UNIVERSAL SPACES FOR UNRAMIFIED GALOIS COHOMOLOGY

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Abstract. We construct and study universal spaces for birational invariants of algebraic varieties over algebraic closures of finite fields.

Introduction

Let ℓ be a prime. Recall that in topology, there exist unique (up to homotopy) topological spaces $\mathrm{K}(\mathbb{Z}/\ell^n, m)$ such that

- $\mathrm{K}(\mathbb{Z}/\ell^n, m)$ is homotopically trivial up to dimension $m-1$, in particular,
$$\mathrm{H}^i(\mathrm{K}(\mathbb{Z}/\ell^n, m), \mathbb{Z}/\ell^n) = 0, \quad \text{for} \quad 0 < i < m;$$
- $\mathrm{H}^m(\mathrm{K}(\mathbb{Z}/\ell^n, m), \mathbb{Z}/\ell^n)$ is cyclic, with a distinguished generator κ_m;
- for every topological space X and every $\alpha \in \mathrm{H}^m(X, \mathbb{Z}/\ell^n)$ there is a unique, up to homotopy, continuous map
$$\mu_{X,\alpha} : X \to \mathrm{K}(\mathbb{Z}/\ell^n, m)$$
such that
$$\mu^*_{X,\alpha}(\kappa_m) = \alpha.$$

This reduces many questions about singular cohomology to the study of these universal spaces (see, e.g., [1, Chapter 2]). Analogous theories exist for other contravariant functors, for example, topological K-theory, or the theory of cobordisms. The study of moduli spaces in algebraic geometry can be viewed, broadly speaking, as an incarnation of the same idea of universal spaces.

Here we propose a similar theory for unramified cohomology, developed in connection with the study of birational properties of algebraic varieties [4], [15]. The Bloch–Kato conjecture proved by Rost and Voevodsky [30], with a patch by Weibel, combined with techniques and results from birational anabelian geometry in [9], implies that an unramified class in the cohomology of the function field $K = k(X)$ of an

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algebraic variety X over an algebraic closure of a finite field $k = \bar{\mathbb{F}}_p$, with finite constant coefficients, is induced from the cohomology of a finite abelian group G^a. This, together with our prior work on centralizers of elements of Galois groups of function fields [8], implies our main result:

Theorem. *Let ℓ and p be distinct primes, $K = k(X)$ the function field of an algebraic variety X of dimension ≥ 2 over $k = \bar{\mathbb{F}}_p$, G_K its absolute Galois group, and $\alpha_K \in \mathrm{H}^i_{nr}(G_K, \mathbb{Z}/\ell^n), i \geq 2$, an unramified class. Then there exists a finite set J of finite-dimensional k-vector spaces $V_j, j \in J$, depending on α_K, such that α_K is induced, via a rational map, from an unramified class in the cohomology of an explicit open subset of the quotient of*

$$\mathbb{P} := \prod_{j \in J} \mathbb{P}(V_j)$$

by a finite abelian ℓ-group G^a, acting projectively on each factor.

Thus, the spaces $\mathbb{P}/G^a$ serve as universal spaces for *all* finite birational invariants of algebraic varieties over $k = \bar{\mathbb{F}}_p$. The theorem fails for H^1_{nr} because all such elements are induced from abelian varieties and H^1_{nr} vanishes for every smooth proper separably rationally connected variety over an algebraically closed field (see e.g., [16, Corollary 3.6]).

Actions of finite abelian groups G^a on products of projective spaces are described by central extensions of G^a, i.e., by subspaces in $\wedge^2(G^a)$. This allows us to present unramified classes of X in terms of configurations of subspaces of skew-symmetric matrices. For example, if the unramified Brauer group of X is trivial, then all finite birational invariants of X are encoded already in the combinatorics of configurations of liftable subgroups in finite abelian quotients of the absolute Galois group G_K (see Section 1 for the definition).

The program towards the construction of universal spaces for unramified cohomology was outlined in [4] and [5]. The recent proof of the Bloch–Kato conjecture allows us to complete this program, in a more precise and constructive form. This approach to birational invariants leads to many new questions:

- Is there a smaller class of configurations with this universal property?
- How does this structure interact with Sylow subgroups of G_K?
- Is there an extension to cohomology with $\mathbb{Z}_\ell$-coefficients? An equally simple description of models for ℓ-adic invariants would provide insights into higher-dimensional Langlands correspondence.

- What are the analogs of universal spaces for varieties over $k = \bar{\mathbb{Q}}$? Counterexamples to our main result arise from bad reduction places, already for abelian varieties [4].

Here is the roadmap of the paper: In Section 1 we recall basic facts about stable and unramified cohomology. In Section 3 we provide some background on valuation theory. In Section 5 we investigate Galois cohomology groups of function fields of higher-dimensional algebraic varieties over $k = \bar{\mathbb{F}}_p$ and their images in cohomology of finite groups. In Section 6 we introduce and study unramified cohomology of algebraic varieties. Section 7 contains the proof of our main theorem, modulo geometric considerations presented in Sections 8 and 9.

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1. Stable cohomology

Let G be a pro-finite group. We will write

$$G^a = G/[G,G] \quad \text{and} \quad G^c = G/[[G,G],G]$$

for the abelianization, respectively, the second lower central series quotient of G; throughout the paper, we write $[G,G]$ and $[[G,G],G]$ for topological closures of algebraic subgroups generated by the corresponding commutators. We have a canonical central extension

$$1 \to Z \to G^c \xrightarrow{\pi_a} G^a \to 1. \tag{1.1}$$

Let M be a topological G-module and $\mathrm{H}^i(G,M)$ its (continuous) i-cohomology group. These groups are contravariant with respect to G and covariant with respect to M. In this paper, G is either a finite group or a Galois group (see [1] for background on group cohomology and [28] for background on Galois cohomology). We will sometimes omit the coefficient module M from the notation.

Our goal is to investigate incarnations of Galois cohomology of function fields in cohomology of finite groups. For example, let $K = k(X)$ be the function field of an algebraic variety X over an algebraically closed field k; varieties birational to X are called *models* of K. We do not assume a model to be proper over k. Let G_K be the absolute Galois group of K and $\hat{\pi}_1(X)$ the étale fundamental group of X, with respect to some basepoint. The choice of a base point will not affect our considerations and we omit it from our notation. When we work with G_K, we take M to be either $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Z}/\ell^n$, for some prime ℓ invertible in k, with trivial G-action.

We have natural homomorphisms

$$\mathrm{H}^*(\hat{\pi}_1(X)) \xrightarrow{\kappa_X^*} \mathrm{H}^*_{et}(X) \xrightarrow{\tilde{\eta}_X^*} \mathrm{H}^*(G_K),$$

where the right arrow arises from the embedding of the generic point $X_\eta \to X$. We will write

$$\eta_X : G_K \to \hat{\pi}_1(X)$$

and

$$\eta_X^* = \tilde{\eta}_X^* \circ \kappa_X^* : \mathrm{H}^*(\hat{\pi}_1(X)) \to \mathrm{H}^*(G_K)$$

for the corresponding map in cohomology.

We say that a class $\alpha_K \in \mathrm{H}^*(G_K)$ is defined (or represented) on a model X of K if there exists a class $\alpha_X \in \mathrm{H}^*_{et}(X)$ such that

$$\alpha_K = \tilde{\eta}_X^*(\alpha_X).$$

Let G be a finite group. A continuous homomorphism

$$\chi : \hat{\pi}_1(X) \to G$$

gives rise to homomorphisms in cohomology

$$\mathrm{H}^*(G) \xrightarrow{\chi^*} \mathrm{H}^*(\hat{\pi}_1(X)) \xrightarrow{\eta_X^*} \mathrm{H}^*(G_K).$$

Conversely, every $\alpha_K \in \mathrm{H}^*(G_K)$ arises in this way: there exist

- a model X of K,
- a continuous homomorphism χ as above,
- and a class $\alpha_G \in \mathrm{H}^*(G)$

such that

$$\alpha_K = \eta_X^*(\chi^*(\alpha_G)).$$

This follows from the description of étale cohomology of points, see [21]. In such situations we say that α_K is defined on X and is induced from χ.

A version of this construction arises as follows: assume that the characteristic of k does not divide the order of G. Let V be a faithful representation of G over k, and X an algebraic variety over k with function field

$$K = k(X) \simeq k(V)^G,$$

the field of invariants; we will write $X = V/G$ and call it a *quotient*. Even more generally, let Y be a quasi-projective algebraic variety over

k with a generically free action of G, and $X = Y/G$ the quotient. This situation gives rise to a natural surjective continuous homomorphism

$$G_K \to G$$

and induced homomorphisms on cohomology

$$s_K^i : \mathrm{H}^i(G) \to \mathrm{H}^i(G_K).$$

The following lemma shows that we have many choices in realizing a class $\alpha_K \in \mathrm{H}^i(G_K)$:

Lemma 1.1. [4] *Assume that $\alpha_K \in \mathrm{H}^i(G_K)$ is represented by a class $\alpha_X \in \mathrm{H}^i_{et}(X)$ on some affine irreducible model X of K and is induced from a surjective continuous homomorphism $\chi : \hat{\pi}_1(X) \to G$ and a class $\alpha_G \in \mathrm{H}^i(G)$. Let V be a faithful representation of G over k and $V^\circ \subset V$ the locus where the action is free. Then, for every $x \in X$ and $v \in V^\circ$ there exists a map*

$$f = f_x : X \to V/G$$

such that

- *$f(x) = v$ and*
- *the restriction of α_X to $X^\circ = f^{-1}(V^\circ/G) \subset X$ is equal to $f^*(\alpha_G)$.*

Proof. We follow the proof in [4]. The homomorphism $\chi : \hat{\pi}_1(X) \to G$ defines a finite étale covering $\pi : \tilde{X} \to X$, by an affine variety $\tilde{X}$. The ring $k[\tilde{X}]$ is a $k[G]$ -module. Every finite-dimensional $k[G]$-submodule $W^* \subset k[\tilde{X}]$ defines a G-equivariant map $\tilde{X} \to W$.

Let $e \in k[G]$ be the unit element of G. For any G-orbit $G \cdot y \in V$ there is a G-linear homomorphism

$$l_y : k[G] \to V,$$

which maps the orbit $G \cdot e$ to $G \cdot y$. Let $\tilde{x} \in \pi^{-1}(x)$. Choose $h \in k[\tilde{X}]$ such that

$$h(\tilde{x}) = 1, \quad h(g \cdot \tilde{x}) = 0, \quad g \neq e.$$

Then h generates a $k[G]$-submodule $W \subset k[\tilde{X}]$ and defines a regular G-map $h : \tilde{X} \to W = k[G]$, with $h(\tilde{x}) = e \in k[G]$. The map $f := l_y \circ h$ is a regular G-map satisfying the first property.

