huy10] have studied the corresponding subgroup of $\mathrm{CH}_0(X_{\bar{F}})$.

Proposition 12. *Let X and Y be derived equivalent K3 surfaces over a field F. Then* $\mathrm{ind}(X) = \mathrm{ind}(Y)$.

The proof will require the notion of a *spherical object* on a K3 surface. This is an object $\mathcal{S} \in D^b(X)$ with

$$\mathrm{Ext}^0(\mathcal{S},\mathcal{S}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{S}) = F, \quad \mathrm{Ext}^i(\mathcal{S},\mathcal{S}) = 0, \quad i \neq 0, 2.$$

These satisfy the following

- $(v(\mathcal{S}), v(\mathcal{S})) = -2$;
- rigid simple vector bundles are spherical;
- each spherical object $\mathcal{S}$ has the associated spherical twist [Huy15, 16.2.4]:
$$T_{\mathcal{S}} : D^b(X) \to D^b(X),$$
an autoequivalence such that the induced homomorphism on the Mukai lattice is the reflection associated with $v(\mathcal{S})$;
- each spherical object $\bar{\mathcal{S}}$ on $X_{\bar{F}}$ is defined over a finite extension F'/F [Huy10, 5.4];
- over $\mathbb{C}$, each $v = (r, D, s) \in \tilde{H}(X,\mathbb{Z}) \cap H^{1,1}$ with $(v, v) = -2$ arises from a spherical object, which may be taken to be a rigid vector bundle E if $r > 0$ [Kul89];
- under the same assumptions, for each polarization h on X there is a *unique* h-slope stable vector bundle E with $v(E) = v$ [Huy12a, 5.1.iii].

Proof. Proposition 6 allows us to express $Y = M_v(X)$ for $v = (r, ah, s)$ where h is a polarization on X and $a^2h^2 = 2rs$. It also yields a Mukai vector $w = (r', bg, s') \in \tilde{H}(X, \mathbb{Z}_\ell)$ with

$$(v, w) = abg \cdot h - rs' - sr' = 1.$$

Thus we have

$$\langle r, s\rangle = \langle 1\rangle \pmod{g \cdot h}. \tag{1.1}$$

Consider a Fourier-Mukai transform realizing the equivalence

$$\Phi : D^b(X) \to D^b(Y)$$

and the induced homomorphism ϕ on the Mukai lattice. Note that

$$\phi(v) = (0, 0, 1)$$

reflecting the fact that a point on Y corresponds to a sheaf on X with Mukai vector v.

Suppose that Y has a rational point over a field of degree n over F; let $Z \subset Y$ denote the corresponding subscheme of length n. Applying Φ^{-1} to $\mathcal{O}_Z$ gives an element of the derived category with Mukai vector (nr, nah, ns) and

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) = \frac{c_1(\Phi^{-1}(\mathcal{O}_Z))^2}{2} - \chi(\Phi^{-1}(\mathcal{O}_Z)) + 2\,\mathrm{rank}(\Phi^{-1}(\mathcal{O}_Z))$$

which equals $n(nrs + r - s)$. Following the proof of Lemma 10, we compute

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) \pmod{\mathrm{inddec}(X)}.$$

First suppose that r and s have different parity, so that

$$\gcd(nrs + r - s, 2rs) = \gcd(nrs + r - s, rs).$$

Then using (1.1) we obtain

$$\begin{aligned} \langle nrs + r - s, rs \rangle &= \langle r - s, rs \rangle = \langle r - s, r \rangle \langle r - s, s \rangle \\ &= \langle -s, r \rangle \langle r, s \rangle = \langle r, s \rangle^2 = \langle 1 \rangle \pmod{g \cdot h}. \end{aligned}$$

If r and s are both even then $g \cdot h$ must be odd and

$$\langle nrs + r - s, 2rs \rangle = \langle nrs + r - s, rs \rangle \pmod{g \cdot h}$$

and repeating the argument above gives the desired conclusion.

Now suppose that r and s are both odd. It follows that $h^2 \equiv 2 \pmod 4$ and we write $h^2 = 2\gamma - 2$ for some even integer γ. Let $\mathcal{S}$ denote the spherical object associated with h so that $v(\mathcal{S}) = (1, h, \gamma)$. Applying $T_{\mathcal{S}}$ to the Mukai vector

$$(r, ah, s) \mapsto (r, ah, s) + ((r, ah, s), (1, h, \gamma))\,(1, h, \gamma),$$

we obtain a new vector with rank $r + (ah^2 - s - r\gamma)$, which is even. This reduces us to the previous situation.

In each case, we find

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) \equiv n \pmod{\mathrm{inddec}(X)},$$

whence $\mathrm{ind}(X) | n$. Varying over all degrees n, we find

$$\mathrm{ind}(X) | \mathrm{ind}(Y)$$

and the Proposition follows. □

The last result raises the question of whether spherical objects are defined over the ground field:

Question 13. Let X be a K3 surface over a field F. Suppose that $\bar{\mathcal{S}}$ is a spherical object on $X_{\bar{F}}$ such that $c_1(\bar{\mathcal{S}}) \in \mathrm{Pic}(X_{\bar{F}})$ is a divisor defined over X. When does $\bar{\mathcal{S}}$ come from an object $\mathcal{S}$ on X?

Kuleshov [Kul89, Kul90] gives a partial description of how to generate all exceptional bundles on K3 surfaces of Picard rank one through 'restructuring' operations and 'dragons'. It would be worthwhile to analyze which of these operations could be defined over the ground field.

Example 14. We give an example of a K3 surface X over a field F with

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{F}}) = \mathbb{Z}h$$

and a rigid sheaf E over $X_{\bar{F}}$ that fails to descend to F.

Choose (X, h) to be a degree fourteen K3 surface defined over $\mathbb{R}$ with $X(\mathbb{R}) = \emptyset$. This may be constructed as follows: Fix a smooth conic C and quadric threefold Q with

$$C \subset Q \subset \mathbb{P}^4, \quad Q(\mathbb{R}) = \emptyset.$$

Let X' denote a complete intersection of Q with a cubic containing C; we have $X'(\mathbb{R}) = \emptyset$ and X' admits a lattice polarization

$$\begin{array}{c|cc} & g & C \\ \hline g & 6 & 2 \\ C & 2 & -2 \end{array}.$$

Write $h = 2g - C$ so that (X', h) is a degree 14 K3 surface containing a conic. Let X be a small deformation of X' with $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}h$.

The K3 surface X is Pfaffian if and only if it admits a vector bundle E with $v(E) = (2, h, 4)$ corresponding to the classifying morphism $X \to \mathrm{Gr}(2, 6)$. However, note that

$$c_2(E) = 5$$

which would mean that $\mathrm{ind}(X) = 1$. On the other hand, if $X(\mathbb{R}) = \emptyset$ then $\mathrm{ind}(X) = 2$.

2. Examples of derived equivalence

2.1. **Elliptic fibrations.** The paper [AKW14] has a detailed discussion of derived equivalences among genus one curves over function fields.

In this section we work over a field F of characteristic zero.

A K3 surface X is *elliptic* if it admits a morphism $X \to C$ to a curve of genus zero with fibers of genus one. We allow $C = \mathbb{P}^1$ or a non-split conic over F.

Lemma 15. *A K3 surface X is elliptic if and only if it admits a non-trivial divisor D with $D^2 = 0$.*

Proof. If X is elliptic then the pull back of a non-trivial divisor from C gives a square-zero class; we focus on the converse.

