MORI CONES OF HOLOMORPHIC SYMPLECTIC VARIETIES OF K3 TYPE

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Abstract. We determine the Mori cone of holomorphic symplectic varieties deformation equivalent to the punctual Hilbert scheme on a K3 surface. Our description is given in terms of Markman's extended Hodge lattice.

Introduction

Let X be an irreducible holomorphic symplectic manifold. Let $(,)$ denote the Beauville-Bogomolov form on $H^2(X,\mathbb{Z})$; we may embed $H^2(X,\mathbb{Z})$ in $H_2(X,\mathbb{Z})$ via this form. Fix a polarization h on X; by a fundamental result of Huybrechts [Huy99], X is projective if it admits a divisor class H with $(H,H)>0$. It is expected that finer birational properties of X are also encoded by the Beauville-Bogomolov form and the Hodge structure on $H^2(X)$, along with appropriate extension data. In particular, natural cones appearing in the minimal model program—the moving cone, the nef cone, the pseudo-effective cone—should have a description in terms of this form.

Now assume X is deformation equivalent to the punctual Hilbert scheme $S^{[n]}$ of a K3 surface S. Recall that

$$H^2(S^{[n]},\mathbb{Z})_{(,)} = H^2(S,\mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta, \quad (\delta,\delta) = -2(n-1) \tag{1}$$

where the restriction of the Beauville-Bogomolov form to the first factor is just the intersection form on S, and 2δ is the class of the locus of non-reduced subschemes. Recall that for K3 surfaces S, the cone of (pseudo-)effective divisors is the closed cone generated by

$$\{D \in \operatorname{Pic}(S) : (D,D) \geq -2, (D,h) > 0\}.$$

The first attempt to extend this to higher dimensions was [HT01]. Further work on moving cones was presented in [HT09, Mar13], which built on Markman's analysis of monodromy groups. The characterization of extremal rays arising from Lagrangian projective spaces $\mathbb{P}^n \hookrightarrow X$ has been addressed in [HT09, HHT12] and [BJ11]. The paper [HT10] proposed a general framework describing all types of extremal rays; however, Markman found counterexamples in dimensions ≥ 10, presented in [BM12].

The formalism of Bridgeland stability conditions [Bri07, Bri08] has led to breakthroughs in the birational geometry of moduli spaces of sheaves on surfaces. The case of punctual Hilbert schemes of $\mathbb{P}^2$ and del Pezzo surfaces was investigated by Arcara, Bertram, Coskun, and Huizenga [ABCH13, Hui12, BC13, CH13]. The effective cone on $(\mathbb{P}^2)^{[n]}$ has a beautiful and complex structure as n increases, which only becomes transparent in the language of stability conditions. Bayer

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and Macrì resolved the case of punctual Hilbert schemes and more general moduli spaces of sheaves on K3 surfaces [BM12, BM13]. Abelian surfaces, whose moduli spaces of sheaves include generalized Kummer varieties, have been studied as well [YY12, Yos12].

In this note, we extend the results obtained for moduli spaces of sheaves over K3 surfaces to all holomorphic symplectic manifolds arising as deformations of punctual Hilbert schemes of K3 surfaces. Our principal result is Theorem 1 below.

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1. Statement of Results

Let X be deformation equivalent to the Hilbert scheme of length-n subschemes of a K3 surface. Markman [Mar11, Cor. 9.5] describes an extension of lattices

$$H^2(X,\mathbb{Z}) \subset \widetilde{\Lambda}$$

and weight-two Hodge structures

$$H^2(X,\mathbb{C}) \subset \widetilde{\Lambda}_{\mathbb{C}}$$

characterized as follows:

- the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$ has rank one, and is generated by a primitive vector of square $2n-2$;
- as a lattice
$$\widetilde{\Lambda} \simeq U^4 \oplus (-E_8)^2$$
where U is the hyperbolic lattice and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram;
- there is a natural extension of the monodromy action on $H^2(X,\mathbb{Z})$ to $\widetilde{\Lambda}$; the induced action on $\widetilde{\Lambda}/H^2(X,\mathbb{Z})$ is encoded by a character *cov* (see [Mar08, Sec. 4.1]);
- we have the following Torelli-type statement: X_1 and X_2 are birational if and only if there is Hodge isometry
$$\widetilde{\Lambda}_1 \simeq \widetilde{\Lambda}_2$$
taking $H^2(X_1,\mathbb{Z})$ isomorphically to $H^2(X_2,\mathbb{Z})$;
- if X is a moduli space $M_v(S)$ of sheaves over a K3 surface S with Mukai vector v then there is an isomorphism from $\widetilde{\Lambda}$ to the Mukai lattice of S taking $H^2(X,\mathbb{Z})$ to $v^\perp$.

Generally, we use v to denote a primitive generator for the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$. Note that $v^2 = (v,v) = 2n-2$. When $X \simeq M_v(S)$ we may take the Mukai vector v as the generator.

There is a canonical homomorphism

$$\theta^\vee : \widetilde{\Lambda} \twoheadrightarrow H_2(X,\mathbb{Z})$$

which restricts to an inclusion

$$H^2(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$$

of finite index. By extension, it induces a $\mathbb{Q}$-valued Beauville-Bogomolov form on $H_2(X,\mathbb{Z})$.

Assume X is projective. Let $H^2(X)_{alg} \subset H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}_{alg} \subset \widetilde{\Lambda}$ denote the algebraic classes, i.e., the integral classes of type $(1,1)$. The Beauville-Bogomolov form on $H^2(X)_{alg}$ has signature $(1,\rho(X)-1)$, where $\rho(X) = \dim(H^2_{alg}(X))$. The *Mori cone* of X is defined as the closed cone in $H_2(X,\mathbb{R})_{alg}$ containing the classes of algebraic curves in X. The *positive cone* (or more accurately, non-negative cone) in $H^2(X,\mathbb{R})_{alg}$ is the closure of the connected component of the cone

$$\{D \in H^2(X,\mathbb{R})_{alg} : D^2 > 0\}$$

containing an ample class. The dual of the positive cone in $H^2(X,\mathbb{R})_{alg}$ is the positive cone.

Theorem 1. *Let (X,h) be a polarized holomorphic symplectic manifold as above. The Mori cone in $H_2(X,\mathbb{R})_{alg}$ is generated by classes in the positive cone and the images under $\theta^\vee$ of the following:*

$$\{a \in \widetilde{\Lambda}_{alg} : a^2 \geq -2, |(a,v)| \leq v^2/2, (h,a) > 0\}.$$

This generalizes [BM13, Theorem 12.2], which treated the case of moduli spaces of sheaves on K3 surfaces. As another application of our methods, we can bound the length of extremal rays of the Mori cone with respect to Beauville-Bogomolov pairing:

Proposition 2. *Let X be a projective holomorphic symplectic manifold as above. Then any extremal ray of its Mori cone contains an effective curve class R with*

$$(R,R) \geq -\frac{n+3}{2}.$$

The value $-\frac{n+3}{2}$ had been conjectured in [HT10]. Proposition 2 has been obtained independently by Mongardi [Mon13]. His proof is based on Twistor deformations, and also applies to non-projective manifolds.

2. Deforming extremal rational curves

In this section, we consider general irreducible holomorphic symplectic manifolds, not necessarily of K3 type. Recall the definition of a *parallel transport operator* $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ between manifolds of a fixed deformation type: there is a smooth proper family $\mathcal{X} \to B$ over a connected analytic space, points $b, b' \in B$ with $\mathcal{X}_b \simeq X$ and $\mathcal{X}_{b'} \simeq X'$, and a continuous path $\gamma : [0,1] \to B, \gamma(0) = b, \gamma(1) = b'$, such that parallel transport along γ induces ϕ.