Let $X^\circ = f^{-1}(V^\circ/G) \subset X$. It is a nonempty affine subvariety. We have a compatible diagram of G-maps

$$\begin{array}{ccccc} V^\circ & \longleftarrow & \tilde{X}^\circ & \subset & \tilde{X} \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\ V^\circ/G & \xleftarrow[f]{} & X^\circ & \subset & X \end{array}$$

and the maps π_G and π induce the same cover π_0. This implies the second claim. □

We can achieve even more flexibility for G-maps, under a *projectivity* condition on V: we say that a G-module V is projective if for every finite-dimensional representation W of G with a G-surjection

$$\mu : W \to V$$

there exists a G-section

$$\theta : V \to W \quad \text{with} \quad \mu \circ \theta = \mathrm{id}.$$

This condition holds, for example, for regular representations over arbitrary fields or when the order of G is invertible in k.

Now let $\{S_j\}_{j\in J}$ be a finite set of G-orbits in a generically free G-variety Y with stabilizers H_j so that $S_j \simeq G/H_j$. Consider a faithful representation V of G and a subset $\{T_j\}_{j\in J}$ of G-orbits in V with stabilizers Q_j, with $H_j \subset Q_j$, $T_j = G/Q_j$. Consider regular G-maps $f_j : S_j \to T_j$, for $j \in J$. Applying the argument above to finite sets of orbits, we obtain:

Lemma 1.2. *Assume that V is a projective G-module. Then there is a regular G-map $f : Y \to V$ such that $f = f_j$, for all $j \in J$.*

We return to our setup: $X = Y/G$, $K = k(X)$, and $\chi : G_K \to G$, inducing

$$s^i_K : \mathrm{H}^*(G) \to \mathrm{H}^*(G_K).$$

The groups

$$\mathrm{H}^i_{s,K}(G) := \mathrm{H}^i(G)/\mathrm{Ker}(s^i_K)$$

are called *stable cohomology groups* with respect to $K = k(X)$. Let

$$\mathrm{Ker}(s^i) := \bigcap_K \mathrm{Ker}(s^i_K),$$

over all function fields $K = k(X)$ as above. In fact,

$$\mathrm{Ker}(s^i) = \mathrm{Ker}(s^i_{k(V/G)}),$$

for some faithful representation V of G over k, in particular, this is independent of the choice of V (see [7, Proposition 4.3]). The groups

$$\mathrm{H}^i_s(G) := \mathrm{H}^i(G)/\mathrm{Ker}(s^i)$$

are called *stable cohomology groups* of G (with coefficients in $M = \mathbb{Z}/\ell^n$ or $\mathbb{Q}/\mathbb{Z}$); they depend on the ground field k. These define contravariant functors in G. For example, for a subgroup $H \subset G$ we have a restriction homomorphism

$$\mathrm{res}_{G/H} : \mathrm{H}^*_s(G) \to \mathrm{H}^*_s(H).$$

Furthermore:

- While usual group cohomology $\mathrm{H}^i(G)$ can be nontrivial for infinitely many i (even for cyclic groups), *stable* cohomology groups $\mathrm{H}^i_s(G)$ vanish for $i > \dim(V)$, where V is a faithful representation.
- We have

$$\mathrm{H}^i_s(G) \subseteq \mathrm{H}^i_s(\mathrm{Syl}_\ell(G))^{N_G(\mathrm{Syl}_\ell(G))},$$

  where the coefficient group M is $\mathbb{Z}/\ell^n$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, $N_G(H)$ is the normalizer of H in G, and $\mathrm{Syl}_\ell(G)$ an ℓ-Sylow subgroup of G.

The determination of the stable cohomology ring

$$\mathrm{H}^*_s(G, \mathbb{Z}/\ell^n) := \oplus_i \mathrm{H}^i_s(G, \mathbb{Z}/\ell^n)$$

is a nontrivial problem, see, e.g., [6] for a computation of stable cohomology of alternating groups. For finite abelian groups G, we have

$$\mathrm{H}^*_s(G, \mathbb{Z}/\ell^n) \subset \wedge^*(\mathrm{H}^1(\mathbb{Z}^m, \mathbb{Z}/\ell^n)), \tag{1.2}$$

induced by a surjection $\mathbb{Z}^m \to G$. For central extensions of finite groups as in (1.1), the kernel of

$$\pi_a^* : \mathrm{H}^*_s(G^a) \to \mathrm{H}^*_s(G^c)$$

contains the ideal $\mathrm{I} = \mathrm{I}(G^c)$ generated by

$$\mathrm{R}^2 = \mathrm{R}^2(G^c) := \mathrm{Ker}\left(\mathrm{H}^2_s(G^a) \to \mathrm{H}^2_s(G^c)\right).$$

(see, for example, [11, Section 8]). An important role in the computation of this subring of $\mathrm{H}^*_s(G^c)$ is played by the *fan*

$$\Sigma = \Sigma(G^c) = \{\sigma\},$$

the set of *noncyclic* liftable subgroups σ of G^a, and the *complete fan*

$$\bar{\Sigma} = \bar{\Sigma}(G^c) = \{\sigma\},$$

consisting of *all* nontrivial liftable subgroups $\sigma \subset G^a$: a subgroup σ is *liftable* if and only if the full preimage $\tilde{\sigma}$ of σ in G^c is abelian. The fan Σ defines a subgroup $\mathrm{R}^2(\Sigma) \subseteq \mathrm{H}^2_s(G^a)$ as the set of all elements which vanish upon restriction to every $\sigma \in \Sigma$. Note that for any $\tilde{\sigma}$ and σ as above, the natural homomorphism of cohomology groups

$$\mathrm{H}^i_s(\sigma) \to \mathrm{H}^i(\tilde{\sigma})$$

is injective; indeed, stable cohomology of any finite abelian group G^a with any finite coefficients coindices with the image of the group cohomology of G^a in the group cohomology of any finite rank free abelian group surjecting onto G^a. Using this fact, we have

$$\mathrm{R}^2 \subseteq \mathrm{R}^2(\Sigma).$$

Lemma 1.3. *For every $\alpha \in \mathrm{I}(G^c) \subseteq \mathrm{H}_s^*(G^a)$ and every $\sigma \in \Sigma(G^c)$ the restriction of α to σ is trivial.*

Definition 1.4. Let

$$1 \to Z \to G^c \to G^a \to 1$$

be a central extension of finite groups, with G^a abelian. A Δ-pair (I, D) of G^a is a set of subgroups

$$I \subseteq D \subseteq G^a$$

such that

- $I \in \bar{\Sigma}(G^c)$,
- D is noncyclic,
- for every $\delta \in D$, the subgroup $\langle I, \delta \rangle \in \bar{\Sigma}(G^c)$.

This definition depends on G^c. Assume we have a commutative diagram of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a & \longrightarrow & 1. \end{array}$$

Definition 1.5. A Δ-pair $(\tilde{I}, \tilde{D})$ of $\tilde{G}^a$ *surjects* onto a Δ-pair (I, D) of G^a if $\gamma(\tilde{I}) = I$ and $\gamma(\tilde{D}) = D$.

Definition 1.6. A class $\alpha \in \mathrm{H}_s^i(G^a)$ is *unramified* with respect to a Δ-pair if its restriction to D is induced from D/I, i.e., there exists a $\beta \in \mathrm{H}_s^i(D/I)$ such that $\phi(\alpha) = \psi(\beta)$, for the natural homomorphisms in the diagram:

$$\mathrm{H}_s^*(G^a) \xrightarrow{\phi} \mathrm{H}_s^*(D) \xleftarrow{\psi} \mathrm{H}_s^*(D/I)$$

Recall that a cohomology class $\beta \in \mathrm{H}^*(G_K)$ is *unramified* if for every divisorial valuation ν of K the restriction of β to G_{K_ν}, the Galois group of the completion of K with respect to ν, is induced from the quotient G_{K_ν}/I_ν, where $I_\nu \subset G_K$ is the corresponding inertia subgroup.

When G is a finite group and $K = k(V/G)$, the field of functions of V/G for some faithful representation V of G, the stable cohomology groups of G are naturally subgroups of the corresponding cohomology groups of G_K, and unramified stable cohomology classes are those which are unramified when considered as classes in $\mathrm{H}^*(G_K)$. In particular, the images of unramified classes in G^a with respect to Δ-pairs are mapped to unramified classes in $\mathrm{H}^*(G^c)$, as the following lemma shows.

Lemma 1.7. *Consider a homomorphism*

$$\gamma : \tilde{G}^a \to G^a$$

and a class $\alpha \in \mathrm{H}^i(G^a)$, *for* $i \geq 2$. *Let* $\tilde{\alpha} := \gamma^*(\alpha) \in \mathrm{H}^i(\tilde{G}^a)$ *be the induced class. Let* $(\tilde{I}, \tilde{D})$ *be a* Δ*-pair in* $\tilde{G}^a$. *Assume that one of the following holds:*

- $\gamma(\tilde{I}) = 0$,
- $\gamma(\tilde{D})$ *is cyclic,*
- γ *induces a surjection of* Δ*-pairs*

$$(\tilde{I}, \tilde{D}) \to (I, D)$$

and $\alpha \in \mathrm{H}^i_s(G^a)$ *is unramified with respect to* (I, D).

Then $\tilde{\alpha} \in \mathrm{H}^i_s(\tilde{G}^a)$ *is unramified with respect to* $(\tilde{I}, \tilde{D})$.

Proof. The first case is evident. In the second case, the stable cohomology of $\tilde{D}$ is trivial. Consider the third condition. By assumption, γ induces a homomorphism $\tilde{D}/\tilde{I} \to D/I$. Passing to cohomology we get a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*_s(D/I) & \longrightarrow & \mathrm{H}^*_s(D) \\ \downarrow & & \downarrow \\ \mathrm{H}^*_s(\tilde{D}/\tilde{I}) & \longrightarrow & \mathrm{H}^*_s(\tilde{D}), \end{array}$$

and thus the claim. ∎

2. Central extensions and isoclinism

Let G^a and Z be finite abelian ℓ-groups. Central extensions of G^a by Z are parametrized by $\mathrm{H}^2(G^a, Z)$; for $\alpha \in \mathrm{H}^2(G^a, Z)$ we let G^c_α be the corresponding central extension:

$$1 \to Z \to G^c_\alpha \xrightarrow{\pi_a} G^a \to 1 \tag{2.1}$$

Fix an embedding $Z \hookrightarrow (\mathbb{Q}/\mathbb{Z})^r$, consider the exact sequence

$$1 \to Z \to (\mathbb{Q}/\mathbb{Z})^r \to (\mathbb{Q}/\mathbb{Z})^r \to 1,$$

and the induced long exact sequence in cohomology

$$\mathrm{H}^1(G^a, (\mathbb{Q}/\mathbb{Z})^r) \stackrel{\delta}{\longrightarrow} \mathrm{H}^2(G^a, Z) \to \mathrm{H}^2(G^a, (\mathbb{Q}/\mathbb{Z})^r).$$

We say that $\alpha, \tilde{\alpha} \in \mathrm{H}^2(G^a, Z)$ and the corresponding extensions are *isoclinic* if

$$\alpha - \tilde{\alpha} \in \delta(\mathrm{H}^1(G^a, (\mathbb{Q}/\mathbb{Z})^r)).$$

This notion does not depend on the chosen embedding $Z \hookrightarrow (\mathbb{Q}/\mathbb{Z})^r$ and is equivalent to the standard definition of isoclinic in the theory of ℓ-groups (as in [17]).

