GALOIS THEORY AND PROJECTIVE GEOMETRY

FEDOR BOGOMOLOV AND YURI TSCHINKEL

ABSTRACT. We explore connections between birational anabelian geometry and abstract projective geometry. One of the applications is a proof of a version of the birational section conjecture.

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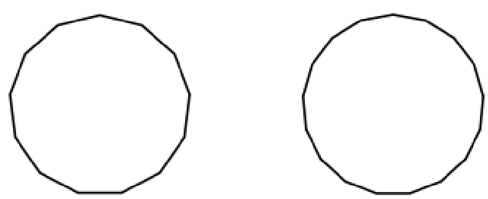
1. INTRODUCTION

What is the difference between a regular 13-gon and a 17-gon?



The discovery by Gauss in 1796 that the 17-gon can be constructed with compass and straightedge, and the 13-gon cannot, and that *internal, hidden* symmetries of number fields are ultimately responsible for this discrepancy, triggered a major transformation of mathematics, shifting the emphasis from the mechanical, Descartian *coordinatization* of space to the study of its internal *symmetries.* Around 1830, Galois developed this idea, giving a criterion for solvability of polynomial equations in one variable in radicals. Both Gauss and Galois solved 2000 year old problems; but more importantly, both realized that rather than untangling how things really work, i.e., the specifics of the construction, or which radicals to use, one has to focus on the *group* of symmetries. This insight from plane geometry and number theory slowly penetrated mathematics and physics, via Klein's 1872 unification of geometry and group theory in his Erlangen Program, works by Poincaré, Hilbert, Minkowski,

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and many others. One of its culminations is Einstein's thesis: natural symmetries in many physical problems provide such strong constraints on possible types of dynamic equations that it becomes a matter of elementary mathematical and physical considerations to find their exact form.

These developments stimulated investigations in abstract group theory, which by itself is very rich, with many open problems. Only some instances of this theory are developed to an extent that they can be applied: Lie groups, finite groups, some discrete groups. One of the most transparent chapters is the theory of abelian groups, and even here there are unpleasant features arising from passage to projective or injective limits, taking duals, and combinations of the above.

A related algebraic object is a *field*, a structure that combines *two* compatible abelian group laws, the basic example being the field of rational numbers $\mathbb{Q}$. Other examples are *finite fields* $\mathbb{F}_p$, *function fields* of algebraic varieties $k(X)$ over some field of definition k, and their completions with respect to *valuations*. *Galois groups* are symmetries of fields respecting these structures: by definition the Galois group $\mathrm{Gal}(E/K)$ of a field extension E/K is the group of automorphisms of E fixing every element of K. We write $G_K := \mathrm{Gal}(\bar{K}/K)$ for the *absolute* Galois group of a field K. For example,

$$G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z},$$

generated by complex conjugation. In general, Galois groups are *profinite* groups, i.e., projective limits of their finite index quotients. We have

$$G_{\mathbb{F}_p} = \hat{\mathbb{Z}},$$

the (uncountable) profinite completion of the additive group of integers $\mathbb{Z}$, topologically generated by the *Frobenius*

$$\mathrm{Fr} : x \mapsto x^p.$$

Typically, Galois groups are large, noncommutative, and complicated objects. For example, the Galois group $G_{\mathbb{C}(t)}$ is a free profinite group on infinitely many generators, and every finite group appears as its quotient. Very little is known about $G_{\mathbb{Q}}$, in particular, the *inverse Galois problem*, i.e., the realization of every finite group as a Galois group over $\mathbb{Q}$, is still open. On the other hand, the abelianization

$$G^a_{\mathbb{Q}} = G_{\mathbb{Q}}/[G_{\mathbb{Q}}, G_{\mathbb{Q}}]$$

is well-understood: the corresponding field

$$\bar{\mathbb{Q}}^{\mathrm{ab}} = \cup_n \mathbb{Q}(\sqrt[n]{1}) \subset \bar{\mathbb{Q}}$$