Proposition 3. *Let X be a projective holomorphic symplectic variety and R the class of an extremal rational curve $\mathbb{P}^1 \subset X$ with $(R,R) < 0$. Suppose that X' is deformation equivalent to X and $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is a parallel transport operator for a family $\pi : \mathcal{X} \to B$ such that R remains algebraic. If $R' := \phi(R) \in H_2(X',\mathbb{Z})_{alg}$ then a multiple of R' is effective and represented by a cycle of rational curves.*

Proof. Fix a proper holomorphic family $\pi : \mathcal{X} \to B$ over a connected complex curve B with $X = \pi^{-1}(b_0)$. We claim that R deforms to rational curves in fibers $\mathcal{X}_{b''} = \pi^{-1}(b'')$ for b'' near b_0.

Let ω denote the holomorphic symplectic form on X, $f : X \to Y$ the birational contraction associated with R, E an irreducible component of the exceptional locus of f, Z its image in Y, and F a generic fiber of $E \to X$. Let $f \colon \mathbb{P}^1 \to F$ be a generic free rational curve of minimal degree in F. Then the normal bundle of $\ell := f(\mathbb{P}^1) \subset X$ was determined completely in [CMSB02]; we briefly review the key points:

- ω restricts to zero on F [Kal06, Lemma 2.7];
- the smooth locus of Z is symplectic with two-form pulling back to $\omega|E$ [Kal06, Thm. 2.5] [Nam01, Prop. 1.6];
- the dimension r of F equals the codimension of E [Wie03, Thm. 1.2];
- by bend-and-break, the pull-back $f^*\mathcal{T}_E$ of the tangent bundle of E has only one summand of the form $\mathcal{O}(2)$, all others being of the form $\mathcal{O}(k)$ for $k \in \{0, 1\}$ [CMSB02, Theorem 2.8].

Using standard exact sequences for normal bundles, and the fact that $f^*\mathcal{T}_Z$ is trivial, one can then show (cf. [CMSB02, Lemma 9.4])

$$N_{\ell/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}^{2(n-r)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-1}.$$

The crucial point is that $h^1(N_{\ell/X}) = 1$. Thus we may apply [Ran95, Cor. 3.2] to deduce that the deformation space of ℓ in X has dimension $2n-2$; [Ran95, Cor. 3.3] then implies that ℓ persists in deformations of X for which R remains algebraic. This proves our claim.

Example. The extremality assumption is essential, as shown by an example suggested by Voisin: Let S be a K3 surface arising as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve of bidegree $(4, 4)$ and $X = S^{[2]}$. We may regard $\mathbb{P}^1 \times \mathbb{P}^1 \subset X$ as a Lagrangian surface. Consider a smooth curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset X$ of bidegree $(1, 1)$. The curve C persists only in the codimension-*two* subspace of the deformation space of X where $\mathbb{P}^1 \times \mathbb{P}^1$ deforms (see [Voi92]); note that $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^2$.

We return to the proof of Proposition 3. For arbitrary fibers $\mathcal{X}_b$, suppose a parallel-transport operator takes R to R'' in the lattice $\widetilde{\Lambda}'$ associated with $\mathcal{X}_b$. Then an application of Remmert's Proper Mapping theorem [Rem57, Satz 23] implies there exists a cycle of rational curves in $\mathcal{X}_b$ that is a specialization of R. (This should be applied to a suitable Hilbert scheme/Douady space parametrizing curves in the fibers of fixed degree.) In particular, R'' is in the Mori cone.

To apply this reasoning, we need to construct an appropriate chain of families joining X and X':

Lemma 4. *There exists a smooth proper family $\varpi : \mathcal{X}^+ \to B^+$ over a connected analytic space with finitely many components, points $b, b' \in B^+$ with $\mathcal{X}^+_b \simeq X$ and $\mathcal{X}^+_{b'} \simeq X'$, and a section*

$$\rho : B^+ \to \mathbb{R}^2\varpi_*\mathbb{Z}$$

of type $(1, 1)$, such that $\rho(b) = R$ and $\rho(b') = R'$.

Proof. This proof is essentially the same as the argument for Proposition 5.12 of [Mar13]. We summarize the key points.