Lemma 2.1. *If $\alpha, \tilde{\alpha} \in \mathrm{H}^2(G^a, Z)$ are isoclinic then the corresponding extensions of G^a define the same set of Δ-pairs in G^a.*

Proof. A pair of subgroups (I, D) is a Δ-pair in G^a, with respect to a central extension G^c, if their preimages commute in G^c, i.e.,

$$[\pi_a^{-1}(I), \pi_a^{-1}(D)] = 0 \quad \text{in} \quad Z.$$

Consider the homomorphism

$$\pi_a^* : \mathrm{H}^2(G^a, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(G^c, \mathbb{Q}/\mathbb{Z}),$$

and note that $\mathrm{Ker}(\pi_a^*)$ only depends on the isoclinism class of the extension. Furthermore, $\mathrm{H}^2(G^a, \mathbb{Q}/\mathbb{Z})$ is dual to $\wedge^2(G^a)$. Let $R \subset \wedge^2(G^a)$ be the subgroup which is dual to $\mathrm{Ker}(\pi_a^*)$. It remains to observe that (I, D) is a Δ-pair for G^c if and only if $\pi_a^{-1}(I) \wedge \pi_a^{-1}(D)$ intersects R trivially; thus the notion of a Δ-pair is an invariant of the isoclinism class of the extension. ∎

Lemma 2.2. *If $\alpha, \tilde{\alpha} \in \mathrm{H}^2(G^a, Z)$ are isoclinic then there exist faithful representations $V, \tilde{V}$ of G^c_α and $G^c_{\tilde{\alpha}}$ over k such that V/G^c_α and $\tilde{V}/G^c_{\tilde{\alpha}}$ are birational.*

Proof. Explicit construction: Let $\chi_1, \ldots, \chi_r$ be a basis of $\mathrm{Hom}(Z, k^\times)$ and put

$$V := \oplus_{j=1}^r V_j \qquad \text{and} \qquad \tilde{V} = \oplus_{j=1}^r \tilde{V}_j,$$

where

$$V_j = \mathrm{Ind}_Z^{G^c_\alpha}(\chi_j) \quad \text{and} \quad \tilde{V}_j = \mathrm{Ind}_Z^{G^c_{\tilde{\alpha}}}(\chi_j).$$

Note that the *projectivizations* $\mathbb{P}(V_j) := (V_j \setminus 0)/k^\times$ and $\mathbb{P}(\tilde{V}_j)$ are canonically isomorphic as G^a-representations. The group $(k^\times)^r$ acts on

V and $\tilde{V}$, and *both* V/G^c_α and $V/G^c_{\tilde{\alpha}}$ are birational to

$$\left(\prod_{j=1}^{r} \mathbb{P}(V_j)\right)/G^a \times \left(\prod_{j=1}^{r} k^\times/\chi_j(Z)\right).$$

□

Lemma 2.3. *Consider a central extension of finite groups*

$$1 \to Z \to G^c \xrightarrow{\pi_a} G^a \to 1$$

and let $V = \oplus_j V_j$ *be a faithful representation of* G^c *as in Lemma 2.2, i.e., each* $V_j = \mathrm{Ind}_Z^{G^c}(\chi_j)$*, where* $\{\chi_j\}_{j\in J}$ *is a basis of* $\mathrm{Hom}(Z, k^\times)$*. Let* $\mathbb{P} := \prod_{j\in J} \mathbb{P}(V_j)$*. Then:*

1. G^a *acts faithfully on* $\mathbb{P}$.
2. *For any subgroup* $\sigma \subset G^a$ *the subset of* σ*-fixed points* $\mathbb{P}^\sigma \subset \mathbb{P}$ *is nonempty if and only if* $\sigma \in \bar{\Sigma}(G^c)$.
3. *Each irreducible component of* $\mathbb{P}^\sigma$ *is a product of projective subspaces of* $\mathbb{P}(V_j)$*, corresponding to different eigenspaces of* σ *in* V_j*, and distinct irreducible components are disjoint.*
4. *Each irreducible component of* $\mathbb{P}^\sigma$ *is stable under the action of* $H_\sigma \subset G^c$*, the maximal subgroup such that* $[H_\sigma, \pi_a^{-1}(\sigma)] = 1$ *in* G^c*; the action of* G^c/H_σ *on the set of components of* $\mathbb{P}^\sigma$ *is free.*
5. *The action of* G^a *on* $\mathbb{P}^\circ := \mathbb{P} \setminus \cup_{\sigma\in\bar{\Sigma}\setminus 0}\mathbb{P}^\sigma$ *is free.*

Proof. Since the order of G^c is coprime to the characteristic of k, every $g \subset G^c$ is semi-simple and we can decompose

$$V_j = \oplus_i V_j(\lambda_i(g)),$$

as a sum of eigenspaces. The subset of g-fixed points splits as a product $\prod_j \coprod_i \mathbb{P}(V_j(\lambda_i(g)))$, where the product runs over different eigenvalues in different V_j. It follows that the subset of g-fixed points $\mathbb{P}^g \subset \mathbb{P}$ is a union of products of projective subspaces of $\mathbb{P}(V_j)$.

If $\sigma \in \bar{\Sigma}$ then its elements can be simultaneously diagonalized. Hence the subset of fixed points in $\mathbb{P} = \prod_j \mathbb{P}(V_j)$ is a union of products of projective subspaces, and there is a Zariski open subvariety of $\mathbb{P}$ on which the action of σ is free.

Let $\sigma := \langle g, h\rangle \subset G^a$ be a subgroup such that $\sigma \notin \bar{\Sigma}$. Then the same holds for the images of g, h in $\mathrm{GL}(V_j)$, for at least one $j \in J$. Thus the commutator $[g, h] \in \mathrm{GL}(V_j)$ is a nontrivial scalar matrix, hence they have no common eigenvectors, i.e., no common fixed points in $\mathbb{P}(V_j)$. Thus if $\sigma \notin \bar{\Sigma}$ then σ has no fixed points in $\prod_j \mathbb{P}(V_j)$. Note that projective subspaces corresponding to different eigenvalues of g do not intersect in $\mathbb{P}(V_j)$ and hence $\mathbb{P}^\sigma$ splits into a disjoint union of products of projective subspaces of different $\mathbb{P}(V_j)$.

Assume that $[h, \tilde{\sigma}] = 1$ in G^c for some $h \in G^c$. Then $\langle h, \tilde{\sigma}\rangle$ has a fixed point in each component of $\mathbb{P}^\sigma$ and h maps every component of $\mathbb{P}^\sigma$ into itself. Thus a subgroup $H \subset G^c$, with $[H, \tilde{\sigma}] = 1$ maps every component of $\mathbb{P}^\sigma$ into itself.

Assume that $\langle h, \gamma\rangle \notin \bar{\Sigma}$, for some $\gamma \in \sigma$. Then for some j, the images of h, γ in $\mathrm{GL}(V_j)$ have nonintersecting invariant subvarieties in $\mathbb{P}(V_j)$. In particular, h does not preserve any component of $\mathbb{P}^\sigma$. □

Lemma 2.4. *Let $K = k(X)$ be a function field with Galois group G_K. Given a surjection $G_K \to G^c$, onto some finite central extension of an abelian group G^a, let $\mathbb{P} = \prod_j \mathbb{P}(V_j)$ be the space constructed in Lemma 2.3. Then there is a rational map $\varrho : X \dashrightarrow \mathbb{P}/G^a$ such that*

- *ϱ maps the generic point of X into $\mathbb{P}^\circ/G^a$;*
- *the homomorphism*

$$s_K : \mathrm{H}^*_s(G^a, \mathbb{Z}/\ell^n) \to \mathrm{H}^*(G_K, \mathbb{Z}/\ell^n)$$

  factors through the cohomology of $\mathbb{P}^\circ/G^a$.

Proof. Let $X^\circ \subset X$ be an open affine subvariety such that $\pi_1(X^\circ)$ surjects onto G^c. Let $\tilde{X}^\circ \to X^\circ$ be the induced unramified G^ccovering. Then $k[\tilde{X}^\circ]$ decomposes into an infinite direct sum of G^c-representations. Fix a point $\tilde{X}^\circ$ and consider its orbit. The restriction of $k[\tilde{X}^\circ]$ to this orbit defines a regular quotient G^c-representation isomorphic to $k[G]$. Since the order of G is coprime to p, we have a direct summand of $k[\tilde{X}^\circ]$ projecting isomorphically to $k[G]$ under the above homomorphism. This subspace of regular functions on $\tilde{X}^\circ$ defines a G-equivariant map to V and hence a map $X \dashrightarrow V/G^c$ with desired properties. □

3. Basic valuation theory

Let X be a variety over $k = \bar{\mathbb{F}}_p$, $K = k(X)$ its function field, and G_K the absolute Galois group of K. A valuation of K is a homomorphism

$$\nu : K^\times \to \Gamma_\nu$$

onto a totally ordered abelian group Γ_ν such that its extension to K, via $\nu(0) = \infty$, satisfies the nonarchimedean triangle inequality. A divisorial valuation measures the order of a rational function along a divisor on some model X of K. Let $\mathfrak{o}_\nu$ denote the valuation ring and $\mathfrak{m}_\nu$ the corresponding maximal ideal. The residue field will be denoted by $\boldsymbol{K}_\nu$; in general, it need not be finitely generated over k, see Example 9 in [29]. We write $\mathcal{V}_K$ for the set of (equivalence classes of) valuations of K and $\mathcal{DV}_K$ for the subset of divisorial valuations.

Let $Z \subset X$ be an affine subset, $Z = \mathrm{Spec}(\mathfrak{o}_Z)$, and $\nu \in \mathcal{V}_K$. A valuation ν is said to have a center on Z,

$$\mathfrak{c}_X(\nu)^\circ \subseteq Z$$

if and only if $\nu(f) \geq 0$, for all $f \in \mathfrak{o}_Z$; the center is the closed subvariety of Z corresponding to the prime ideal defined by $\nu(f) > 0$.

For $\nu \in \mathcal{V}_K$, let $D_\nu \subset G_K$ denote a decomposition group of ν and $I_\nu \subset D_\nu$ the inertia subgroup; we have $G_{\boldsymbol{K}_\nu} = D_\nu / I_\nu$ (see, e.g., [19, Section 5] for the description of the inertia subgroup in terms of the value group and the description of the Galois group of the residue field). The pro-ℓ-quotients of these groups will be denoted by $\mathcal{G}_K$, $\mathcal{D}_\nu$, and $\mathcal{I}_\nu$, respectively. We will always assume that $p \neq \ell$. The corresponding abelianizations will be denoted by $\mathcal{G}^a_K, \mathcal{D}^a_\nu$, and $\mathcal{I}^a_\nu$; their canonical central extensions by $\mathcal{G}^c_K, \mathcal{D}^c_K$, and $\mathcal{I}^c_K$. Under our assumptions, $\mathcal{G}^a_K$ is a free abelian pro-ℓ-group.