is obtained by adjoining all roots of unity. By Kummer theory, given any field K, containing all roots of unity, its extensions with cyclic Galois group $\mathbb{Z}/n\mathbb{Z}$ are given by adjoining $\sqrt[n]{f}$, for some $f \in K^\times$, the multiplicative subgroup of K. This recipe gives a constructive solution to the inverse Galois problem for abelian groups. More precisely, by Kummer theory, we have a canonical pairing

$$\begin{array}{rcl} [\cdot,\cdot]_n \,:\, G^a_K/n \times K^\times/(K^\times)^n & \to & \boldsymbol{\mu}_n \\ (\gamma, f) & \mapsto & [\gamma, f]_n := \gamma(\sqrt[n]{f})/\sqrt[n]{f}, \end{array}$$

for every $n \in \mathbb{N}$. Here $\boldsymbol{\mu}_n$ is the (multiplicative) group of roots of unity of order n. This pairing extends to a nondegenerate pairing

$$[\cdot,\cdot] \,:\, G^a_K \times \hat{K}^\times \to \lim_{n\to\infty} \boldsymbol{\mu}_n, \quad \boldsymbol{\mu}_n := \{\sqrt[n]{1}\},$$

where $\hat{K}^\times$ is the profinite completion of $K^\times$ and the target can be identified with $\hat{\mathbb{Z}}$. Thus

$$G^a_K = \mathrm{Hom}(K^\times, \hat{\mathbb{Z}}), \qquad \text{and} \qquad \hat{K}^\times = \mathrm{Hom}(G^a_K, \hat{\mathbb{Z}}), \tag{1.1}$$

so that the *double-dual* of $K^\times$ is actually $\hat{K}^\times$. We find that G^a_K "encodes", in the weak sense, the multiplicative structure of K.

A major open problem today is to identify classes of fields *characterized* by their absolute Galois groups. There exist genuinely different fields with isomorphic Galois groups, e.g., $\mathbb{F}_p$ and $\mathbb{C}((t))$. However, Neukirch and Uchida showed that Galois groups of maximal *solvable* extensions of number fields or function fields of curves over finite fields determine the corresponding field, up-to isomorphism [22], [30].

This result is the first instance of *birational anabelian geometry*, which is, in some sense, an algebraic incarnation of Einstein's postulate: it aims to show that Galois groups of function fields of algebraic varieties over an algebraically closed ground field determine the function field, in a functorial way. The version proposed by Grothendieck in [13] introduces a class of *anabelian* varieties, functorially characterized by their étale fundamental groups; with prime examples being hyperbolic curves and varieties successively fibered into hyperbolic curves. For representative results, see [20], [32], [31], [29], as well as [15], [21], [25], [24], [19].

However, absolute Galois groups are simply too large. It turns out that there are intermediate groups, whose description involves some *projective* *geometry*, most importantly, geometry of lines and points in the projective plane, bridging Gauss and Galois. These groups are just minimally different from abelian groups; they encode the geometry of simple configurations. On the other hand, their structure is already sufficiently rich so that the corresponding objects in the theory of fields allow to capture *all* invariants and individual properties of large fields, i.e., function fields of transcendence degree at least two over algebraically closed ground fields. This insight of the first author [3], [5], [4], was developed in the series of papers written at the Courant Institute [6], [7], [8], [9] over the last decade. One of our main results is that function fields $K = k(X)$ over $k = \bar{\mathbb{F}}_p$ are determined by

$$\mathcal{G}^c_K := \mathcal{G}_K/[\mathcal{G}_K, [\mathcal{G}_K, \mathcal{G}_K]],$$

where $\mathcal{G}_K$ is the maximal pro-ℓ-quotient of G_K, and $\mathcal{G}^c$ is the canonical central extension of its abelianization $\mathcal{G}^a_K$ (see also [26]).