This is well-known if F is algebraically closed [PŠŠ71, §6, Th. 1]. Indeed, let $\mathcal{C}_+ \subset H^2(X, \mathbb{R})$ denote the component of the positive cone $\{\eta : (\eta, \eta) > 0\}$ containing an ample divisor and $\Gamma \subset \mathrm{Aut}(H^2(X, \mathbb{Z}))$ the group generated by Picard-Lefschetz reflections ρ_R associated with (-2)-curves $R \in \mathrm{Pic}(X)$. Then the Kähler cone of X is a fundamental domain for the action of Γ on $\mathcal{C}_+$, in the sense that no two elements of the cone are in the same orbit and each orbit in $\mathcal{C}_+$ meets the closure of the Kähler cone. Cohomology classes in the interior of the cone have trivial Γ-stabilizer; classes on walls of the boundary associated with (-2)-curves are stabilized by the associated reflections.

Thus if $\mathrm{Pic}(X)$ represents zero then there exists a non-zero divisor D in the closure of the Kähler cone with $D^2 = 0$, which induces an elliptic fibration $X \to \mathbb{P}^1$. Given a divisor $D \in \mathcal{C}_+$, we can be a bit more precise about the $\gamma \in \Gamma$ required to take D to a nef divisor. We can write

$$\gamma = \rho_{R_1} \cdots \rho_{R_m}$$

where each R_j is the class of an irreducible rational curve contained in the fixed part of the linear series $|D|$ (cf. proof in [PŠŠ71, §6], [SD74, §2]).

Now suppose F is not algebraically closed and D is defined over F. The group $\mathrm{Gal}(\bar{F}/F)$ acts on the (-2)-curves on $\bar{X}$ and thus on Γ and the orbit $\Gamma \cdot D$. The Galois action on $R_1, \ldots, R_m$ may be nontrivial. However, the fundamental domain description guarantees a unique $f \in \Gamma \cdot D$ in the nef cone of X, which is necessarily Galois invariant; some multiple of this divisor is defined over F. This semiample divisor induces our elliptic fibration. □

Proposition 16. *Let X be an elliptic K3 surface over F and Y another K3 surface derived equivalent to X over F. Then Y is elliptic over F.*

Proof. By Proposition 9, the Picard groups of X and Y are stably isomorphic as lattices

$$\mathrm{Pic}(X) \oplus U \simeq \mathrm{Pic}(Y) \oplus U, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ share the same discriminant and p-adic invariants. A rank-two indefinite lattice represents zero if and only if its discriminant is a square. In higher ranks, an indefinite lattice represents zero if and only if it represents zero p-adically for each p.

Thus $\mathrm{Pic}(Y)$ admits a square-zero class and is elliptic by Lemma 15. □

An elliptic K3 surface $J \to C$ is *Jacobian* if it admits a section $C \to J$. It has geometric Picard group containing

(2.1)

	f	Σ
f	0	1
Σ	1	−2

where f is a fiber and Σ is the section. Jacobian elliptic surfaces admit numerous autoequivalences [Bri98, §5]. Let $a, b \in \mathbb{Z}$ with $a > 0$ and $(a, b) = 1$. The moduli space of rank a degree b indecomposable vector bundles on fibers of $J \to C$ is an elliptic fibration with section, isomorphic to J over C. This reflects Atiyah's classification of vector bundles over elliptic curves [Ati57, Th. 7]. The associated Fourier-Mukai transform induces an autoequivalence of J acting on $\tilde{H}(J, \mathbb{Z})$ via an element of

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

obtained as in [Bri98, Th. 3.2,5.3]. This acts on the Mukai vectors (or their ℓ-adic analogues)

$$(1, 0, 0), (0, 0, 1) \in H^0(J, \mathbb{Z})(-1) \oplus H^2(J, \mathbb{Z}) \oplus H^4(J, \mathbb{Z})(1)$$

by the formula

$$(1, 0, 0) \mapsto (0, af, c), \qquad (0, 0, 1) \mapsto (b, d(f + \Sigma), 0).$$

The action is transitive on the (-2)-vectors in the lattice of algebraic classes

$$H^0(J, \mathbb{Z})(-1) \oplus H^4(J, \mathbb{Z})(1) \oplus \langle f, \Sigma \rangle \simeq \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \qquad (2.2)$$

Thus we have established the following:

Proposition 17. *Let $J \to C$ be a Jacobian elliptic K3 surface over F. Then autoequivalences of J defined over F induce a representation of* $\mathrm{SL}_2(\mathbb{Z})$ *on (2.2) as above.*

The point is that here autoequivalences are defined over the ground field.

Over $\mathbb{C}$, the availability of autoequivalences has strong consequences for Jacobian elliptic K3 surfaces. For instance, orientation preserving automorphisms of the Mukai lattice of a complex K3 surface arise from autoequivalences [HLOY04a, Th. 1.6]. This implies that derived equivalent Jacobian elliptic K3 surfaces are isomorphic [HLOY04b, Cor. 2.7.3]. The autoequivalences are generated via spherical twists associated with rigid objects.

As we have seen (e.g., in Example 14), rigid objects over non-closed fields need not descend. It is natural to ask the following question:

Question 18. Let J_1 and J_2 denote Jacobian elliptic K3 surfaces over a field F of characteristic zero. If J_1 and J_2 are derived equivalent does it follow that $J_1 \simeq J_2$?

If the geometric Picard group has rank two then the isomorphism follows from Proposition 17.

We recall the classical Ogg-Shafarevich theory for elliptic fibrations, following [Căl00, 4.4.1,5.4.5]: Let F be algebraically closed of characteristic zero and $J \to \mathbb{P}^1$ a Jacobian elliptic K3 surface. We may interpret

$$Ш(J/\mathbb{P}^1) \simeq \mathrm{Br}(J)$$

and each α in this group may be realized by an elliptically fibered K3 surface $X \to \mathbb{P}^1$ with Jacobian fibration $J \to \mathbb{P}^1$. If α has order n then we have natural exact sequences

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{Br}(J) \to \mathrm{Br}(X) \to 0$$

and

$$0 \to T(X) \to T(J) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Note that if $Y = \mathrm{Pic}^c(X/\mathbb{P}^1)$ then $[Y] = c[X] \in Ш(J/\mathbb{P}^1)$. If c is relatively prime to n, so there exists an integer b with $bc \equiv 1 \pmod{n}$, then we have $\mathrm{Pic}^b(Y) = X$ as well. It follows that X and Y are derived equivalent with the universal sheaves inducing the Fourier-Mukai transform [Bri98, Th. 1.2].

Proposition 19. *Let F be algebraically closed of characteristic zero. Let $\phi : X \to \mathbb{P}^1$ be an elliptic K3 surface with Jacobian fibration $J(X) \to \mathbb{P}^1$. Let $\alpha \in \mathrm{Br}(J(X))$ denote the Brauer class associated with $[X]$ in the Tate-Shafarevich group of $J(X) \to \mathbb{P}^1$. Then X is derived equivalent to the pair $(J(X), \alpha)$.*

This follows from the proof of Căldăraru's conjecture; see [HS06, 1.vi] as well as [Căl00, 4.4.1] for the fundamental identification between the twisting data and the Tate-Shafarevich group.

Question 20. Let X and Y be K3 surfaces derived equivalent over a field F. Suppose we have an elliptic fibration $X \to C$ over F. Does Y admit a fibration $Y \to C$ such that

$$\begin{array}{ccc} X & \stackrel{\sim}{\to} & \mathrm{Pic}^b(Y/C) \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

for some integer b?