Lemma 3.1. *For $\nu \in \mathcal{V}_K$ consider the commutative diagram*

$$\begin{array}{ccc} D_\nu & \longrightarrow & G_K \\ {\scriptstyle \pi_\nu}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{D}^a_\nu & \xrightarrow[\delta^a_\nu]{} & \mathcal{G}^a_K, \end{array}$$

where π_v and π are the canonical projections and δ^a_ν is the induced homomorphism. Then δ^a_ν is injective with primitive (i.e., nondivisible) image. In particular, δ^a_ν embeds $\mathcal{I}^a_\nu$ as a primitive subgroup of $\mathcal{D}^a_\nu$.

Proof. We have $\mathcal{G}^a_K = \mathrm{Hom}(K^\times, \mathbb{Z}_\ell)$ and $\mathcal{D}^a_\nu = \mathrm{Hom}(K^\times_\nu, \mathbb{Z}_\ell)$. We have exact sequences

$$1 \to \mathfrak{o}^\times_\nu \to K^\times \to \Gamma_\nu \to 1$$

and

$$1 \to (1 + \mathfrak{m}_\nu)^\times \to \mathfrak{o}^\times_\nu \to \boldsymbol{K}^\times_{nu} \to 1.$$

Note that the elements of $\bar{\mathbb{F}}_p(X)$ with $\mathbb{Q}$-independent values of $\nu(x)$ are algebraically independent. Thus the $\mathbb{Q}$-rank of Γ_ν is $\leq n$ and the $\mathbb{Z}_\ell$ rank of $\mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell)$ is also $\leq n$; it is a free $\mathbb{Z}_l$-module of finite rank. Taking a finitely generated subgroup $S \subset K^\times$ of the same rank, with an isomorphism $\mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell) = \mathrm{Hom}(S, \mathbb{Z}_\ell)$, we obtain a direct splitting (depending on S):

$$\mathrm{Hom}(K^\times, \mathbb{Z}_\ell) = \mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell) \oplus \mathrm{Hom}(\mathfrak{o}^\times_\nu, \mathbb{Z}_\ell).$$

The right summand contains $\mathrm{Hom}(\boldsymbol{K}^*_\nu, \mathbb{Z}_\ell)$ as a primitive subgroup. This implies that

$$\mathcal{D}^a_\nu = \mathcal{I}^a_\nu \oplus \mathcal{G}^a_{\boldsymbol{K}_\nu},$$

where $\mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell) = \mathcal{I}^a_\nu$. □

Remark 3.2. If the residue field $\boldsymbol{K}_\nu$ is finitely generated over k then there is a model X of K such that the center of ν is realized by a subvariety $X_\nu \subset X$.

Indeed, in this case there is a finite subset of elements $f_i \in \mathfrak{o}_\nu$ which generate K and reduce to a generating subset of $\boldsymbol{K}_\nu$. The subring $k[f_1, \dots, f_n]$ defines an affine model X of K and its image B in $\boldsymbol{K}_\nu$ a finitely generating subring of $\boldsymbol{K}_\nu$; hence we have an inverse embedding of affine varieties $X_B \subset X$ with desired properties.

Let $\Sigma(\mathcal{G}^c_K)$ be the set of primitive topologically noncyclic subgroups of $\mathcal{G}^a_K$ whose preimage in $\mathcal{G}^c_K$ is abelian. By [8, Section 6], we have:

Theorem 3.3. *Assume that* $\dim(X) \geq 2$. *Then*

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\sigma) \leq \dim(X), \quad \textit{for all} \quad \sigma \in \Sigma(\mathcal{G}^c_K).$$

The following key result gives a valuation-theoretic interpretation of liftable subgroups in $\mathcal{G}^a_K$; it is crucial for the reconstruction of function fields in [9] and [10].

Theorem 3.4. [8, Corollary 6.4.4] *Assume that* $\dim(X) \geq 2$ *and let* $\sigma \in \Sigma(\mathcal{G}^c_K)$. *Then there exists a valuation* $\nu \in \mathcal{V}_K$ *such that* $\mathcal{I}^a_\nu$ *is a subgroup of* σ *of* $\mathbb{Z}_\ell$*-corank at most one and* $\sigma \subseteq \mathcal{D}^a_\nu$.

4. Liftable subgroups and their configurations

Let $K = k(X)$ be the function field of an algebraic variety over $k = \bar{\mathbb{F}}_p$. In this section, we compare the structure of the fan $\Sigma(\mathcal{G}^c_K)$ with fans in its finite quotients. Consider the canonical central extension

$$1 \to \mathcal{Z}_K \to \mathcal{G}^c_K \to \mathcal{G}^a_K \to 1. \tag{4.1}$$

Lemma 4.1. *We have*

$$\mathcal{Z}_K = [\mathcal{G}^c_K, \mathcal{G}^c_K].$$

Proof. This holds for function fields of curves since the corresponding pro-ℓ-quotients of their absolute Galois groups are free. In higher dimensions, $\mathcal{G}^a_K$ embedds into the product $\prod_E \mathcal{G}^a_E$, where E ranges over function fields of curves $E \subset K$. Under the projection to $\mathcal{G}^c_K \to \mathcal{G}^a_E$, the center of $\mathcal{G}^c_K$ maps to zero, hence the claim. □

Lemma 4.2. *Consider commutative diagrams of continuous homomorphisms*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma_K^c} & & \downarrow{\scriptstyle \gamma_K} & & \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a & \longrightarrow & 1, \end{array}$$

where G^c is finite, with fixed surjective γ_K and surjective γ_K^c. Assume that $Z \subset G^c$ is a quotient of $\mathcal{Z}_K$ such that

$$\mathrm{Ker}\left(\mathrm{H}^2(G^a) \to \mathrm{H}^2(G^c)\right) = \mathrm{Ker}\left(\mathrm{H}^2(G^a) \to \mathrm{H}^2(\mathcal{G}_K^c)\right),$$

with $\mathbb{Q}/\mathbb{Z}$-coefficients. Then G^c is unique modulo isoclinism.

Proof. Assume that G_1^c, G_2^c are two such extensions of G^a with Z_1, Z_2, respectively, and put $G := G_1^c \times_{G^a} G_2^c$. We have a natural surjection $G \to G^a$ and an inclusion $Z_1 \times Z_2 \hookrightarrow G$. Moreover, $[G, G] \subseteq Z_1 \times Z_2$. By Lemma 4.1, $\mathcal{Z}_K$ is generated by commutators in $\mathcal{G}_K^c$. There is natural diagonal projection $\mathcal{G}_K^c \to G$ which maps $\mathcal{Z}_K$ onto $[G, G]$. The image of $\mathcal{G}_K^c$ in G is a subgroup $\tilde{G}^c$ with $\tilde{G}^a \twoheadrightarrow G^a$ and $[\tilde{G}^c, \tilde{G}^c] = [G, G]$. By the maximality assumption, we obtain that both projections of $[G, G]$ into Z_1 and Z_2 are are isomorphisms; this implies isoclinism. $\square$

We proceed to investigate the properties of *fans* under such factorizations. Let

$$\gamma_K : \mathcal{G}_K^a \to G^a$$

be a continuous surjective homomorphism onto a finite group. We choose a maximal finite central extension G^c of G^a as in Lemma 4.2.

Corollary 4.3. *Given continuous surjective homomorphisms*

$$\mathcal{G}_K^a \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a, \tag{4.2}$$

with $\tilde{G}^a$ a finite group, there is a unique (modulo isoclinism of lower rows) diagram of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \tilde{\gamma}_K^c} & & \downarrow{\scriptstyle \tilde{\gamma}_K} & & \\ 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma^c} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a & \longrightarrow & 1 \end{array}$$

with surjective $\tilde{\gamma}_K^c$, γ^c and maximal $\tilde{Z}$, Z.

Proof. Evident. $\square$

We will use the following observation:

Lemma 4.4. *Let $\mathcal{G}^a$ be a profinite abelian group and*

$$\mathcal{G}^a \xrightarrow{\gamma_j} \tilde{G}^a_j, \quad j = 1, \dots, n,$$

a collection of continuous surjective homomorphisms onto finite groups. Then there exists a continuous surjection

$$\gamma : \mathcal{G}^a \to \tilde{G}^a$$

onto a finite group such that each γ_j factors through γ:

$$\gamma_j : \mathcal{G}^a \xrightarrow{\gamma} \tilde{G}^a \to \tilde{G}^a_j.$$

Proof. We can choose $\tilde{G}^a$ to be the image of $\mathcal{G}^a$ in the direct product

$$\tilde{G}_1 \times \cdots \times \tilde{G}_n.$$

□

We are interested in factorizations (4.2), with finite $\tilde{G}^a$, preserving liftable subgroups and their configurations. Throughout we will be working with the canonical, modulo isoclinism, diagram as in Corollary 4.3, i.e., a factorization as in Equation (4.2) will canonically determine $\Sigma(\tilde{G}^c)$ and the set of Δ-pairs in $\tilde{G}^a$, by Lemma 2.1. Let

$$\Sigma_E(G^c) := \{\sigma \in \Sigma(G^c) \,|\, \sigma = \gamma_K(\sigma_K), \quad \text{for some} \quad \sigma_K \in \Sigma(\mathcal{G}^c_K)\}$$

be the subset of *extendable* subgroups.

Lemma 4.5. *Given a continuous surjective homomorphism*

$$\gamma_K : \mathcal{G}^a_K \to G^a$$

onto a finite abelian group there exists a factorization

$$\mathcal{G}^a_K \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a, \quad \gamma_K = \gamma \circ \tilde{\gamma}_K,$$

with finite $\tilde{G}^a$, such that for all $\sigma \in \Sigma(G^c)$ we have: if σ is nonextendable then there is no $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$ with $\gamma(\tilde{\sigma}) = \sigma$.

Proof. First we prove the statement for one nonextendable σ. Write

$$\mathcal{G}^a_K = \operatorname{proj\,lim}_{\iota \in I} G^a_\iota, \qquad \gamma_{\iota\iota'} : G^a_\iota \longrightarrow G^a_{\iota'}, \quad \iota' \preceq \iota, \tag{4.3}$$

where the limit is over finite continuous quotients of $\mathcal{G}^a_K$. Assume that for all ι, there is some $\sigma_\iota \in \Sigma(G^c_\iota)$ surjecting onto σ; this implies that there exist such $\sigma_{\iota'}$, for all $\iota' \preceq \iota$, with $\gamma_{\iota\iota'}(\sigma_\iota) = \sigma_{\iota'}$.