Let $\mathfrak{M}$ denote the moduli space of marked holomorphic symplectic manifolds of K3 type [Huy99, Sec. 1]. Essentially, this is obtained by gluing together all the local Kuranishi spaces of the relevant manifolds. It is non-Hausdorff. Let $\mathfrak{M}^\circ$ denote a connected component of $\mathfrak{M}$ containing X equipped with a suitable marking.

Consider the subspace $\mathfrak{M}^\circ_R$ such that R is type $(1,1)$ and $(\kappa, R) > 0$ for *some* Kähler class, which may vary from point to point of the moduli space. This coincides with an open subset of the preimage of the hyperplane $R^\perp$ under the period map P [Mar13, Claim 5.9]. Furthermore, for general periods τ—those for which R is the unique integral class of type $(1,1)$—the preimage $P^{-1}(\tau)$ consists of a single marked manifold [Mar13, Cor. 5.10]. The proof of this in [Mar13] only requires that $(R,R) < 0$. (The Torelli Theorem implies two manifolds share the same period point only if they are bimeromorphic [Mar11, Th. 1.2], but if R is the only algebraic class, the only other bimeromorphic model would not admit a Kähler class κ' with $(\kappa', R) > 0$.) Finally, $\mathfrak{M}^\circ_R$ is path-connected by [Mar13, Cor. 5.11].

Choose a path $\gamma : [0,1] \to \mathfrak{M}^\circ_R$ joining X and X' equipped with suitable markings, taking R and R' to the distinguished element R in the reference lattice. Cover the image with a finite number of small connected neighborhoods U_i admitting Kuranishi families. We claim there exists an analytic space B^+

$$\gamma([0,1]) \subset B^+ \subset \cup_{i=1}^m U_i$$

with a universal family. Indeed, we choose B^+ to be an open neighborhood of $\gamma([0,1])$ admitting a deformation retract onto the path, but small enough so it is contained in the union of the U_i's. The topological triviality of B^+ means there is no obstruction to gluing local families. Applying the proper mapping theorem as above, we conclude that each fiber over B^+ contains a cycle of rational curves with class R. □

□

3. Specializing extremal rays

We continue to address the case of general irreducible holomorphic symplectic varieties.

Proposition 5. *Let X be a projective holomorphic symplectic manifold. Let $R \subset X$ denote an extremal rational curve with $R^2 < 0$. Consider a projective family $\pi : \mathcal{X} \to B$ over a connected curve B with $\pi^{-1}(b) \simeq X$ and $[R]$ remaining algebraic in the fibers of π. Then the specialization of R in $\pi^{-1}(b_0)$ is extremal for all but finitely many $b_0 \in B$.*

The results of Ran and Voisin mentioned above imply that R deforms to nearby fibers of π. Note that $\mathrm{rank}(H_2(\mathcal{X}_b, \mathbb{Z})_{alg})$ typically jumps for infinitely many b—the Proposition asserts most of these do not affect whether R is extremal in the cone of curves.

We require two lemmas. The first one is an a priori (but not effective) version of Proposition 2.

Recall that a *quasi-polarization* g of X is a primitive Cartier divisor on X that is nef and big. Each multiple $Ng, N > 0$ has vanishing higher cohomology by Kawamata-Viehweg vanishing; some multiple is globally generated by Kawamata basepoint freeness.

Let $\mathcal{T}$ denote a deformation equivalence class of quasi-polarized irreducible holomorphic symplectic manifolds (X, g). Write $H^2(\mathbb{Z})$ and $H_2(\mathbb{Z})$ to denote the cohomology and homology lattices of an unspecified manifold of type $\mathcal{T}$.