By symmetry we have $X \simeq \mathrm{Pic}^b(Y/C)$ and $Y \simeq \mathrm{Pic}^c(X/C)$ for $b, c \in \mathbb{Z}$. As before, if $n = \mathrm{ord}([X]) = \mathrm{ord}([Y])$ in the Tate-Shafarevich group then $bc \equiv 1 \pmod{n}$.

A positive answer to Question 20 would imply:

- X dominates Y over F and *vice versa*.
- If X and Y are elliptic K3 surfaces derived equivalent over a field F of characteristic zero then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$.

2.2. **Rank one K3 surfaces.** We recall the general picture:

Proposition 21. [Ogu02, Prop. 1.10], [Ste04] *Let $X/\mathbb{C}$ be a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}h$, where $h^2 = 2n$. Then the number m of isomorphism classes of K3 surfaces Y derived equivalent to X is given by*

$$m = 2^{\tau(n)-1}, \quad \textit{where} \quad \tau(n) = \textit{number of prime factors of } n.$$

Example 22. The first case where there are multiple isomorphism classes is degree twelve. Let (X, h) be such a K3 surface and $Y = M_{(2,h,3)}(X)$ the moduli space of stable vector bundles $E \to X$ with

$$\mathrm{rk}(E) = 2, \quad c_1(E) = h, \quad \chi(E) = 2 + 3 = 5,$$

whence $c_2(E) = 5$. Note that if $Y(F) \neq \emptyset$ then X admits an effective zero-cycle of degree five and therefore a zero-cycle of degree one. Indeed, if $E \to X$ is a vector bundle corresponding to $[E] \in Y(F)$ then a generic $\sigma \in \Gamma(X, E)$ vanishes at five points on X. As we vary σ, we get a four-parameter family of such cycles. Moreover, the cycle h^2 has degree twelve, relatively prime to five.

Is $X(F) \neq \emptyset$ when $Y(F) \neq \emptyset$?

2.3. **Rank two K3 surfaces.** Exhibiting pairs of non-isomorphic derived equivalent complex K3 surfaces of rank two is a problem on quadratic forms [HLOY04b, §3]. Suppose that $\mathrm{Pic}(X_{\mathbb{C}}) = \Pi_X$ and $\mathrm{Pic}(Y_{\mathbb{C}}) = \Pi_Y$ and X and Y are derived equivalent. Orlov's Theorem implies $T(X) \simeq T(Y)$ which means that Π_X and Π_Y have isomorphic discriminant groups/p-adic invariants. Thus we have to exhibit p-adically equivalent rank-two even indefinite lattices that are not equivalent over $\mathbb{Z}$. (Proposition 9 asserts this is a necessary condition over general fields of characteristic zero.)

Example 23. We are grateful to Sho Tanimoto and Letao Zhang for assistance with this example. Consider the lattices

$$\Pi_X = \begin{array}{c|cc} & C & f \\ \hline C & 2 & 13 \\ f & 13 & 12 \end{array} \qquad \Pi_Y = \begin{array}{c|cc} & D & g \\ \hline D & 8 & 15 \\ g & 15 & 10 \end{array}$$

which both have discriminant 145. Note that Π_X represents -2

$$(2f - C)^2 = (25C - 2f)^2 = -2$$

but that Π_Y fails to represent -2.

Let X be a K3 surface over F with split Picard group Π_X over a field F. We assume that C and f are ample. The moduli space

$$Y = M_{(2,C+f,10)}(X)$$

has Picard group

$$\begin{array}{c|cc} & 2C & (C+f)/2 \\ \hline 2C & 8 & 15 \\ (C+f)/2 & 15 & 10 \end{array} \simeq \Pi_Y$$

while $M_{(2,D,2)}(Y)$ has Picard group

$$\begin{array}{c|cc} & D/2 & 2g \\ \hline D/2 & 2 & 15 \\ 2g & 15 & 40 \end{array} \simeq \Pi_X$$

and is isomorphic to X.

These surfaces have the following properties:

- X and Y admit decomposable zero cycles of degree one over F;

- $X(F) \neq \emptyset$: the rational points arise from the smooth rational curves with classes $2f - C$ and $25C - 2f$, both of which admit zero-cycles of odd degree and thus are $\simeq \mathbb{P}^1$ over F;
- $Y(F')$ is dense for some finite extension F'/F, due to the fact that $|\operatorname{Aut}(Y_{\mathbb{C}})| = \infty$.

We do not know whether

- $X(F')$ is dense for any finite extension F'/F;
- $Y(F) \neq \emptyset$.

3. Finite and real fields

The ℓ-adic interpretation of the Fourier-Mukai transform yields

Theorem 24. [LO11] [Huy15, 16.4.3] *Let X and Y be K3 surfaces derived equivalent over a finite field F. Then for each finite extension F'/F we have*

$$|X(F')| = |(Y(F')|.$$

For the case of general surfaces see [Hon13].

We have a similarly complete picture over the real numbers. We review results of Nikulin [Nik79, §3] [Nik08, §2] on real K3 surfaces.

Let X be a K3 surface over $\mathbb{R}$, $X_{\mathbb{C}}$ the corresponding complex K3 surface, and φ the action of the anti-holomorphic involution (complex conjugation) of $X_{\mathbb{C}}$ on $H^2(X_{\mathbb{C}}, \mathbb{Z})$. Let $\Lambda_{\pm} \subset H^2(X_{\mathbb{C}}, \mathbb{Z})$ denote the eigenlattices where φ acts via ± 1. If D is a divisor on X defined over $\mathbb{R}$ then

$$\varphi([D]) = -D;$$

the sign reflects the fact that complex conjugation reverses the sign of $(1,1)$ forms. In Galois-theoretic terms, the cycle class of a divisor lives naturally $H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ and twisting by -1 accounts for the sign change. Let $\tilde{\Lambda}_{\pm}$ denote the eigenlattices of the Mukai lattice; note that $\tilde{\Lambda}_{-}$ contains the degree zero and four summands. Again, the sign change reflects the fact that these are twisted in the Mukai lattice.

We introduce the key invariants: Let $\mathbf{r}$ denote the rank of Λ_{-}. The discriminant groups of $\Lambda_{\pm}$ are two-elementary groups of order 2^a where a is a non-negative integer. Note that $\tilde{\Lambda}_{\pm}$ have discriminant groups of the same order. Finally, we set

$$\delta_{\varphi} = \begin{cases} 0 \text{ if } (\lambda, \varphi(\lambda)) \equiv 0 (\operatorname{mod} 2) \text{ for each } \lambda \in \Lambda \\ 1 \text{ otherwise.} \end{cases}$$

Note that δ_φ can be computed via the Mukai lattice

$$\delta_\varphi = 0 \text{ iff } (\lambda, \varphi(\lambda)) \equiv 0 (\text{mod } 2) \text{ for each } \lambda \in \tilde{\Lambda},$$

as the degree zero and four summands always give even intersections.

We observe the following:

Proposition 25. *Let X and Y be K3 surfaces over $\mathbb{R}$, derived equivalent over $\mathbb{R}$. Then*

$$(\mathbf{r}(X), a(X), \delta_{\varphi,X}) = (\mathbf{r}(Y), a(Y), \delta_{\varphi,Y}).$$

Proof. The derived equivalence induces an isomorphism

$$\tilde{H}(X_{\mathbb{C}}, \mathbb{Z}) \simeq \tilde{H}(Y_{\mathbb{C}}, \mathbb{Z})$$

compatible with the conjugation actions. Since $(\mathbf{r}, a, \delta_\varphi)$ can be read off from the Mukai lattice, the equality follows. □

The topological type of a real K3 surface is governed by these invariants. Let T_g denote a compact orientable surface of genus g.