By compactness of $\mathcal{G}^a_K$, there exists a closed liftable $\sigma_K \subset \mathcal{G}^a_K$ surjecting onto σ. This contradicts our assumption that σ is nonextendable. Thus there is a required factorization

$$\mathcal{G}^a_K \to \tilde{G}^a \xrightarrow{\gamma} G^a.$$

Let

$$\{\sigma_1, \dots, \sigma_n\} = \Sigma(G^c) \setminus \Sigma_E(G^c).$$

For each j, let

$$\mathcal{G}^a_K \longrightarrow \tilde{G}^a_j \xrightarrow{\gamma_j} G^a$$

be the factorization constructed above. Now we apply Lemma 4.4, combined with Corollary 4.3, and obtain factorizations of γ_j:

$$\mathcal{G}^a_K \to \tilde{G}^a \to \tilde{G}_j \to G^a, \qquad \gamma : \tilde{G}^a \to G^a,$$

Assume that there is some j for which there exists a $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$ surjecting onto σ_j. Then image $\tilde{\sigma}$ in $\tilde{G}_j$ must be liftable, contradicting the construction in the first part. □

Let

$$I \subseteq D \subseteq G^a.$$

be a Δ-pair (see Definition 1.4). Throughout, we assume that G^a arises as a finite quotient of the Galois group $\mathcal{G}_K$ of some function field K, in particular, the corresponding G^c is determined as in Lemma 4.2, up to isoclinism. We say that (I, D) is *extendable* if there exists a valuation $\nu \in \mathcal{V}_K$ and subgroups

$$I^a \subseteq \mathcal{I}^a_\nu, \quad D^a \subseteq \mathcal{D}^a_\nu, \quad I^a \subset D^a,$$

such that

$$\gamma_K(I^a) = I, \quad \gamma_K(D^a) = D.$$

Recall that a Δ-pair $(\tilde{I}, \tilde{D})$ is said to surject onto (I, D) if

$$\gamma(\tilde{I}) = I, \quad \gamma(\tilde{D}) = D.$$

We will need the following strengthening of Lemma 4.5

Proposition 4.6. *Given a continuous surjective homomorphism*

$$\mathcal{G}^a_K \to G^a$$

onto a finite abelian group there exists a factorization

$$\mathcal{G}^a_K \to \tilde{G}^a \to G^a,$$

with finite $\tilde{G}^a$, such that for all Δ-pairs (I, D) in G^a we have: if (I, D) nonextendable and D is not liftable then there is no Δ-pair $(\tilde{I}, \tilde{D})$ in $\tilde{G}^a$ surjecting onto (I, D).

Proof. As in the proof of Lemma 4.5, it suffices to establish the statement for one nonextendable Δ-pair; indeed, there are only finitely many Δ-pairs in G^a and the same application of Lemma 4.4 will then establish it for all.

Assume that there is no finite quotient of $\mathcal{G}^a_K$ with the desired property. We start with a factorization

$$\mathcal{G}^a_K \to \tilde{G}^a \xrightarrow{\gamma} G^a$$

such that $\tilde{G}^a$ satisfies the conclusions of Lemma 4.5, i.e., no $\sigma \in \Sigma(G^c) \setminus \Sigma_E(G^c)$ is the image of a $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$.

Let (I, D) be a nonextendable Δ-pair. If D/I is cyclic or trivial then, in fact, $D \in \Sigma(G^c)$ and by assumption on $\tilde{G}^c$ it is not liftable to $\Sigma(\tilde{G}^c)$. Thus if a Δ-pair $(\tilde{I}, \tilde{D})$ surjects onto (I, D) then $\langle \tilde{I}, g\rangle \subset \Sigma(\tilde{G}^c)$ surjects onto D, for some $g \in \tilde{D}$, and hence D lifts to $\Sigma(\tilde{G}^c)$, contradicting our assumption. Thus it suffices to consider Δ-pairs (I, D) with D/I noncyclic.

By our assumption, there exists a Δ-pair $(\tilde{I}, \tilde{D})$ in $\tilde{G}^a$ surjecting onto (I, D). Choose representatives $g_1, \dots, g_n \in D \setminus I$ for nontrivial elements of D/I and $\tilde{g}_j \in \tilde{D}$ surjecting onto g_j under γ. Note that for each j,

$$\sigma_j := \langle g_j, I\rangle \in \bar{\Sigma}(G^c), \quad \tilde{\sigma}_j := \langle \tilde{g}_j, \tilde{I}\rangle \in \bar{\Sigma}(\tilde{G}^c).$$

and that $\tilde{\sigma}_j$ surject onto σ_j. By Lemma 4.5 and our choice of $\tilde{G}^a$, all σ_j are extendable. Then

$$\tilde{I} \subseteq \cap_{j=1}^n \tilde{\sigma}_j,$$

and if we replace and rename the original $\tilde{D}$ by

$$\tilde{D} := \sum_{j=1}^n \tilde{\sigma}_j,$$

then $(\tilde{I}, \tilde{D})$ is a Δ-pair in $\tilde{G}^a$ surjecting onto (I, D).

Now we consider a projective system of finite continuous quotients

$$\mathcal{G}^a_K \to \tilde{G}^a_\iota \to \tilde{G}^a \to G^a, \qquad \gamma_{\iota\iota'} : G^a_\iota \longrightarrow G^a_{\iota'}, \quad \iota' \preceq \iota.$$

Assume that for each ι there exists a Δ-pair $(\tilde{I}_\iota, \tilde{D}_\iota)$ in $\tilde{G}^a_\iota$ surjecting onto (I, D). Iterating the construction above, we construct, for each ι, a collection of liftable subgroups

$$\tilde{\sigma}_{\iota,1}, \dots, \tilde{\sigma}_{\iota,n}$$

and a Δ-pair $(\tilde{I}_\iota, \tilde{D}_\iota)$ of the form

$$\tilde{I}_\iota \subseteq \cap_{j=1}^n \tilde{\sigma}_{\iota,j}, \quad \tilde{D}_\iota := \sum_{j=1}^n \tilde{\sigma}_{\iota,j},$$

such that

- $(\tilde{I}_\iota, \tilde{D}_\iota)$ surjects onto $(\tilde{I}_{\iota'}, \tilde{D}_{\iota'})$, for each $\iota' \preceq \iota$, and in particular onto (I, D). By compactness of $\mathcal{G}^a_K$ (see Lemma 4.5), there exist closed subgroups

$$\sigma_{K,1}, \dots, \sigma_{K,n} \in \Sigma(\mathcal{G}^c_K);$$

the closed subgroups

$$\mathcal{I}^a := \cap\, \sigma_{K,j}, \quad \mathcal{D}^a := \langle \sigma_{K,j} \rangle_j$$

of $\mathcal{G}^a_K$ surject onto I, resp. D.

Note that if a pair $(I, D') \subset (I, D)$ is non extendable then (I, D) is also not extendable. Thus we can assume that proper sub-pairs $(I, D') \subset (I, D)$ are extendable. In particular, any liftable subgroup in D^a is equal to $\langle I^a, g\rangle, g \notin I^a$, for some $g \in D^a \setminus I^a$.

By [8, Lemma 6.4.3] and [8, Corollary 6.4.4] any liftable subgroup L_g contains a subgroup I_g of corank ≤ 1 which consists of flag elements and the group I_g is contained in $I^a_{\nu,g}$ and $D^a_{\nu,g}$. If $I_g = L_g$, for some g, then I^a consists of flag elements and hence by [8, Lemma 6.4.3] and [8, Corollary 6.4.4] there is a ν such that $I_g \subset I^a_\nu$ and $D^a \subset D^a_\nu$ and hence (I, D) is liftable, contradicting our assumption.

Thus we can assume that for any L_g the subgroup I_g has corank exactly one. Let us show that $I_g = I^a$. Assume that $h \in I^a$ is not a flag map. By results mentioned above, $h \subset D^a_{\mu_h}$, for some valuation μ_h with the property that any commuting pair $\langle h, x\rangle$ is contained in the image of $\langle h, I_\mu\rangle$. The image of $\langle h, I_\mu\rangle = H^a \subset G^a$ is a liftable subgroup, but then $D^a \subset H^a$ and hence D^a is also liftable, contradicting our assumption on D^a.

Our assumption on (I, D) implies that any closed subgroup containing $\langle I, \delta\rangle$, with $\delta \in D \setminus I$, lifts to an abelian group. The theory developed in [8] describes all pairs (g, h) of topologically independent elements in $\mathcal{G}^a_K$ which lift to commuting pairs in $\mathcal{G}^c_K$: they are realized as $\mathbb{Z}_\ell$-valued maps on $K^\times/k^\times = \mathbb{P}(K)$, a projective space over k, with the property that $g(xy) = g(x) + g(y)$, for all x, y. The so-called flag maps f are maps such that every finite-dimensional subspace $\mathbb{P}^n \subset \mathbb{P}(K)$ admits a flag of projective subspaces $\mathbb{P}_1 \subset \ldots \subset \mathbb{P}_r = \mathbb{P}^n$ so that f is constant on $\mathbb{P}_i \setminus \mathbb{P}_{i-1}$, for all $i = 2, \dots, r$. A flag map defines

(1) a natural scale on K: a sequence of linear subspaces $L_\gamma \subset K$ over k parametrized by an ordered abelian group Γ with the property that $L_\gamma \subset L_\beta$ if $\gamma > \beta$ in Γ,
(2) a map $\nu : K^\times \to \Gamma$, where $\nu(x) = \beta$ if $x \in L_\beta$ and is not contained in $L_\gamma \subset L_\beta$.

Moreover, $x \cdot L_\gamma = L_{\gamma+\nu(x)}$, i.e., the scale is invariant under multiplication in $K^\times$. Thus to any multiplicative flag map f on $\mathbb{P}(K)$ we can associate a nonarchimedean valuation ν of K with value group Γ. We have $f(x) = f_*(\nu(x))$, where f_* is a homomorphism $\Gamma \to \mathbb{Z}_\ell$. Note that a flag map f defines a *unique* order, and hence the value group of the valuation. Of course, similar homomorphisms exist for refinements of this valuation, but the latter is defined intrinsically by the flag map f.

The main result of [8] states that for any pair of (g, h) as above there is a basis (f, δ) of $\langle g, h\rangle$ such that f is a flag map defining (canonically!) a valuation ν_f and δ belongs to the decomposition group of ν_f. This holds for function fields over $\bar{\mathbb{F}}_p$; a slightly more complicated version is valid for function fields over arbitrary algebraically closed fields k. The property of δ to be in the decomposition group of ν is also described in terms of projective geometry of the level sets in $\mathbb{P}(K)$. In particular, for any such δ there is a maximal valuation ν such that $\delta \in \mathcal{D}^a_\nu$ and every $\sigma \in \Sigma(\mathcal{G}^c_K)$ containing δ is contained in $\langle \mathcal{I}^a_\nu, \delta\rangle$. The above general description of commuting pairs provides also a description of pairs $(\mathcal{I}^a, \mathcal{D}^a)$ in $\mathcal{G}^a_K$. Since by assumption $I \neq D$, the same holds for $\mathcal{I}^a \neq \mathcal{D}^a$ in $\mathcal{G}^a_K$ and hence $\mathcal{D}^a/\mathcal{I}^a$ has topologically independent elements g_1, g_2, since we assumed that $\mathcal{D}^a$ is not a liftable subgroup $\mathcal{G}^a_K$.