In [10] we survey the development of the main ideas merging into this *almost abelian anabelian geometry* program. Here we formulate our vision of the future directions of research, inspired by this work. In particular, in Section 4 we prove a new result, a version of the *birational section conjecture*. In Sections 5 and 6 we discuss cohomological properties of Galois groups closely related to the Bloch–Kato conjecture, proved by Voevodsky, Rost, and Weibel, and focus on connections to anabelian geometry.

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2. Projective geometry and K-theory

The introduction of the projective plane essentially trivialized plane geometry and provided simple proofs for many results concerning configurations of lines and points, considered difficult before that. More importantly, the axiomatization efforts in late 19th and early 20th century revealed that *abstract* projective structures capture *coordinates*, a "triumph of modern mathematical thought" [33, p. v]. The axioms can be found in many books, including Emil Artin's lecture notes from a course he gave at the Courant Institute in the Fall of 1954 [2, Chapters VI and VII]. The classical result mentioned above is that an abstract projective space (subject to Pappus' axiom) is a projectivization of a vector space over a *field*. This can be strengthened as follows:

Theorem 1. *Let K/k be an extension of fields. Then $K^\times/k^\times$ is simultaneously an abelian group and a projective space. Conversely, an abelian group with a compatible projective structure corresponds to a field extension.*

Proof. See [10, Section 1]. □

In *Algebraic geometry*, projective spaces are the most basic objects. Over nonclosed fields K, they admit nontrivial forms, called *Brauer-Severi* varieties. These forms are classified by the *Brauer group* $\mathrm{Br}(K)$, which admits a Galois-cohomological incarnation:

$$\mathrm{Br}(K) = \mathrm{H}^2(G_K, \mathbb{G}_m).$$

The theory of Brauer groups and the local-global principle for *Brauer–Severi* varieties over number fields are cornerstones of arithmetic geometry. Much less is known over more complicated ground fields, e.g., function fields of surfaces. Brauer groups, in turn, are closely related to Milnor's K_2-groups, and more generally K-theory, which emerged in algebra in the study of matrix groups.

We recall the definition of Milnor K-groups. Let K be a field. Then

$$\mathrm{K}_1^M(K) = K^\times$$

and the higher K-groups are spanned by symbols:

$$\mathrm{K}_n^M(K) = (K^\times)^{\otimes n}/\langle \cdots x \otimes (1-x) \cdots \rangle,$$

the relations being symbols containing $x \otimes (1-x)$. For $i = 1, 2$, Milnor K-groups of fields coincide with those defined by Quillen, and we will often omit the superscript.

Throughout, we work with function fields of algebraic varieties over algebraically closed ground fields; by convention, the dimension of the field is its transcendence degree over the ground field.

Theorem 2. [8] *Assume that K and L are function fields of algebraic varieties of dimension ≥ 2, over algebraically closed fields k and l, and that there exist compatible isomorphisms of abelian groups*

$$\mathrm{K}_1(K) \xrightarrow{\psi_1} \mathrm{K}_1(L) \quad \textit{and} \quad \mathrm{K}_2(K) \xrightarrow{\psi_2} \mathrm{K}_2(L).$$

Then there exists an isomorphism of fields

$$\psi : K \to L$$

such that the induced map on $K^\times$ *coincides with* $\psi_1^{\pm 1}$.

The proof exploits the fact that $\mathrm{K}_2(K)$ encodes the canonical projective structure on $\mathbb{P}_k(K) = K^\times/k^\times$. It is based on the following observations:

- The multiplicative groups $k^\times$ and $l^\times$ are characterized as *infinitely-divisible* elements in $\mathrm{K}_1(K)$, resp. $\mathrm{K}_1(L)$. This leads to an isomorphism of abelian groups (denoted by the same symbol):
$$\mathbb{P}_k(K) \xrightarrow{\psi_1} \mathbb{P}_l(L).$$
- rational functions $f_1, f_2 \in K^\times$ are algebraically dependent in K if and only if their symbol (f_1, f_2) is *infinitely-divisible* in $\mathrm{K}_2(K)$. This allows to characterize $\mathbb{P}_k(E) \subset \mathbb{P}_k(K)$, for one-dimensional $E \subset K$ and we obtain a *fan* of infinite-dimensional projective subspaces in $\mathbb{P}_k(K)$. The compatibility of ψ_1 with ψ_2 implies that the corresponding structures on $\mathbb{P}_k(K)$ and $\mathbb{P}_l(L)$ coincide.
- By Theorem 1, it remains to show that ψ_1 (or $1/\psi_1$) maps projective lines $\mathbb{P}^1 \subset \mathbb{P}_k(K)$ to projective lines in $\mathbb{P}_l(L)$. It turns out that projective lines can be intrinsically characterized as *intersections* of well-chosen infinite-dimensional $\mathbb{P}_k(E_1)$ and $\mathbb{P}_k(E_2)$, for 1-dimensional subfields $E_1, E_2 \subset K$ (see [8, Theorem 22] or [10, Proposition 9]).

The theorem proved in [8] is stronger, it addresses the case when ψ_1 is an *injective* homomorphism.

3. Projective geometry and Galois groups

Let K be a function field over $k = \bar{\mathbb{F}}_p$. In Section 2 we considered the abelian group / projective space $\mathbb{P}_k(K)$ and its relationship to the K-theory of the field K. Here we focus on a *dual* picture.

Let R be one of the following rings:

$$R = \mathbb{Z},\ \mathbb{Z}/\ell^n,\ \mathbb{Z}_\ell, \quad \ell \neq p.$$

In fact, the theory below extends almost *verbatim* to quite general commutative rings such that the order of all torsion elements $r \in R$ is coprime to p and without infinitely divisible elements.

Define

$$\mathrm{W}^a_K(R) := \mathrm{Hom}(K^\times/k^\times, R) = \mathrm{Hom}(K^\times, R), \tag{3.1}$$

the R-module of continuous homomorphisms, where $\mathbb{P}_k(K) = K^\times/k^\times$ is endowed with discrete topology. We call $\mathrm{W}^a_K(R)$ the *abelian Weil group* of K with values in R.

The abelian Weil group carries a collection of distinguished subgroups, corresponding to various *valuations*. Recall that a valuation is a surjective homomorphism

$$\nu : K^\times \to \Gamma_\nu$$

onto an ordered abelian group, subject to a nonarchimedean triangle inequality

$$\nu(f+g) \geq \min(\nu(f), \nu(g)). \tag{3.2}$$

Let $\mathcal{V}_K$ be the set of all nontrivial valuations of K, for $\nu \in \mathcal{V}_K$ let $\mathfrak{o}_\nu$ be the valuation ring, $\mathfrak{m}_\nu \subset \mathfrak{o}_\nu$ its maximal ideal, and $\boldsymbol{K}_\nu$ the corresponding function field. We have the following fundamental diagram:

$$\begin{array}{ccccccccc} & & 1 & \longrightarrow & \mathfrak{o}_\nu^\times & \longrightarrow & K^\times & \xrightarrow{\nu} & \Gamma_\nu \longrightarrow 1 \\ & & & & \| & & & & \\ 1 & \longrightarrow & (1+\mathfrak{m}_\nu)^\times & \longrightarrow & \mathfrak{o}_\nu^\times & \xrightarrow{\rho_\nu} & \boldsymbol{K}_\nu^\times & \longrightarrow & 1 \end{array}$$

Every valuation of $k = \bar{\mathbb{F}}_p$ is trivial, and valuation theory over function fields $K = k(X)$ is particularly *geometric*: all $\nu \in \mathcal{V}_K$ are trivial on $k^\times$ and define Γ_ν-valued functions on $\mathbb{P}_k(K)$. Throughout, we restrict our attention to such ground fields. In this context, nontrivial valuations on $k(t)$ are in bijection with points $\mathfrak{p} \in \mathbb{P}^1$, they measure the order of a function at $\mathfrak{p}$. There are many more valuations on higher-dimensional varieties.