Proposition 26. [Nik79, Th. 3.10.6] [Nik08, 2.2] *Let X be a real K3 surface with invariants $(\mathbf{r}, a, \delta_\varphi)$. Then the manifold $X(\mathbb{R})$ is orientable and*

$$X(\mathbb{R}) = \begin{cases} \emptyset & \text{if } (\mathbf{r}, a, \delta_\varphi) = (10, 10, 0) \\ T_1 \sqcup T_1 & \text{if } (\mathbf{r}, a, \delta_\varphi) = (10, 8, 0) \\ T_g \sqcup (T_0)^k & \text{otherwise, where} \\ & \qquad g = (22 - \mathbf{r} - a)/2, k = (\mathbf{r} - a)/2. \end{cases}$$

Corollary 27. *Let X and Y be K3 surfaces defined and derived equivalent over $\mathbb{R}$. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are diffeomorphic. In particular, $X(\mathbb{R}) \neq \emptyset$ if and only if $Y(\mathbb{R}) \neq \emptyset$.*

The last statement also follows from Proposition 12: A variety over $\mathbb{R}$ has a real point if and only if its index is one. (This was pointed out to us by Colliot-Thélène.)

Example 28. Let X and Y be derived equivalent K3 surfaces, defined over $\mathbb{R}$; assume they have Picard rank one. Then $Y = M_v(X)$ for some isotropic Mukai vector $v = (r, f, s) \in \tilde{H}(X(\mathbb{C}), \mathbb{Z})$ with $(r, s) = 1$. For a vector bundle E of this type note that

$$c_2(E) = c_1(E)^2/2 + r\chi(\mathcal{O}_X) - \chi(E) = rs + r - s,$$

which is odd as r and s are not both even. Then a global section of E gives an odd-degree cycle on X over $\mathbb{R}$, hence an $\mathbb{R}$-point.

4. Geometric case: local fields with complex residue field

We start with a general definition: Let R be a discrete valuation ring with quotient field F. Given a K3 surface X over F, a model is a flat proper algebraic space $\mathcal{X} \to \Delta = \mathrm{Spec}(R)$ with generic fiber X. We say X has *good reduction* if there exists a smooth model and *ADE reduction* if there is a model with (at worst) rational double points in the central fiber $\mathcal{X}_0$.

Let $F = \mathbb{C}((t))$ with algebraic closure $\bar{F}$, obtained by adjoining all the roots of t. Let X be a projective K3 surface over $F = \mathbb{C}((t))$; write $\bar{X}$ for its base-change to $\bar{F}$. Consider the monodromy action

$$T : H^2(\bar{X}, \mathbb{Z}) \to H^2(\bar{X}, \mathbb{Z})$$

associated with a counterclockwise loop about $t = 0$. This is *quasi-unipotent*, i.e., there exist $e, f \in \mathbb{N}$ such that $(T^e - I)^f = 0$ [Lan73]; we choose e, f minimal with this property.

If X and Y are derived equivalent over F then Orlov's Theorem and the discussion preceding Proposition 6 implies that their Mukai lattices admit a monodromy equivariant isomorphism.

Write $\Delta = \mathrm{Spec}\,\mathbb{C}[[t]]$ and fix a projective model

$$\mathcal{X} \to \Delta$$

and a resolution

$$\varpi : \mathcal{X}' \to \Delta$$

such that the central fiber $\varpi^{-1}(0)$ is a normal crossings divisor, perhaps with multiplicities along some components. Let $\mathcal{X}'_\circ \subset \varpi^{-1}(0)$ denote the smooth locus of the central fiber, i.e., the points of mulitiplicity one. A'Campo [A'C75, Th. 1] proved that

$$\chi(\mathcal{X}'_\circ) = 2 + \mathrm{trace}(T).$$

This may be interpreted as the alternating sum of the traces of the monodromy matrices on *all* the cohomology groups of X.

An application of Hensel's Lemma yields

Proposition 29. *If* $\mathrm{trace}(T) \neq -2$ *then* $\mathcal{X} \to \Delta$ *admits a section, i.e.,* $X(F) \neq \emptyset$.

This result was previously obtained by Nicaise [Nic11, Cor. 6.6]; his techniques are also applicable in mixed characteristic under appropriate tameness assumptions.

Proposition 29 applies when T is unipotent ($e = 1$). This is the case when there exists a resolution $\mathcal{X}' \to \Delta$ with central fiber *reduced* normal crossings. (See the Appendix and Theorem 38 for further analysis.)

If X and Y are derived equivalent over F then Orlov's Theorem and the discussion preceding Proposition 6 implies that their Mukai lattices admit a monodromy equivariant isomorphism. Thus the characteristic polynomials of their monodromy matrices are equal.

Corollary 30. *Suppose that X and Y are derived equivalent K3 surfaces with monodromy satisfying* $\mathrm{trace}(T) \neq -2$. *Then both $X(F)$ and $Y(F)$ are nonempty.*

Let $\mathcal{X} \to \Delta$ denote a nonsingular model with central fiber $\mathcal{X}_0$. Inclusion $\bar{X} \hookrightarrow \mathcal{X}$ and retraction of $\mathcal{X}$ to $\mathcal{X}_0$ induce

$$\varrho : H_2(\bar{X}, \mathbb{Z}) \to H_2(\mathcal{X}, \mathbb{Z}) \to H_2(\mathcal{X}_0, \mathbb{Z});$$

elements of the kernel are called *vanishing cycles*. The local invariant cycle theorem [CEZGT14, 5.3.4] gives an exact sequence

$$H^2(\mathcal{X}_0, \mathbb{Q}) \xrightarrow{\varrho^*} H^2(\bar{X}, \mathbb{Q}) \xrightarrow{T-I} H^2(\bar{X}, \mathbb{Q}).$$

The kernel of ϱ is dual to the cokernel of ϱ^*; we may identify this with the image of $(T - I)$. Thus we define the *lattice of vanishing classes* of X as saturation of the image of $(T - I)$ in $H^2(\bar{X}, \mathbb{Z})$. This carries the structure of a representation of a cyclic group, i.e., the action of T. Since $(T - I)$ acts trivially on H^0 and H^4, the lattice of vanishing cycles may also be in interpreted as the saturation of the image of $(T - I)$ on the full Mukai lattice $\widetilde{H}(X, \mathbb{Z})$. Again, Orlov's Theorem implies

Proposition 31. *For K3 surfaces over $F = \mathbb{C}((t))$, lattices of vanishing classes are a derived invariant.*

We have results in some cases where T is semisimple, i.e., when $T^e = I$ for some $e \in \mathbb{N}$.

Definition 32. Let X be a K3 surface over $F = \mathbb{C}((t))$ with monodromy T. It is of *ADE* type if

- the lattice of vanishing classes admits a finite-index sublattice isomorphic to a direct sum of negative definite lattices associated with root systems of types A_n, D_n, E_6, E_7, or E_8;
- the action of T on the vanishing classes may be expressed as a product of reflections in roots in these lattices.

Roughly, the monodromy is in the product of Weyl groups associated with the vanishing classes.