Therefore, all elements in $\mathcal{I}^a$ are flag and hence $\mathcal{I}^a \subseteq \mathcal{I}^a_\nu$, for some ν, and $\mathcal{D}^a \subseteq \mathcal{D}^a_\nu$. In particular, the initial pair (I, D) was extendable which completes the proof of the proposition. $\square$

5. Galois cohomology of function fields

In [9], [10] we proved that if $k = \bar{\mathbb{F}}_p$, with $p \neq \ell$, and X is an algebraic variety over k of dimension ≥ 2 then $K = k(X)$ is encoded, up to purely inseparable extensions, by $\mathcal{G}^c_K$, the second lower series quotient of $\mathcal{G}_K$. Related reconstruction results have been obtained in [24], [22], [25].

The proof of the Bloch–Kato conjecture by Voevodsky, Rost, and Weibel, substantially advanced our understanding of the relations between fields and their Galois groups, in particular, their Galois cohomology. Indeed, consider the diagram

$$\begin{array}{ccc} & G_K & \\ {}^{\pi_c}\swarrow & & \searrow{}^{\pi} \\ \mathcal{G}^c_K & \xrightarrow[\pi_a]{} & \mathcal{G}^a_K. \end{array}$$

The following theorem relates the Bloch–Kato conjecture to statements in Galois-cohomology, with coefficients in $\mathbb{Z}/\ell^n$ (see also [12], [13], [26]).

Theorem 5.1. [3], [11, Theorem 11] *Let $k = \bar{\mathbb{F}}_p$, $p \neq \ell$, and $K = k(X)$ be the function field of an algebraic variety of dimension ≥ 2. The Bloch–Kato conjecture for K is equivalent to:*

(1) *The map*
$$\pi^* \colon \mathrm{H}^*(\mathcal{G}^a_K, \mathbb{Z}/\ell^n) \to \mathrm{H}^*(G_K, \mathbb{Z}/\ell^n)$$
is surjective and

(2) $\mathrm{Ker}(\pi^*_a) = \mathrm{Ker}(\pi^*)$.

This implies that the Galois cohomology of the pro-ℓ- quotient $\mathcal{G}_K$ of the absolute Galois group G_K encodes important birational information of X. For example, in the case above, $\mathcal{G}^c_K$, and hence K, modulo purely-inseparable extensions, can be recovered from the cup-products
$$\mathrm{H}^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \cup \mathrm{H}^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \to \mathrm{H}^2(\mathcal{G}_K, \mathbb{Z}/\ell^n), \quad n \in \mathbb{N}.$$
From now on, we will frequently omit the coefficient ring $\mathbb{Z}/\ell^n$ from notation.

The first part of the Bloch-Kato theorem says that every $\alpha_K \in \mathrm{H}^i(G_K)$ is induced from a cohomology class $\alpha^a \in \mathrm{H}^i(G^a)$ of some finite *abelian quotient* $G_K \to G^a$. An immediate application of this is the following proposition:

Proposition 5.2. *Let $\alpha_K \in \mathrm{H}^i(G_K)$ be defined on a model X of K and induced from a continuous surjective homomorphism $\chi : \hat{\pi}_1(X) \to G$ onto a finite group. Let $\alpha = \alpha_X \in \mathrm{H}^i_{et}(X)$ be the class representing α_K on X. Then there exists a finite cover $X = \cup_j X_j$ by Zariski open subvarieties such that, for each j, the restriction $\alpha_j := \alpha|_{X_j}$ is induced from a continuous surjective homomorphism $\chi_j : \hat{\pi}_1(X_j) \to G^a$ onto a finite abelian group and a class $\alpha^a \in \mathrm{H}^i_s(G^a) = \wedge^i(\mathrm{H}^1(G^a))$.*

Proof. We first apply the Bloch-Kato theorem to V°/G and find a Zariski open subset $U := U^\circ_\alpha$ of V°/G such that the restriction α_U is as claimed, i.e., induced from a class $\alpha^a \in \mathrm{H}^i_s(G^a) = \wedge^i(\mathrm{H}^1(G^a))$, for some homomorphism $\chi_a : G_{k(V^\circ/G)} \to G^a$ to a finite abelian group. Note that this homomorphism is unramified on U.

By Lemma 1.1, for every $x \in X$ there exists a map $f : X \to V/G$ such that $f(x) \subset U$ and the restriction of α to $f^{-1}(U) \subset X$ equals $f^*(\chi^*_a(\alpha^a))$. The claim follows by choosing a finite cover by open subvarieties with these properties. □

The second part implies the following:

Corollary 5.3. *Let $\alpha_K \in \mathrm{H}^i(G_K)$. Assume that we are given finitely many quotients*
$$\chi_j : G_K \to G^a_j$$

onto finite abelian groups and classes

$$\alpha_j^a \in \mathrm{H}^i(G_j^a)$$

with $\chi_j^*(\alpha_j^a) = \alpha_K$*, for all* j*. Then there exists a continuous finite quotient* $G_K \to G^c$ *onto a finite central extension of an abelian group* G^a *such that*

- χ_j *factor through* G^c*, i.e., there exist surjective homomorphisms* $\psi_j : G^c \to G_j^a$ *, for all* j*;*
- *there exists a class* $\alpha^c \in \mathrm{H}^i(G^c)$ *with*

$$\alpha^c = \psi_j^*(\alpha_j^a), \quad \text{for all} \quad j.$$

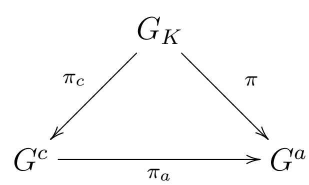
Lemma 5.4. *Let* X *be a normal variety with function field* K*. Assume that* $\alpha_K \in \mathrm{H}^i(G_K)$ *is defined on* X *and induced from a homomorphism* $\chi : \hat{\pi}_1(X) \to G$ *to a finite group* G*. Consider the sequence*

$$\chi_K : G_K \to \hat{\pi}_1(X) \xrightarrow{\chi} G.$$

Then $\chi_K(I_\nu) = 0$*, for every* ν *such that* $\mathfrak{c}_X(\nu)^\circ \subset X$*.*

Proof. An étale cover of X induces an étale cover of the generic point of $\mathfrak{c}_X(\nu)$, thus the cover is unramified in ν, i.e., $\chi_K(I_\nu) = 0$. □

Corollary 5.5. *Let* $\alpha_K \in \mathrm{H}^i(G_K)$*. Let* X *be a normal projective model of* K *and* $\cup_j X_j$ *a finite cover by open subvarieties such that* α_K *is defined on* X_j*, for each* j*, and is induced from a class* $\alpha_j^a \in \mathrm{H}^i(G_j^a)$*, via a homomorphism* $\chi_j : \hat{\pi}_1(X_j) \to G_j^a$ *to some finite abelian group. Then there exist a diagram*



where G^c *is a finite* ℓ*-group which is a central extension of* G^a*, and a class* $\alpha \in \mathrm{H}^i(G^a)/\mathrm{I}(G^c)$ *such that* α *induces* α_K *and for any extendable* Δ*-pair* $(I, D) \subset G^a$ α *has a representative in* $\mathrm{H}^i(G^a)$ *which is unramified with respect to* (I, D)*.*

Proof. Each α_j^a is unramified on all ν such that the generic point $\mathfrak{c}_X(\nu)^\circ \subset X_j$, by Lemma 5.4. Since α_j are induced from a finite number of finite abelian quotients G_j^a of G_K there exists an abelian quotient G^a of G_K with surjections $G_K \to G^a \to G_i^a$; it follows that all classes α_j are simultaneously induced from G^a. Note that α_j define the same class already on G_K^c and hence on some finite quotient $\tilde{G}^c$ of G_K^c with a abelian quotient $\tilde{G}^a$ which surjects onto G^a. For each ν such that the center of ν is in X_j, the image of I_ν in G^a_j is trivial, and the restriction of α_j to the image of D_ν in $\tilde{G}^a$ is induced from the image of D_ν/I_ν.

For any extendable Δ-pair $(I, D) \subset G^a$ there exists a j and a projection $\tilde{G}^a \to G^a_j$ which maps I to a trivial group. Since on the corresponding central extension $\tilde{G}^c$ all α_j define the same class α, we obtain that the image of α_j in $\mathrm{H}^j(\tilde{G}^a)/\mathrm{I}(\tilde{G}^c)$ is induced from D/I, for all extendable Δ-pairs in $\tilde{G}^a$. □

6. Unramified cohomology

An important class of birational invariants of algebraic varieties are *unramified* cohomology groups, with finite constant coefficients (see [4], [15]). These are defined as follows: Let ν be a divisorial valuation of K. We have a natural homomorphism

$$\partial_\nu \,:\, \mathrm{H}^i(G_K) \to \mathrm{H}^{i-1}(G_{\boldsymbol{K}_\nu}).$$

Classes in $\ker(\partial_\nu)$ are called *unramified with respect to* ν. The *unramified* cohomology is

$$\mathrm{H}^i_{nr}(G_K) := \bigcap_{\nu \in \mathcal{DV}_K} \mathrm{Ker}(\partial_\nu) \subset \mathrm{H}^i(G_K).$$

For $i = 2$ this is the *unramified Brauer group* which was used to provide counterexamples to Noether's problem, i.e., nonrational varieties of type V/G, where V is a faithful representation of a finite group G (see [27], [2]).

Generally, for $\nu \in \mathcal{V}_K$ and $\alpha \in \mathrm{H}^i(G_K)$ let

$$\alpha_\nu \in \mathrm{H}^i(D_\nu)$$

be the restriction of α to the decomposition subgroup $D_\nu \subset G_K$ of ν.

Lemma 6.1. *A class α is in $\mathrm{Ker}(\partial_\nu) \subseteq \mathrm{H}^i(G_K)$, for $\nu \in \mathcal{DV}_K$, if and only if α_ν is induced from the quotient $G_{\boldsymbol{K}_\nu} = D_\nu/I_\nu$. In particular, α_ν is well-defined as an element in $\mathrm{H}^i(G_{\boldsymbol{K}_\nu})$.*

Proof. Since ν is divisorial, the exact sequence

$$1 \to I_\nu \to D_\nu \to G_{\boldsymbol{K}_\nu} \to 1$$

where I_ν and D_ν are quotients of the inertia, respectively decomposition, subgroups, by wild inertia, admits a noncanonical splitting, i.e., D_ν is noncanonical direct product of $G_{\boldsymbol{K}_\nu} = D_\nu/I_\nu$ with the corresponding inertia group, which is a torsion-free central procyclic subgroup of D_ν. This follows from Lemma 3.1, using that I_ν is abelian and $\mathcal{G}^a_{\boldsymbol{K}_\nu}$ is a free abelian pro-ℓ group.