Lemma 6. *Let $\mathcal{T}$ denote a deformation equivalence class of quasi-polarized irreducible holomorphic symplectic manifolds (X, g). Up to the action on $H_2(\mathbb{Z})$ of polarization-preserving monodromies, there are only finitely many classes of extremal rays of the Mori cones occurring in the deformation class $\mathcal{T}$. In particular, there exists a constant $c_{\mathcal{T}}$ such that, for every $X \in \mathcal{T}$ and every primitive $R \in g^{\perp} \subset H_2(X, \mathbb{Z})$ generating an extremal ray in the cone of effective curves of X,*

$$(R, R) \geq c_{\mathcal{T}}.$$

Proof. We let $\Gamma_g \subset \mathrm{Aut}\left(H^2(\mathbb{Z})\right)$ be the image of the monodromy group preserving g. Assume there exists a sequence $(R_j, X_j)_{j=1,2,\dots}$ where R_j is extremal on X_j, $(R_j, g) = 0$, and the Γ_g-orbits of all R_j, considered as elements in $H_2(\mathbb{Z})$, are disjoint. We will proceed to obtain a contradiction to the openness of the subset of g-polarized varieties in the moduli space of quasi-polarized varieties:

By the Cone Theorem (see [HT09]) we may assume that some multiple $n_j R_j$, $n_j \in \mathbb{N}$, is the class of an extremal rational curve in X_j. Consider the sequence of rank-two saturated sublattices

$$\Pi_j := \mathbb{Z}g + \mathbb{Z}R_j \subset H_2(\mathbb{Z}), \quad (g, R_j) = 0,$$

where R_j is one of the rays enumerated above. Local Torelli, surjectivity of the period map [Huy99, Sec. 8] and Proposition 3 imply there exist codimension-one families of manifolds X_j in $\mathcal{T}$ satisfying:

- $H_2(X_j, \mathbb{Z})_{alg} \supset \mathbb{Z}g + \mathbb{Z}R_j$;
- R_j is effective, and g is nef but not ample.

Some further geometric observations:

- A general deformation of (X_j, g) yields a polarized holomorphic symplectic variety (Z, g) that is a deformation of Z_j.
- There exists a universal constant $d_{\mathcal{T}}$ such that $d_{\mathcal{T}} g$ is basepoint free and birational onto its image for any variety in $\mathcal{T}$ [Kol93].

We now use the global Torelli theorem for quasi-polarized varieties. Let $\mathcal{P}_{g^{\perp}}$ denote the period domain of the lattice $g^{\perp}$. Since Γ_g is of finite index in the group $\mathrm{Aut}(g^{\perp})$ of lattice automorphisms, the quotient $\mathcal{P}_{g^{\perp}}/\Gamma_g$ has a natural algebraic structure as a quasi-projective variety by Baily-Borel.

Consider the irreducible component $\mathcal{H}$ of the Hilbert scheme containing the deformed (Z, g), and the dense open subset $U \subset \mathcal{H}$ parametrizing the smooth such varieties. The period map $U \to \mathcal{P}_{g^{\perp}}/\Gamma_g$ is algebraic (see [GHS10], proof of Thm. 1.5). The period map from the moduli space to $\mathcal{P}_{g^{\perp}}/\Gamma_g$ identifies pairs (X, g) and (X', g) if and only if there is a birational morphism $X \dashrightarrow X'$ preserving g, see [Ver09, Cor. 1.25]; in particular, the image of U is disjoint from the divisor of (g, R_j)-lattice polarized varieties.

This is a contradiction, as the image of a dominant algebraic morphism cannot omit a countably infinite collection of divisors. $\square$

Remark. We would like a stronger version of Lemma 6, with $\mathcal{T}$ a deformation-equivalence class of irreducible holomorphic symplectic manifolds, without specifying a quasi-polarization. This is available where the monodromy action on $H^2(X, \mathbb{Z})$

is sufficiently large, e.g., for X deformation equivalent to $S^{[n]}$ where S is a K3 surface.