We call

$$\mathrm{I}^a_\nu(R) := \{\gamma \in \mathrm{W}^a_K(R) \,|\, \gamma \text{ is trivial on } \mathfrak{o}_\nu^\times\} \subseteq \mathrm{W}^a_K(R)$$

the *abelian inertia group* and

$$\mathrm{D}^a_\nu(R) := \{\gamma \in \mathrm{W}^a_K(R) \,|\, \gamma \text{ is trivial on } (1+\mathfrak{m}_\nu)^\times\} \subseteq \mathrm{W}^a_K(R)$$

the *abelian decomposition group*, we have

$$\mathrm{D}^a_\nu(R)/\mathrm{I}^a_\nu(R) = \mathrm{W}^a_{\boldsymbol{K}_\nu}(R).$$

Example 3. Let G_K be the absolute Galois group of a field K, G^a_K its abelianization, and $\mathcal{G}^a_K$ the ℓ-completion of G^a_K. By *Kummer theory*,

$$\mathcal{G}^a_K = \mathrm{W}^a_K(\mathbb{Z}_\ell).$$

Moreover, $\mathrm{I}^a_\nu(\mathbb{Z}_\ell)$ and $\mathrm{D}^a_\nu(\mathbb{Z}_\ell)$ *are* the standard abelian inertia and decomposition subgroups corresponding to ν.

A valuation ν defines a simple geometry on the projective space $\mathbb{P}_k(K)$; equation (3.2) implies that each finite dimensional subspace $\mathbb{P}^n \subset \mathbb{P}_k(K)$ admits a *flag*

$$\mathbb{P}_1 \subset \mathbb{P}_2 \subset \ldots \tag{3.3}$$

of projective subspaces, such that

$$\nu : \mathbb{P}_k(K) \to \Gamma_\nu$$

is constant on $\mathbb{P}_j \setminus \mathbb{P}_{j-1}$, for all j, and this flag structure is preserved under multiplicative shifts by any $f \in K^\times/k^\times$.

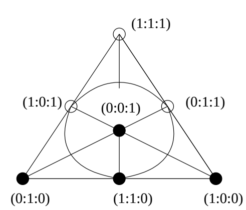
Let $\mathbb{P}$ be a projective space over k, e.g., $\mathbb{P} = \mathbb{P}_k(K)$. We say that a map $\iota : \mathbb{P} \to R$ to an *arbitrary* ring R is a *flag map* if every finite-dimensional $\mathbb{P}^n \subset \mathbb{P}$ admits a flag as in (3.3) such that ι is constant on each $\mathbb{P}_j \setminus \mathbb{P}_{j-1}$; a subset $S \subset \mathbb{P}$ will be called a *flag subset* if its set-theoretic characteristic function is a flag map on $\mathbb{P}$.

Example 4. A nonempty flag subset of $\mathbb{P}^1(k)$ is either a point, the complement to a point, or all of $\mathbb{P}^1(k)$. Nonempty proper flag subsets of $\mathbb{P}^2(k)$ are one of the following:

- a point $\mathfrak{p}$, a line $\mathfrak{l}$, $\mathfrak{l}^\circ := \mathfrak{l} \setminus \mathfrak{p}$,
- $\mathbb{P}^2 \setminus \mathfrak{p}$, $\mathbb{P}^2 \setminus \mathfrak{l}$, $\mathbb{P}^2 \setminus \mathfrak{l}^\circ$.

Proposition 5. [6, Section 2] *Let $\mathbb{P}$ be a projective space over $k = \bar{\mathbb{F}}_p$. A map $\iota : \mathbb{P} \to R$ is a flag map if and only if its restriction to every $\mathbb{P}^1 \subset \mathbb{P}$ is a flag map, i.e., is constant on the complement to one point.*

Example 6. This fails when $k = \mathbb{F}_2$, for the Fano plane:



By [6, Section 6.3], flag maps are closely related to valuations: given a flag homomorphism $\iota : \mathbb{P}_k(K) \to R$ there exists a *unique* $\nu \in \mathcal{V}_K$ and a homomorphism $\chi : \Gamma_\nu \to R$ such that

$$\iota = \chi \circ \nu. \tag{3.4}$$

This means that $\iota \in \mathrm{I}^a_\nu(R)$.