Thus
$$\mathrm{H}^*(D_\nu) = \mathrm{H}^*(G_{\boldsymbol{K}_\nu}) \otimes \wedge^* \mathrm{H}^1(I_\nu).$$
We have
$$\mathrm{H}^1(I_\nu, \mathbb{Z}/\ell^n) = \mathrm{H}^0(I_\nu, \mathbb{Z}/\ell^n) = \mathbb{Z}/\ell^n$$
and
$$\wedge^*(\mathrm{H}^1(I_\nu, \mathbb{Z}/\ell^n)) = \mathrm{H}^1(I_\nu, \mathbb{Z}/\ell^n) \oplus \mathrm{H}^0(I_\nu, \mathbb{Z}/\ell^n).$$
Thus
$$\mathrm{H}^i(D_\nu) = \mathrm{H}^{i-1}(G_{\boldsymbol{K}_\nu}) \otimes \mathrm{H}^1(I_\nu) \oplus \mathrm{H}^i(G_{\boldsymbol{K}_\nu})$$
and the differential ∂_ν coincides with the projection onto the first summand. Hence $\partial_\nu(\alpha) = 0$ is equivalent to α_ν being induced from $G_{\boldsymbol{K}_\nu} = D_\nu/I_\nu$. □

Combining the considerations above we obtain the notion of *unramified stable cohomology*
$$\mathrm{H}^*_{s,nr}(G)$$
of a finite group G: a stable cohomology class $\alpha \in \mathrm{H}^i_s(G)$ is unramified if and only if it is contained in the kernel of the composition
$$\mathrm{H}^i_s(G) \to \mathrm{H}^i(G_K) \xrightarrow{\partial_\nu} \mathrm{H}^{i-1}(G_{\boldsymbol{K}_\nu}),$$
for every valuation $\nu \in \mathcal{DV}_K$, where $K = k(V/G)$ for some faithful representation of G. This does not depend on the choice of V, provided $\ell \neq \mathrm{char}(k)$. These groups are contravariant in G and form a subring
$$\mathrm{H}^*_{s,nr}(G) \subset \mathrm{H}^*_s(G).$$
Furthermore:

- If V/G is stably rational then $\mathrm{H}^i_{s,nr}(G) = 0$, for all $i \geq 2$.
- We have
$$\mathrm{H}^i_{s,nr}(G) \subseteq \mathrm{H}^i_{s,nr}(\mathrm{Syl}_\ell(G))^{N_G(\mathrm{Syl}_\ell(G))}.$$

Remark 6.2. In [7], we proved that $\mathrm{H}^i_{s,nr}(G) = 0$, $i \geq 1$, for most quasi-simple groups of Lie type. A complete result for quasi-simple groups and $i = 2$ was obtained in [20].

Note that the ℓ-Sylow-subgroups of finite simple groups often have stably-rational fields of invariants; this provides an alternative approach to our vanishing theorem.

Lemma 6.3. *Let*
$$\gamma_K : \mathcal{G}^a_K \to G^a$$
be a continuous surjective homomorphism and $\alpha^a_K = \gamma^*_K(\alpha^a) \in \mathrm{H}^i(\mathcal{G}^a_K)$, *for some* $\alpha^a \in \mathrm{H}^i_s(G^a)$. *If* α^a *is unramified with respect to every extendable* Δ*-pair in* G^a *then* $\alpha^a_K \in \mathrm{H}^i_{nr}(\mathcal{G}^a_K)$.

Proof. For $\nu \in \mathcal{DV}_K$, let $D := \gamma_K(\mathcal{D}^a_\nu)$ and $I := \gamma_K(\mathcal{I}^a_\nu)$. Then either D is cyclic or (D, I) is an extendable Δ-pair in G^a. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^i_s(G^a) & \xrightarrow{\phi} & \mathrm{H}^i_s(D) & \xleftarrow{\psi} & \mathrm{H}^i_s(D/I) \\ \downarrow{\scriptstyle \gamma_K^*} & & \downarrow & & \downarrow \\ \mathrm{H}^i(\mathcal{G}^a_K) & \longrightarrow & \mathrm{H}^i(\mathcal{D}^a_\nu) & \longleftarrow & \mathrm{H}^i(\mathcal{D}^a_\nu/\mathcal{I}^a_\nu) \end{array}$$

In either case, α^a_K is unramified with respect to ν, by Lemma 1.7. □

7. Main theorem

Theorem 7.1. *Let $K = k(X)$ be a function field over $k = \bar{\mathbb{F}}_p$ of $\operatorname{tr\,deg}_k(K) \geq 2$ and $\alpha_K \in \mathrm{H}^i_{nr}(G_K)$, with $\ell \neq p$ and $i > 1$. Then there exist a continuous homomorphism $G_K \to \tilde{G}^a$ onto a finite abelian ℓ-group, fitting into a diagram*

$$\begin{array}{ccccccccc} & & & & G_K & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a & \longrightarrow & 1 \end{array}$$

and a class $\tilde{\alpha}^a \in \mathrm{H}^i(\tilde{G}^a)$ such that

(1) *α_K is induced from $\tilde{\alpha}^a$,*
(2) *$\tilde{\alpha}^c := \pi_a^*(\tilde{\alpha}^a) \in \mathrm{H}^*_{s,nr}(\tilde{G}^c)$.*

Conversely, every $\alpha_K \in \mathrm{H}^i(G_K)$ induced from $\wedge^*(\mathrm{H}^1(\tilde{G}^a))$ and unramified on some $\tilde{G}^c$ as above is in $\mathrm{H}^i_{nr}(G_K)$.

In this section we begin the proof of Theorem 7.1, reducing it to geometric statements addressed in Sections 8 and 9.

Fix an unramified class

$$\alpha_K \in \mathrm{H}^i_{nr}(G_K) \subset \mathrm{H}^i(G_K).$$

By Theorem 5.1, we have a surjection

$$\pi^* : \mathrm{H}^i(\mathcal{G}^a_K) \to \mathrm{H}^i(G_K),$$

let $\alpha^a_K \in \mathrm{H}^i(\mathcal{G}^a_K)$ be a class such that $\pi^*(\alpha^a_K) = \alpha_K$. Let

$$\gamma_K : \mathcal{G}^a_K \to G^a$$

be a continuous quotient onto a finite abelian ℓ-group such that α^a_K is induced from a class $\alpha^a \in \mathrm{H}^i_s(G^a) = \wedge^i(\mathrm{H}^1(G^a))$. We have a diagram of central extensions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}^c_K & \longrightarrow & \mathcal{G}^a_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \gamma_K & & \\
1 & \longrightarrow & Z & \longrightarrow & G^c & \xrightarrow{\pi_a} & G^a & \longrightarrow & 1
\end{array}$$

where the lower row is uniquely defined, up to isoclinism, as in Lemma 4.2. The group G^a might be too small, i.e., it may happen that

$$\pi_a^*(\alpha^a) \notin \mathrm{H}^i_{s,nr}(G^c).$$

Our goal is to pass to an intermediate finite quotient $\mathcal{G}^a_K \to \tilde{G}^a$ fitting into a commutative diagram below

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}^c_K & \longrightarrow & \mathcal{G}^a_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \tilde{\gamma}_K & & \\
1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \gamma & & \\
1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a & \longrightarrow & 1
\end{array}$$

where the vertical arrows are surjections onto finite ℓ-groups, and such that α^a_K is induced from a class $\tilde{\alpha}^a \in \mathrm{H}^*(\tilde{G}^a)$ with

$$\tilde{\pi}_a^*(\tilde{\alpha}^a) \in \mathrm{H}^i_{s,nr}(\tilde{G}^c).$$

There are two possibilities:

(1) There exists a finite quotient $G_K \to G^a$ such that α_K is induced from $\alpha^a \in \mathrm{H}^i_s(G^a)$ which is unramified with respect to every extendable Δ-pair (I, D) in G^a. This case is treated in Lemma 7.2.

(2) On *every* finite quotient, the class α^a inducing α_K is ramified on *some* extendable Δ-pair (I, D). This possibility is eliminated by Lemma 7.3.

Lemma 7.2. *Assume that $\alpha^a \in \mathrm{H}^i_s(G^a)$ is unramified with respect to every extendable Δ-pair (I, D) in G^a. Then there is a factorization*

$$\mathcal{G}^a_K \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a, \quad \gamma_K = \gamma \circ \tilde{\gamma}_K, \tag{7.1}$$

with finite $\tilde{G}^a$, such that

$$\pi_a^*(\tilde{\alpha}^a) \in \mathrm{H}^i_{s,nr}(\tilde{G}^c), \qquad \tilde{\alpha}^a := \gamma^*(\alpha^a).$$

Proof. Let $\tilde{G}^a$ be the quotient constructed in Proposition 4.6, i.e., if (I, D) is not an extendable Δ-pair in G^a then *no* Δ-pair $(\tilde{I}, \tilde{D})$ surjects onto (I, D). Thus, for each Δ-pair $(\tilde{I}, \tilde{D})$ in $\tilde{G}^a$ one of the following holds:

- either $\gamma(\tilde{I}) = 0$ or $\gamma(\tilde{D})$ cyclic,
- or $(\gamma(\tilde{I}), \gamma(\tilde{D}))$ is an extendable Δ-pair in G^a.

Applying Lemma 1.7 and Lemma 6.3 to $V/\tilde{G}^c$, we obtain that $\tilde{\alpha}^a := \gamma^*(\alpha^a)$ is unramified with respect to $(\tilde{I}, \tilde{D})$. □

Lemma 7.3. *There exists a finite quotient $G_K \to G^a$ such that $\alpha^a \in \mathrm{H}^i(G^a)$ induces α_K and is unramified on every extendable Δ-pair (I, D) in G^a.*

The proof of this lemma is presented in Section 8, in the case when K admits a smooth projective model; a reduction to the smooth case is postponed until Section 9.

8. The smooth case

Let X be a smooth projective irreducible variety over an algebraically closed field k with function field $K = k(X)$. By the Bloch-Ogus theorem, there is an isomorphism

$$\mathrm{H}^i_{nr}(G_K) = \mathrm{H}^0_{Zar}(X, \mathcal{H}^i_{et}(X)),$$

where $\mathcal{H}^i_{et}$ is an étale cohomology sheaf (see also Theorem 4.1.1 in [14]). In particular, a class $\alpha_K \in \mathrm{H}^i_{nr}(G_K)$ can be represented by a finite collection of classes $\{\alpha_n\}_{n\in N}$, with α_n defined on some Zariski open affine $X_n \subset X$, with $X = \cup_n X_n$, such that the restrictions of α_n to some common open affine subvariety $X^\circ \subset \cap_n X_n$ coincide. We will need the following strengthening:

Lemma 8.1. *Let X be a smooth variety with function field K and $S = \{x_1, \dots, x_r\} \subset X$ a finite set of points. Given a class $\alpha_K \in \mathrm{H}^i_{nr}(G_K)$ there exist*

- *a Zariski open subset $U_S \subset X$, containing S, and*
- *a class $\alpha_S \in \mathrm{H}^i_{et}(U_S)$*

such that α_K and α_S coincide on some dense Zariski open subset $U_S^\circ \subset U_S$, i.e., for every representation of α_K by $\{\alpha_n\}_{n\in N}$ as above there exists some dense Zariski open $U_S^\circ \subset U_S$, containing S, such that the restrictions of α_S and all α_n to U_S° coincide.