Given an extremal ray R and a positive integer d, there exists a vector

$$g \in R^{\perp} \subset H^2(X,\mathbb{Z})$$

satisfying

$$(g,g) = 2d, \quad \big(g, H^2(X,\mathbb{Z})\big) = \mathbb{Z}. \tag{2}$$

Indeed, using (1) we may write

$$H^2(X,\mathbb{Z})_{(,)} \simeq (-E_8)^{\oplus 2} \oplus U^{\oplus 3} \oplus (-2(n-1)).$$

Markman's characterization of the monodromy group [Mar11, Lemma 9.2] and classical results of Eichler (cf. [GHS10, Lemma 3.5]) allow us to choose the isomorphism such that R lies in the last two summands. Thus we may take g to be any primitive vector of length $2d$ in the remaining summands.

Moreover, by [GHS10, Cor. 3.7] (or a second application of the quoted results of Markman and Eichler) there is an *irreducible* moduli space of polarized holomorphic symplectic manifolds (X,g) where g satisfies (2). Thus each monodromy orbit of extremal rays arises at the boundary of this moduli space. More precisely, Markman's analysis of monodromy reflections [Mar11, Sec. 1.2] and Huybrechts' interpretation of the moving cone via the birational Kähler cone [HT09, Huy03] implies there exists a quasi-polarized (X,g) with $(g,R)=0$, where R is the specialization of our extremal ray.

Proof of Proposition 5. Let X be a very general fiber of π. The Mori cone of X near R is locally polyhedral; see [HT09, Cor. 18] for a discussion of this in the context of the Cone Theorem and the Log Minimal Model program. It follows that $R^{\perp}$ is a supporting hyperplane of a facet of the nef cone; let g be an integral divisor in this facet. Then g is semiample by Kawamata basepoint freenesss and satisfies $(g,R)=0$. The locus $U \subset B$ over which g remains semiample is Zariski dense, with finite complement.

Now consider the Baily-Borel quotient $\mathcal{P}_{g^{\perp}}/\Gamma_g$ parametrizing polarized weight two-Hodge structures on $H^2(X,\mathbb{Z})_{prim} \cong g^{\perp}$ up to the action of the monodromy group Γ_g preserving g. By Lemma 6, it admits finitely many divisors corresponding to Hodge structures such that there exists a $R' \in H_2(X,\mathbb{Z})_{alg}$ with R' possibly generating an extremal ray, and $(R',g)=0$.

For our family $\pi : \mathcal{X} \to B$, there are finitely many such divisors not identically containing the image of B under the period mapping, which cut out finitely many points on B. This allows us to control fibers $\mathcal{X}_u, u \in U$, in which R fails to be extremal: In these cases, we may express

$$R = \sum_{i=1}^{r} c_i R_i, \quad c_i \in \mathbb{Q}_{>0}$$

where the R_i are extremal in $\mathcal{X}_u$ and satisfy $(g,R_i)=0$. □

4. Proof of Theorem 1

Note that a projects to a negative class in $v^{\perp}$ if and only if

$$(a,a)\,(v,v) < (a,v)^2.$$

The autoduality of the positive cone and the fact that nef divisors have non-negative Beauville-Bogomolov squares imply that the Mori cone contains the positive cone. Thus we restrict our attention to negative classes.

We first claim the enumerated elements are all in the Mori cone. This follows from [BM13, Thm. 12.2] in the case where $X \simeq M_v(S)$. Moreover, the proof of Proposition 12.6 of this paper shows there exist examples of moduli spaces and stability conditions where $\theta^\vee(a)$ is the class of a contractible smooth rational curve R. Typically, the contraction entails identifying all extensions of two stable objects of equal slope.

Markman [Mar11, Lemma 9.2] shows that the image of the monodromy representation consists of the orientation-preserving automorphisms of the lattice $H^2(X,\mathbb{Z})$ acting via ± 1 on the discriminant group $H^2(X,\mathbb{Z})^*/H^2(X,\mathbb{Z})$. A classical result of Eichler [Eic74] (see also [GHS10, Lemma 3.5]) shows that there is a unique orbit in $\widetilde{\Lambda}$ of elements a' such that

$$(a',v) = (a,v)\,, (a',a') = (a,a)\,, \tag{3}$$

and the divisibility of a' and a in $\widetilde{\Lambda}$ are equal. By definition, the divisibility of a non-zero vector $\lambda \in \widetilde{\Lambda}$ is the largest $d \in \mathbb{N}$ such that $\lambda/d \in \widetilde{\Lambda}$.