Since the lattice of vanishing classes is a derived invariant, being of ADE type is as well.

Proposition 33. *X is of ADE type if and only if it has ADE reduction.*

An application of Hensel's Lemma gives

Corollary 34. *If X is of ADE type then $X(F)$ is non-empty.*

Proof. Suppose X admits a model $\mathcal{X} \to \Delta$ with ADE singularities in the central fiber. Replacing $\mathcal{X}$ with a small modification, we may assume that it has $\mathbb{Q}$-factorial terminal singularities [KM98, Th. 6.25]. Such a model is maximal among small partial resolutions of the original family; see [Rei83, §1,8] and [KM92] for interpretations in the context of families of surfaces with ADE singularities. The point of passing to this model is to ensure that the cycles collapsed by the specialization $\bar{X} \rightsquigarrow \mathcal{X}_0$ are in fact vanishing cycles.

After a suitable basechange

$$\Delta_1 \to \Delta \quad t = t_1^e,$$

our family admits a simultaneous resolution [Art74, Bri71]

$$\widetilde{\mathcal{X}} \to \mathcal{X} \times_\Delta \Delta_1,$$

i.e., $\rho_0 : \widetilde{\mathcal{X}}_0 \to \mathcal{X}_0$ is the minimal resolution. As $\widetilde{\mathcal{X}}_0$ is a smooth degeneration of X it is also a K3 surface. Moreover, the vanishing cycles of X are spanned (over $\mathbb{Q}$) by ρ_0-exceptional curves and the monodromy of X is a product of reflections in the product of Weyl groups indexed by the singularities of $\mathcal{X}_0$. Thus X is of ADE type.

Conversely, assume that X satisfies the requisite monodromy condition. After a basechange $\Delta_1 \to \Delta$ as above, the Torelli Theorem gives a smooth (Type I Kulikov) model

$$\widetilde{\mathcal{X}} \to \Delta_1.$$

We analyze the central fiber $\widetilde{\mathcal{X}}_0$. The lattice of vanishing classes specializes to the saturation of a direct sum of ADE lattices in the Picard group of $\widetilde{\mathcal{X}}_0$. Moreover, we may interpret T as acting birationally on $\widetilde{\mathcal{X}}$, resolved after a flop of vanishing smooth rational curves in $\widetilde{\mathcal{X}}_0$ [Rei83, Thm. 8.2]. This action is over the action $t_1 \to \zeta t_1$ with $\zeta = \exp(2\pi i/e)$.

Contracting the vanishing (-2)-curves, we obtain a model

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \rightarrow & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & \Delta_1 & \end{array}$$

with ADE singularities in the central fiber $\mathcal{Y}_0$, over which T is regular. The quotient

$$\mathcal{X} := \mathcal{Y}/T \rightarrow \Delta_1/T = \Delta$$

is the desired model of X. □

5. Semistable models over p-adic fields

Let F be a p-adic field with ring of integers R.

We start with the case of good reduction, which follows from Theorem 24 and Hensel's Lemma:

Corollary 35. *Let X and Y be K3 surfaces over F, which are derived equivalent and have good reduction over F. Then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$.*

We can extend this as follows:

Proposition 36. *Assume that the residue characteristic $p \geq 7$. Let X and Y be K3 surfaces defined and derived equivalent over F. Assume X and Y admit models*

$$\mathcal{X}, \mathcal{Y} \rightarrow \mathrm{Spec}(R)$$

that are regular and have ADE singularities in the central fiber. Then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$.

Proof. Let k be the finite residue field, $\mathcal{X}_0$ and $\mathcal{Y}_0$ the resulting reductions, and $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{Y}}_0$ their minimal resolutions over $\bar{k}$.

Our assumptions on p guarantee that the classification and deformations of rational double points over k coincides with the classification in characteristic 0 [Art77]. Applying Artin's version of Brieskorn simultaneous resolution [Art74, Th. 2], there exists a finite extension

$$\mathrm{Spec}(R_1) \rightarrow \mathrm{Spec}(R)$$

and proper models

$$\widetilde{\mathcal{X}} \rightarrow \mathcal{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R'),$$

$$\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R'),$$

in the category of algebraic spaces, with central fibers $\widetilde{\mathcal{X}}_0$ and $\widetilde{\mathcal{Y}}_0$.

The Fourier-Mukai transform induces isomorphisms

$$\tilde{H}_{\text{ét}}(\bar{X}, \mathbb{Z}_\ell) \to \tilde{H}_{\text{ét}}(\bar{Y}, \mathbb{Z}_\ell)$$

compatible with monodromy and the action of Frobenius. Specializing yields an isomorphism

$$\psi : H^2_{\text{ét}}(\widetilde{\mathcal{X}}_0, \mathbb{Z}_\ell) \to H^2_{\text{ét}}(\widetilde{\mathcal{Y}}_0, \mathbb{Z}_\ell).$$

Note that since $\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}$ are not projective over $\operatorname{Spec}(R')$, there is not an evident interpretation of this as a derived equivalence over $\operatorname{Spec}(R')$. (See [BM02] for such interpretations for K3 fibrations over complex curves.) Furthermore ψ is far from unique, as we may compose with reflections arising from exceptional curves in either $\widetilde{\mathcal{X}}_0 \to \mathcal{X}_0$ or $\widetilde{\mathcal{Y}}_0 \to \mathcal{Y}_0$ associated with vanishing cycles of $\mathcal{X}$ or $\mathcal{Y}$.

Let L (resp. M) denote the lattice of vanishing classes in $H^2(\bar{X}, \mathbb{Z}_\ell)$ (resp. $H^2(\bar{Y}, \mathbb{Z}_\ell)$). As in the proof of Proposition 31, $L \simeq M$, compatibly with the monodromy and Frobenius actions. (Since X and Y have ADE reduction, the local invariant cycle theorem still applies to their models.) Their orthogonal complements in the Mukai lattice $L^\perp$ and $M^\perp$ are isomorphic as well.

The central fibers $\mathcal{X}_0$ and $\mathcal{Y}_0$ are obtained from $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{Y}}_0$ by blowing down the (-2)-curves classes associated with L and M respectively. Let $\mathcal{X}_\circ \subset \mathcal{X}_0$ and $\mathcal{Y}_\circ \subset \mathcal{Y}_0$ denote the smooth loci, i.e., the complements of the rational curves associated with L and M respectively.

We can relate compactly supported cohomology to our lattices:

$$L^\perp \otimes \mathbb{Q}_\ell \supset H^2_{c,\text{ét}}(\bar{\mathcal{X}}_\circ, \mathbb{Q}_\ell), \quad M^\perp \otimes \mathbb{Q}_\ell \supset H^2_{c,\text{ét}}(\bar{\mathcal{Y}}_\circ, \mathbb{Q}_\ell),$$

with the difference reflecting contributions from H^0 and H^4. Thus ψ induces an isomorphism

$$H^2_{c,\text{ét}}(\bar{\mathcal{X}}_\circ, \mathbb{Q}_\ell) \simeq H^2_{c,\text{ét}}(\bar{\mathcal{Y}}_\circ, \mathbb{Q}_\ell),$$

compatible with Galois actions. The Weil conjectures yield then that

$$|\mathcal{X}_\circ(k)| = |\mathcal{Y}_\circ(k)|.$$

Rational points of X and Y over F correspond to sections of $\mathcal{X} \to \operatorname{Spec}(R)$ and $\mathcal{Y} \to \operatorname{Spec}(R)$. Since $\mathcal{X}$ and $\mathcal{Y}$ are regular, these reduce to points of $\mathcal{X}_\circ(k)$ and $\mathcal{Y}_\circ(k)$. Hensel's Lemma then implies our result. □

Question 37. Is admitting a model with good or ADE reduction a derived invariant?