Proof. By the argument in [14, Theorem 4.1.1], α_K has a representative $\alpha = \alpha_X \in \mathrm{H}^0_{Zar}(X, \mathcal{H}^i(\mathbb{Z}/\ell^n))$. Hence, there is a covering of X by Zariski open subsets X_n with α_n representing α in X_n. Moreover, using the refinement of the Bloch-Ogus exact sequence for semi-local rings as in [23, Theorem 1.1], we can assume that one of the subsets contains the finite set S. □

Fix a representation of $\alpha_K \in \mathrm{H}^i_{nr}(G_K)$ by $\{\alpha_n\}_{n\in N}$ as above. Each class $\alpha_n \in \mathrm{H}^i_{et}(X_n)$ is represented by a finite collection $\{X_{nm}\}$ of affine charts X_{nm}, with $\cup X_{nm} = X_n$ and finite étale covers

$$\psi_{nm} : \tilde{X}_{nm} \to X_{nm},$$

such that the restrictions $\alpha_{nm} := \alpha_n|_{X_{nm}}$ are induced from homomorphisms $\chi_{nm} : \hat{\pi}_1(X_{nm}) \to G_{nm}$ onto finite groups. Proposition 5.2 implies that there is further refinement of the cover by affine subcovers

$$X = \cup_j X_j,$$

such that for each j there exist

- a finite *abelian* ℓ-group G^a_j
- a surjection $\chi_{K,j} : G_K \to G^a_j$, unramified over X_j, and
- a class $\alpha_j \in \mathrm{H}^i(G^a_j)$ inducing α via $\chi_{K,j}$.

Corollary 5.3 implies that there exists a finite quotient $\pi_c : G_K \to G^c$ onto a central extension of an abelian group G^a such that the projections $\chi_{K,j}$ factor through G^c and the images of α_j in $\mathrm{H}^i(G^c)$ coincide. In particular, α_K is induced from $\alpha^c \in \mathrm{H}^i(G^c)$.

We claim that α^c is unramified on every pair $(\pi_c(I_\nu), \pi_c(D_\nu))$, for $\nu \in \mathcal{V}_K$. Indeed, for each ν, $\chi_{K,j}(I_\nu) = 0$ on at least *one* of the charts X_j, thus the restriction of α_j to $\chi_{K,j}(D_\nu)$ is induced from $\chi_{K,j}(D_\nu)/\chi_{K,j}(I_\nu)$; since $\alpha^c \in \mathrm{H}^i(G^c)$ is induced from α_j, we have the same property for α^c, with respect to the pair $(\pi_c(I_\nu), \pi_c(D_\nu))$. The description of the action in Lemma 2.3 identifies subgroups of G^a acting on products of projective spaces with fixed points with images of inertia subgroups of some valuations, and images of their decomposition subgroups with subgroups preserving the corresponding components, see Lemmas 2.2 and 2.3.

The rest of the argument is similar to the proof of Lemma 7.2. Let $G_K \to G^a$ be an intermediate quotient surjecting onto each G^a_j and

$$G_K \to \tilde{G}^a \to G^a$$

the intermediate finite quotient constructed in Proposition 4.6. In particular, the projection of every Δ-pair in $\tilde{G}^a$ to G^a_j is either of the form $(\pi_{a,j}(I_\nu), \pi_{a,j}(D_\nu))$, for some $\nu \in \mathcal{V}_K$, or *cyclic*. Let $\tilde{G}^c$ be a central extension as in Corollary 4.3, surjecting onto G^c. We have classes $\tilde{\alpha}^a_j \in \mathrm{H}^i(\tilde{G}^a)$, induced by α^a_j constructed above, and mapping to the same class $\tilde{\alpha}^c \in \mathrm{H}^i(\tilde{G}^c)$.

We claim that $\tilde{\alpha}^c \in \mathrm{H}^i_{s,nr}(\tilde{G}^c)$. Indeed, for every Δ-pair $(\tilde{I}, \tilde{D})$ in $\tilde{G}^a$ either $\tilde{D}$ projects to a cyclic group in G^a or it is extendable, i.e., image of some (I_ν, D_ν). In the first case, all elements $\tilde{\alpha}_j$ are unramified on $(\tilde{I}, \tilde{D})$. In the second case, at least one of the $\tilde{\alpha}_j$ is unramified on it.

9. Reduction to the smooth case

In absense of resolution of singularities in positive characteristic, we reduce to the smooth case via the de Jong-Gabber alterations theorem (see [18]): The Galois group G_K contains a closed subgroup $G_{\tilde{K}}$ of finite index, coprime to ℓ, such that

- The function field $\tilde{K}$ corresponding to $G_{\tilde{K}}$ admits a smooth proper model, i.e., there exists a generically finite morphism of proper varieties
$$\rho : \tilde{X} \to X,$$
of degree $|G_K/G_{\tilde{K}}|$ with $\tilde{X}$ smooth and $\tilde{K} = k(\tilde{X})$.

Let $\alpha_K \in \mathrm{H}^n_{nr}(G_K)$ be an unramified class. Its restriction $\alpha_{\tilde{K}}$ to a class in $\mathrm{H}^n(G_{\tilde{K}})$ is also unramified. By results in Section 8, there exists a surjection

$$G_{\tilde{K}} \to \tilde{G}^c \tag{9.1}$$

onto a finite abelian ℓ-group such that $\alpha_{\tilde{K}}$ is unduced from a class in $\tilde{\alpha}^c \in \mathrm{H}^n_{s,nr}(\tilde{G}^c)$.

Lemma 9.1. *There exists a diagram*

$$\begin{array}{ccccc} G_{\tilde{K}} & \longrightarrow & \tilde{G} & \xrightarrow{\tilde{\pi}_c} & \tilde{G}^c \\ \downarrow & & \downarrow & & \\ G_K & \longrightarrow & G & & \end{array}$$

where the vertical arrows are injections, with image of index coprime to ℓ, $\tilde{G}$ and G are finite groups, and α_K is induced from an element $\alpha_G \in \mathrm{H}^n_{nr}(G)$. In particular, $\mathrm{Syl}_\ell(G) \simeq \mathrm{Syl}_\ell(\tilde{G})$.

Proof. Fix a finite continuous quotient $G_K \to G'$ such that α_K is induced from some $\alpha_{G'} \in \mathrm{H}^i(G')$. Note that for every intermediate quotient
$$G_K \to G \to G'$$

there exists an $\alpha_G \in \mathrm{H}^i(G)$ inducing α_K. It suffices to find a sufficiently large G such that the sujection (9.1) factors through a subgroup of G. This is a standard fact in Galois theory.

Since $\tilde{\alpha}^c \in \mathrm{H}^i_{s,nr}(G^c)$ its image $\tilde{\alpha} \in \mathrm{H}^i(\tilde{G})$ is also unramified. Since the index $(G : \tilde{G})$ is coprime to ℓ, and since the unramified $\tilde{\alpha}$ is induced from an element $\alpha_G \in \mathrm{H}^i(G)$, α_G is also unramified, as claimed. □

At this stage, we cannot yet guarantee that G is a central extension of an abelian group, nor that it is an ℓ-group. However, we know that

$$\mathrm{tr}(\mathrm{res}_{G_K/G_{\tilde{K}}}(\alpha_K)) = \alpha_K \in \mathrm{H}^i_{nr}(G_K),$$

modulo multiplication by an element in $(\mathbb{Z}/\ell^n)^\times$.

We need the following version of resolution of singularities in positive characteristic:

Theorem 9.2. [18] *Let G be a finite ℓ-group and Y a smooth variety over a perfect field with a generically free action of G. Then there exists a G-variety $\tilde{Y}$ with a proper G-equivariant birational map $\tilde{Y} \to Y$ such that $\tilde{Y}/G$ is smooth.*

We are very grateful to D. Abramovich for providing the reference and indicating the main steps of the proof in [18]:

- Theorem VIII.1.1 gives an equivariant modification Y' of Y with a regular log structure on Y' such that the action is *very tame*, i.e., the stabilizers in G of points in Y' are abelian and act as subgroups of tori in toroidal charts of the log structure Y'.
- By Theorem VI.3.2, the quotient Y'/G of a log regular variety by a very tame action is log regular.
- By Theorem VIII.3.4.9, which is a step in Theorem VIII.1.1, a log regular variety has a resolution of singularities.

We return to the proof of Theorem 7.1. Start with a suitable faithful representation V of G, and thus of $\mathrm{Syl}_\ell(G) = \mathrm{Syl}_\ell(\tilde{G})$, and construct a diagram

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\pi_Y} & Y = \tilde{Y}/\mathrm{Syl}_\ell(G) \\ {\scriptstyle \pi_G}\downarrow & & \\ \bar{V}/G & & \end{array}$$

where $Y = \tilde{Y}/\mathrm{Syl}_\ell(G)$ is the smooth projective variety from Theorem 9.2, and π_Y is a $\mathrm{Syl}_\ell(G)$-equivariant map from a G-equivariant projective closure of V. Let $L = k(Y)$ be the function field of Y.

Given a class $\alpha_L \in \mathrm{H}^i_{nr}(G_L)$ we have a covering of $Y = \cup_n Y_n$ by affine Zariski open subsets and a finite set of classes $\{\alpha_n\}$ representing α_L, as considered in Section 8.

Pick a point $v \in \bar{V}$. The image $S := \pi_Y(G \cdot b)$ of its G-orbit is a finite set of points. By Lemma 8.1, there exist a dense Zariski open subset U_S and a class $\alpha_S \in \mathrm{H}^i_{et}(U_S)$ coinciding with α_L on some dense Zariski open subset $U_S^\circ \subset U_S$. Its preimage

$$\bar{U}_S^\circ := \pi_Y^{-1}(U_S^\circ) \subset \bar{V}$$

is a dense Zariski open subset containing $G \cdot v$. Put

$$\bar{U}_v := \cap_{g \in G}\ g(\bar{U}_S^\circ) \subset \bar{V},$$

it is a G-stable dense Zariski open subvariety containing v. Its image $\pi_G(\bar{U}_v) \subset \bar{V}/G$ is a Zariski open subset containing $\pi_G(G \cdot v)$. Note that $\pi_v : \bar{U}_v/\mathrm{Syl}_\ell(G) \to U_S$ is a birational morphism to an open subset and $\pi_v^*(\alpha_S)$ is well-defined in étale cohomology of $\bar{U}_v/\mathrm{Syl}_\ell(G)$. It follows that the trace

$$\mathrm{tr}_{\pi_G}(\pi_v^*(\alpha_S)) \in \mathrm{H}^i_{et}(\bar{U}_v/G)$$

is well-defined and coincides with $(G : \mathrm{Syl}_\ell(G)) \cdot \alpha_S$ at the generic point of $\bar{V}/G$.

Thus we have a covering of $\bar{V}/G$ by Zariski open subsets of the form $\bar{U}_v/G$, $v \in \bar{V}$, with cohomology classes representing α_K on each chart. There exists a finite subcovering by $\bar{U}_v/G$ with extensions of α_S to each $\bar{U}_v/G$. We can now apply Proposition 5.2 to produce a finite subcover such that on each chart, the class is induced from homomorphisms onto finite *abelian* groups, and proceed as in Section 8.

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