Suppose X' is arbitrary and $a' \in H_2(X',\mathbb{Z})_{alg}$ a class with the same numerical properties as a (see (3) above). It follows there exists a parallel transport operator

$$\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$$

taking a to a'. Proposition 3 guarantees that $\theta^\vee(a')$ remains effective.

Second, we address the reverse inclusion, i.e., that our list of extremal rays is complete. The key ingredient is the next Proposition, which implies that extremal rays of general X share the same numerical properties as extremal rays of moduli spaces of sheaves over K3 surfaces.

Proposition 7. *Let (X,h) be a holomorphic symplectic manifold of K3 type. Let $R \subset X$ denote an extremal rational curve with $R^2 < 0$. Then there exists a connected curve B, points $b, b_0 \in B$, and a projective family $\pi : \mathcal{X} \to B$ satisfying the following:*

- *$\pi^{-1}(b) \simeq X$;*
- *the class $[R]$ remains algebraic in the fibers of π;*
- *the specialization of R in $\pi^{-1}(b_0)$ is extremal for all but finitely many $b_0 \in B$;*
- *$\pi^{-1}(b_0) \simeq M_v(S)$, the moduli space of sheaves on a K3 surface with Mukai vector v, for infinitely many $b_0 \in B$.*

Proof. Everything except the last assertion follows from Proposition 5. Markman-Mehotra [MM12, Th. 1.1] showed that the locus where the fibers are isomorphic to Hilbert schemes is dense.

Here is another approach: Let $\delta \in H^2(X,\mathbb{Z})$ be a primitive vector such that $(\delta,\delta) = -2(n-1)$ and $\big(\delta, H^2(X,\mathbb{Z})\big) = 2(n-1)\mathbb{Z}$; 2δ 'looks like' the divisor parametrizing non-reduced subschemes of a Hilbert scheme. (Precisely, these are in the same orbit under the monodromy group.) Using the period description for the moduli space, we choose X' with period in $\{h,\alpha,\delta\}^\perp$. It follows that the Hodge structure on $H^2(X', \mathbb{Z})$ is isomorphic to that on some Hilbert scheme of a K3 surface. Indeed, $\delta^\perp$ is isomorphic to the middle cohomology of a K3 surface. Let S be that K3 surface—it exists by surjectivity of Torelli—so we obtain an isomorphism of Hodge structures

$$H^2(X, \mathbb{Z}) \simeq H^2(M_v(S), \mathbb{Z}), \quad v = (1, 0, 1 - n).$$

The Torelli Theorem [Ver09, Huy11] (see [Mar11, Cor. 9.8]) implies that the specialization (X', h, a) is birational to $M_v(S)$ for a suitable Mukai vector v. Theorem 1.2 of [BM13] asserts that all projective birational models of $M_v(S)$ may be interpreted as moduli spaces of sheaves (or objects in the derived category) on a K3 surface satisfying a suitable Bridgeland stability condition. □

This finishes the proof of Theorem 1.

Proof of Proposition 2. We will reduce the statement to the case of moduli spaces of sheaves on K3 surfaces, which is [BM13, Proposition 12.6]. Consider the family $\pi : \mathcal{X} \to B$ constructed in in Proposition 7, and let $b_0 \in B$ be such that $\mathcal{X}_{b_0}$ is a moduli space of sheaves on a K3 surface with $[R]$ extremal. Let R_0 be the rational curve on $\mathcal{X}_{b_0}$ in the ray $[R]$ with $(R_0, R_0) \geq -\frac{n+3}{2}$ given by [BM13, Proposition 12.6]. The curve R_0 is a minimal free curve in a generic fibre of the exceptional locus over B (see [BM13, Section 14]); therefore, the deformation argument in Proposition 3 applies directly to R_0 (rather than a multiple) and implies the conclusion. □

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