Y. Matsumoto and C. Liedtke have recently addressed this. Having *potentially* good reduction is governed by whether $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ is unramified, under technical assumptions [Mat14, Th. 1.1]. These are satisified if there exists a Kulikov model after some basechange [ML14, p. 2]. The ramification condition depends on the ℓ-adic cohomology and thus only on the derived equivalence class. Proposition 33 suggests a monodromy characterization of ADE reduction in mixed characteristic; under technical assumptions, there exists such a model if the cohomology is unramified [ML14, Th. 5.1].

Appendix: Semistability and derived equivalence

To understand the implications of derived equivalence for rational points over local fields, we must first describe how monodromy governs the existence of models with good properties. For K3 surfaces, it is widely expected that unipotent monodromy should suffice to guarantee the existence of a Kulikov model. Over $\mathbb{C}$ this is widely known to experts; the referee advised us that this was addressed in correspondence among R. Friedman, D. Morrison, and F. Scattone in 1983. As we are unaware of a published account, we offer an argument:

Theorem 38. *Let X be a K3 surface over $F = \mathbb{C}((t))$. Then X admits a Kulikov model if and only if its monodromy is unipotent.*

Corollary 39. *Let X and Y be derived equivalent K3 surfaces over F. Then X admits a Kulikov model if and only if Y admits a Kulikov model.*

As we have seen, if X and Y are derived equivalent over F then their Mukai lattices admit a monodromy-equivariant isomorphism; thus the characteristic polynomials of their monodromy matrices are equal.

The remainder of this section is devoted the proof of Theorem 38. We start with a review of basic results on Kulikov models.

Let $R = \mathbb{C}[[t]]$, $\Delta = \mathrm{Spec}(R)$, and $\Delta^\circ = \mathrm{Spec}(F)$. The monodromy T of X over $\mathbb{C}((t))$ satisfies

$$(T^e - I)^f = 0$$

for some $e, f \in \mathbb{N}$. We take e and f minimal with this property.

The semistable reduction theorem [KKMSD73] implies there exists an integer $n \geq 1$ such that after basechange to

$$R_2 = \mathbb{C}[[t_2]], F_2 = \mathbb{C}((t_2)), \quad t_2^n = t,$$

there exists a flat proper

$$\pi_2 : \mathcal{X}_2 \to \Delta_2 = \mathrm{Spec}(R_2)$$

such that

- the generic fiber is the basechange of X to F_2;
- the central fiber $\pi_2^{-1}(0)$ is a reduced normal crossings divisor.

We call this a *semistable model* for X. It is well-known that semistable reductions have unipotent monodromy so $e|n$.

By work of Kulikov and Persson-Pinkham [Kul77, PP81], there exists a semistable modification of $\mathcal{X}_2$

$$\varpi : \widetilde{\mathcal{X}} \to \Delta_2$$

with trivial canonical class, i.e., there exists a birational map $\mathcal{X}_2 \dashrightarrow \widetilde{\mathcal{X}}$ that is an isomorphism away from the central fibers. We call this a *Kulikov model* for X. Furthermore, the structure of the central fiber $\widetilde{\mathcal{X}}_0$ can be described in more detail:

Type I $\widetilde{\mathcal{X}}_0$ is a K3 surface and $f = 1$.

Type II $\widetilde{\mathcal{X}}_0$ is a chain of surfaces glued along elliptic curves, with rational surfaces at the end points and elliptic ruled surfaces in between; here $f = 2$.

Type III $\widetilde{\mathcal{X}}_0$ is a union of rational surfaces and $f = 3$.

We will say more about the Type III case: It determines a combinatorial triangulation of the sphere with vertices indexed by irreducible components, edges indexed by double curves, and 'triangles' indexed by triple points [Mor84]. We analyze this combinatorial structure of $\widetilde{\mathcal{X}}_0$ in terms of the integer m.

Let $\widetilde{\mathcal{X}}_0 = \cup_{i=1}^n V_i$ denote the irreducible components, $\tilde{V}_i$ their normalizations, and $D_{ij} \subset \tilde{V}_i$ the double curves over $V_i \cap V_j$.

Definition 40. $\widetilde{\mathcal{X}}_0$ is in *minus-one* form if for each double curve D_{ij} we have $({D'_{ij}}^2)_{V_i} = -1$ if D'_{ij} is a smooth component of D_{ij} and $(D_{ij}^2)_{V_i} = 1$ if D_{ij} is nodal.

Miranda-Morrison [MM83] have shown that after elementary transformation of $\widetilde{\mathcal{X}}$, we may assume that $\widetilde{\mathcal{X}}$ is in minus-one form.

The following are equivalent [Fri83, §3],[FS85, 0.5,7.1]:

- the logarithm of the monodromy is m times a primitive matrix;
- $\widetilde{\mathcal{X}}_0$ admits a 'special μ_m action', i.e., acting trivially on the sets of components, double/triple points, and Picard groups of the irreducible components;
- $\widetilde{\mathcal{X}}_0$ admits 'special m-bands of hexagons', i.e., the triangulation coming from the components of $\widetilde{\mathcal{X}}_0$ arises as a degree m refinement of another triangulation.

In other words, $\widetilde{\mathcal{X}}_0$ 'looks like' it is obtained from applying semistable reduction to the degree m basechange of a Kulikov model. Its central fiber $\widetilde{\mathcal{X}}'_0$ can readily be described [Fri83, 4.1]—its triangulation is the one with refinement equal to the triangulation of $\widetilde{\mathcal{X}}_0$, and its components are contractions of the corresponding components of $\widetilde{\mathcal{X}}_0$. (When we refer to $\widetilde{\mathcal{X}}'_0$ below in the Type III case, we mean the surface defined by this process.)

For Type II we can do something similar [Fri84, §2]. After elementary modifications, we may assume the elliptic surfaces are minimal. Then following are equivalent:

- the logarithm of the monodromy is m times a primitive matrix;
- $\widetilde{\mathcal{X}}_0 = V_0 \cup_E \ldots \cup_E V_m$ is a chain of $m+1$ surfaces glued along copies of an elliptic curve E, where V_0 and V_m are rational and $V_1, \ldots, V_{m-1}$ are minimal surfaces ruled over E.

Again $\widetilde{\mathcal{X}}_0$ 'looks like' it is obtained from applying semistable reduction to another Kulikov model with central fiber $\widetilde{\mathcal{X}}'_0 = V_0 \cup_E V_m$. Moreover $(E^2)_{V_0} + (E^2)_{V_m} = 0$ and $\widetilde{\mathcal{X}}'_0$ is d-semistable in the sense of Friedman [Fri84, 2.1]. (When we refer to $\widetilde{\mathcal{X}}'_0$ below in the Type II case, we mean the surface defined by this process.)

There are refined Kulikov models taking into account polarizations: Let (X, g) be a polarized K3 surface over F of degree $2d$. Shepherd-Barron [SB83] has shown there exists a Kulikov model $\varpi : \widetilde{\mathcal{X}} \to \Delta_2$ with the following properties:

- there exists a specialization of g to a nef Cartier divisor on the central fiber $\widetilde{\mathcal{X}}_0$;
- g is semi-ample relative to Δ_2, inducing

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \stackrel{|g|}{\to} & \mathcal{Z} \\ & \searrow \quad \swarrow & \\ & \Delta_2 & \end{array}$$

  where $\widetilde{\mathcal{X}}_0 \to \mathcal{Z}_0$ is birational and $\mathcal{Z}_0$ has rational double points, normal crossings, or singularities with local equations

$$xy = zt = 0.$$

These will be called *quasi-polarized Kulikov models* and their central fibers *admissible degenerations of degree* $2d$.

Recall the construction in sections five and six of [FS85]: Let $\mathcal{D}$ denote the period domain for degree $2d$ K3 surfaces and Γ the corresponding arithmetic group—the orientation-preserving automorphisms of the cohomology lattice $H^2(X, \mathbb{Z})$ fixing g. Fix an admissible degeneration $(\mathcal{Y}_0, g)$ of degree $2d$ and its image $(\mathcal{Z}_0, h)$, with deformation spaces $\mathrm{Def}(\mathcal{Y}_0, g) \to \mathrm{Def}(\mathcal{Z}_0, h)$; the morphism arises because g is semiample over the deformation space. Let

$$\overline{\Gamma\backslash\mathcal{D}}_{N_{\mathcal{Y}_0}} \supset \Gamma\backslash\mathcal{D}$$

denote the partial toroidal compactification parametrizing limiting mixed Hodge structures with monodromy weight filtration given by a nilpotent $N_{\mathcal{Y}_0}$ associated with $\mathcal{Y}_0$ (see [FS85, p.27]). We do keep track of the stack structure. Given a holomorphic mapping

$$f : \{t : 0 < |t| < 1\} \to \Gamma\backslash\mathcal{D},$$

that is locally liftable (lifting locally to $\mathcal{D}$), with unipotent monodromy Γ-conjugate to $N_{\mathcal{Y}_0}$, then f extends to

$$f : \{t : |t| < 1\} \to \overline{\Gamma\backslash\mathcal{D}}.$$

The period map extends to an étale morphism [FS85, 5.3.5,6.2]

$$\mathrm{Def}(\mathcal{Y}_0, g) \to \overline{\Gamma\backslash\mathcal{D}}.$$

Thus the partial compactification admits a (local) universal family.

Theorem 38 thus boils down to

> The smallest positive integer n for which we have a Kulikov model equals the smallest positive integer e such that T^e is unipotent.

Proof. We show that a Kulikov model exists provided the monodromy is unipotent. Suppose we have unipotent monodromy over

$$R_1 = \mathbb{C}[[t_1]], t_1^e = t,$$

and semistable reduction

$$\mathcal{X}_2 \to \Delta_2 = \mathrm{Spec}(R_2), \quad R_2 = \mathrm{Spec}(\mathbb{C}[[t_2]]), t_2^{me} = t.$$

Let $\widetilde{\mathcal{X}} \to \Delta_2$ denote a Kulikov model, obtained after applying elementary transformations as specified above. Write

$$mN = \log(T^e) = (T^e - I) - \frac{1}{2}(T^e - I)^2$$

where $m \in \mathbb{N}$ and N is primitive (cf.[FS85, 1.2] for the Type III case).

Let $\widetilde{\mathcal{X}}'_0$ be the candidate for the 'replacement' Kulikov model, i.e., the central fiber of the Kulikov model we expect to find

$$\widetilde{\mathcal{X}}' \to \Delta_1.$$

In the Type I case $\widetilde{\mathcal{X}}'_0 = \widetilde{\mathcal{X}}_0$ by Torelli, so we focus on the Type II and III cases.

Lemma 41. *Suppose that $\widetilde{\mathcal{X}}_0$ admits a degree $2d$ semiample divisor g. Then $\widetilde{\mathcal{X}}'_0$ admits one as well, denoted by g'.*

Proof. In the Type II case, this follows from [Fri84, Th. 2.3]. The discussion there shows how (after elementary modification) the divisor can be chosen to induce the morphism $\widetilde{\mathcal{X}}_0 \to \widetilde{\mathcal{X}}'_0$ collapsing $V_1 \cup_E \ldots \cup_E V_{m-1}$ to E, interpreted as the normal crossings locus of $\widetilde{\mathcal{X}}'_0$.

For Type III, we rely on Proposition 4.2 of [Fri83], which gives an analogous process for modifying the coefficients of h so that it is trivial or a sum of fibers on the special bands of hexagons. However, Friedman's result does not indicate whether the resulting line bundle is nef. This can be achieved after birational modifications of the total space [SB83, Th. 1]. □

We can apply the Friedman-Scattone compactification construction to both $(\widetilde{\mathcal{X}}_0, g)$ and $(\widetilde{\mathcal{X}}'_0, g')$, with $N = N_{\widetilde{\mathcal{X}}'_0}$ and $mN = N_{\widetilde{\mathcal{X}}_0}$. Thus we obtain *two* compactifications

$$\overline{\Gamma\backslash\mathcal{D}}_{mN} \to \overline{\Gamma\backslash\mathcal{D}}_N \supset \Gamma\backslash\mathcal{D},$$

both with universal families of degree $2d$ K3 surfaces and admissible degenerations.

To construct $\widetilde{\mathcal{X}}' \to \Delta_1 = \mathrm{Spec}(R_1)$ we use the diagram

$$\begin{array}{ccc} \Delta_2 & \to & \overline{\Gamma\backslash\mathcal{D}}_{mN} \\ \downarrow & & \downarrow \\ \Delta_1 & & \overline{\Gamma\backslash\mathcal{D}}_N. \end{array}$$

The liftability criterion for mappings to the toriodal compactifications gives an arrow

$$\Delta_1 \to \overline{\Gamma\backslash\mathcal{D}}_N$$

making the diagram commute. The induced universal family on this space induces a family

$$\widetilde{\mathcal{X}}' \to \Delta_1,$$

agreeing with our original family for $t_1 \neq 0$ by the Torelli Theorem. More precisely, the monodromies of the two families over $\Delta_1^\circ =$

$\mathrm{Spec}(R_1) \setminus \{0\}$ are identified and automorphisms of K3 surfaces act faithfully on cohomology, so they are isomorphic over Δ_1°. Thus $\widetilde{\mathcal{X}}' \to \Delta_1$ is the desired model. □

References

[A'C75] Norbert A'Campo. La fonction zêta d'une monodromie. *Comment. Math. Helv.*, 50:233–248, 1975.

[Add13] Nicolas Addington. The Brauer group is not a derived invariant. To appear in the proceedings of the AIM workshop *Brauer groups and obstruction problems: moduli spaces and arithmetic*, 2013. arXiv:1306.6538.

[ADPZ15] Kenneth Ascher, Krishna Dasaratha, Alexander Perry, and Rong Zhou. Derived equivalences and rational points of twisted $K3$ surfaces, 2015. arXiv:1506.01374.

[AKW14] Benjamin Antieau, Daniel Krashen, and Matthew Ward. Derived categories of torsors for abelian schemes, 2014. arXiv:1409.2580.

[Art74] Michael Artin. Algebraic construction of Brieskorn's resolutions. *J. Algebra*, 29:330–348, 1974.

[Art77] Michael Artin. Coverings of the rational double points in characteristic p. In *Complex analysis and algebraic geometry*, pages 11–22. Iwanami Shoten, Tokyo, 1977.

[Ati57] Michael F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[BM02] Tom Bridgeland and Antony Maciocia. Fourier-Mukai transforms for $K3$ and elliptic fibrations. *J. Algebraic Geom.*, 11(4):629–657, 2002.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014. arXiv:1203.4613.

[Bri71] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.

[Bri98] Tom Bridgeland. Fourier-Mukai transforms for elliptic surfaces. *J. Reine Angew. Math.*, 498:115–133, 1998.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[BV04] Arnaud Beauville and Claire Voisin. On the Chow ring of a $K3$ surface. *J. Algebraic Geom.*, 13(3):417–426, 2004.

[Căl00] Andrei Horia Căldăraru. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Cornell University.

[CEZGT14] Eduardo Cattani, Fouad El Zein, Phillip A. Griffiths, and Lê Dũng Tráng, editors. *Hodge theory*, volume 49 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2014.

[Fri83] Robert Friedman. Base change, automorphisms, and stable reduction for type III $K3$ surfaces. In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, volume 29 of *Progr. Math.*, pages 277–298. Birkhäuser Boston, Mass., 1983.

[Fri84] Robert Friedman. A new proof of the global Torelli theorem for $K3$ surfaces. *Ann. of Math. (2)*, 120(2):237–269, 1984.

[FS85] Robert Friedman and Francesco Scattone. Type III degenerations of $K3$ surfaces. *Invent. Math.*, 83(1):1–39, 1985.

[HLOY04a] Shinobu Hosono, Bong H. Lian, Keiji Oguiso, and Shing-Tung Yau. Autoequivalences of derived category of a $K3$ surface and monodromy transformations. *J. Algebraic Geom.*, 13(3):513–545, 2004.

[HLOY04b] Shinobu Hosono, Bong H. Lian, Keiji Oguiso, and Shing-Tung Yau. Fourier-Mukai number of a K3 surface. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 177–192. Amer. Math. Soc., Providence, RI, 2004.

[Hon13] Katrina Honigs. Derived equivalent surfaces and abelian varieties and their zeta functions. To appear in the *Proceedings of the American Mathematical Society*, 2013. arXiv:1310.4721.

[HS06] Daniel Huybrechts and Paolo Stellari. Proof of Căldăraru's conjecture. Appendix: "Moduli spaces of twisted sheaves on a projective variety" [in *moduli spaces and arithmetic geometry*, 1–30, Math. Soc. Japan, Tokyo, 2006;] by K. Yoshioka. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 31–42. Math. Soc. Japan, Tokyo, 2006.

[Huy08] Daniel Huybrechts. Derived and abelian equivalence of $K3$ surfaces. *J. Algebraic Geom.*, 17(2):375–400, 2008.

[Huy10] Daniel Huybrechts. Chow groups of K3 surfaces and spherical objects. *J. Eur. Math. Soc. (JEMS)*, 12(6):1533–1551, 2010.

[Huy12a] Daniel Huybrechts. Chow groups and derived categories of K3 surfaces. In *Current developments in algebraic geometry*, volume 59 of *Math. Sci. Res. Inst. Publ.*, pages 177–195. Cambridge Univ. Press, Cambridge, 2012.

[Huy12b] Daniel Huybrechts. A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]. *Astérisque*, (348):Exp. No. 1040, x, 375–403, 2012. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042.

[Huy15] Daniel Huybrechts. Lectures on $K3$ surfaces, 2015.

[HVA13] Brendan Hassett and Anthony Várilly-Alvarado. Failure of the Hasse principle on general $K3$ surfaces. *J. Inst. Math. Jussieu*, 12(4):853–877, 2013.

[HVAV11] Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly. Transcendental obstructions to weak approximation on general K3 surfaces. *Adv. Math.*, 228(3):1377–1404, 2011.

[KKMSD73] George Kempf, Finn Faye Knudsen, D. Mumford, and Bernard Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.

[KM92] Sheldon Katz and David R. Morrison. Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups. *J. Algebraic Geom.*, 1(3):449–530, 1992.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kul77] Viktor S. Kulikov. Degenerations of $K3$ surfaces and Enriques surfaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(5):1008–1042, 1199, 1977.

[Kul89] Sergej A. Kuleshov. A theorem on the existence of exceptional bundles on surfaces of type $K3$. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(2):363–378, 1989.

[Kul90] Sergej A. Kuleshov. Exceptional bundles on $K3$ surfaces. In *Helices and vector bundles*, volume 148 of *London Math. Soc. Lecture Note Ser.*, pages 105–114. Cambridge Univ. Press, Cambridge, 1990.

[Lan73] Alan Landman. On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities. *Trans. Amer. Math. Soc.*, 181:89–126, 1973.

[LO11] Max Lieblich and Martin Olsson. Fourier-Mukai partners of $K3$ surfaces in positive characteristic. To appear in the *Annales scientifique de l'École normale supérieure*, 2011. arXiv:1112.5114.

[Mar10] Eyal Markman. Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a $K3$ surface. *Internat. J. Math.*, 21(2):169–223, 2010.

[Mat14] Yuya Matsumoto. Good reduction criteria for $K3$ surfaces, 2014. arXiv:1401.1261.

[ML14] Yuya Matsumoto and Christian Liedtke. Good reduction of $K3$ surfaces, 2014. arXiv:1411.4797.

[MM83] Rick Miranda and David R. Morrison. The minus one theorem. In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, volume 29 of *Progr. Math.*, pages 173–259. Birkhäuser Boston, Boston, MA, 1983.

[Mor84] David R. Morrison. The Clemens-Schmid exact sequence and applications. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 101–119. Princeton Univ. Press, Princeton, NJ, 1984.

[MSTVA14] Kelly McKinnie, Justin Sawon, Sho Tanimoto, and Anthony Várilly-Alvarado. Brauer groups on K3 surfaces and arithmetic applications, 2014. arXiv:1404.5460.

[Muk87] Shigeru Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[Nic11] Johannes Nicaise. A trace formula for varieties over a discretely valued field. *J. Reine Angew. Math.*, 650:193–238, 2011.

[Nik79] Viacheslav V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[Nik08] Viacheslav V. Nikulin. On connected components of moduli of real polarized $K3$ surfaces. *Izv. Ross. Akad. Nauk Ser. Mat.*, 72(1):99–122, 2008.

[Ogu02] Keiji Oguiso. K3 surfaces via almost-primes. *Math. Res. Lett.*, 9(1):47–63, 2002.

[Orl97] Dmitri O. Orlov. Equivalences of derived categories and $K3$ surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Algebraic geometry, 7.

[PP81] Ulf Persson and Henry Pinkham. Degeneration of surfaces with trivial canonical bundle. *Ann. of Math. (2)*, 113(1):45–66, 1981.

[PŠŠ71] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič. Torelli's theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:530–572, 1971.

[Rei83] Miles Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.

[SB83] Nicholas I. Shepherd-Barron. Extending polarizations on families of $K3$ surfaces. In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, volume 29 of *Progr. Math.*, pages 135–171. Birkhäuser Boston, Mass., 1983.

[SD74] B. Saint-Donat. Projective models of $K3$ surfaces. *Amer. J. Math.*, 96:602–639, 1974.

[Sos10] Pawel Sosna. Derived equivalent conjugate $K3$ surfaces. *Bull. Lond. Math. Soc.*, 42(6):1065–1072, 2010.

[Ste04] Paolo Stellari. Some remarks about the FM-partners of $K3$ surfaces with Picard numbers 1 and 2. *Geom. Dedicata*, 108:1–13, 2004.

[Ver13] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013. Appendix A by Eyal Markman.

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