We now describe the theory of *commuting pairs* developed in [6]: We say that nonproportional $\gamma, \gamma' \in \mathrm{W}^a_K(R)$ form a *c*-pair if for every normally closed one-dimensional subfield $E \subset K$ the image of the subgroup $R\gamma \oplus R\gamma'$ in $\mathrm{W}^a_E(R)$ is cyclic; a *c*-subgroup is a noncyclic subgroup $\sigma \subset \mathrm{W}^a_K(R)$ whose image in $\mathrm{W}^a_E(R)$ is cyclic, for all E as above. We define the *centralizer* $\mathrm{Z}_\gamma \subset \mathrm{W}^a_K(R)$ of an element $\gamma \in \mathrm{W}^a_K(R)$ as the subgroup of all elements forming a *c*-pair with γ.

The main result of [6] says:

Theorem 7. *Assume that R is one of the following:*

$$\mathbb{Z},\ \mathbb{Z}/\ell^n,\ \mathbb{Z}_\ell.$$

Then

- *every c-subgroup σ has R-rank $\leq \operatorname{tr}\deg_k(K)$;*
- *for every c-subgroup σ there exists a valuation $\nu \in \mathcal{V}_K$ such that*
  - *σ is trivial on $(1+\mathfrak{m}_\nu)^\times \subset K^\times$*
  - *there exists a maximal subgroup $\sigma' \subseteq \sigma$ of R-corank at most one such that*

$$\sigma' \subseteq \mathrm{Hom}(\Gamma_\nu, R) \subset \mathrm{Hom}(K^\times, R) = \mathrm{W}^a_K(R).$$

The groups σ' are, in fact, inertia subgroups $\mathrm{I}^a_\nu(R)$ corresponding to ν. The union of all σ containing an inertia subgroup $\mathrm{I}^a_\nu(R)$ is the corresponding decomposition group $\mathrm{D}^a_\nu(R)$. If $\mathrm{I}^a_\nu(R)$ is cyclic, generated by ι, then $\mathrm{D}^a_\nu(R) = \mathrm{Z}_\iota$, the centralizer of ι.

The proof of Theorem 7 is based on the following geometric observation, linking the abelian Weil group with affine/projective geometry: Let $\gamma, \gamma' \in \mathrm{W}^a_K(R)$ be nonproportional elements forming a *c*-pair and let

$$\begin{array}{ccc} \mathbb{P}_k(K) & \xrightarrow{\phi} & R^2 \\ f & \mapsto & (\gamma(f), \gamma'(f)) \end{array}$$

be the induced map to the affine plane $\mathbb{A}^2(R)$. If follows that

(*) the image of every $\mathbb{P}^1 \subset \mathbb{P}_k(K)$ satisfies a linear relation, i.e., is contained in an affine line in $\mathbb{A}^2(R)$.

Classically, this is called a *collineation.* A simple model is a map

$$\mathbb{P}^2(\mathbb{F}_p) \xrightarrow{\phi} \mathbb{A}^2(\mathbb{F}_2),$$

where $p > 2$ is a prime and where the target is a set with 4 elements. It turns out that when the map ϕ satisfies condition $(*)$ then the target contains only 3 points. Furthermore, on every line $\mathbb{P}^1 \subset \mathbb{P}^2$ the map is constant on the complement of one point! This in turn implies that there is a stratification

$$\mathfrak{p} \subset \mathfrak{l} \subset \mathbb{P}^2,$$

where $\mathfrak{p}$ is a point and $\mathfrak{l} = \mathbb{P}^1$ and $r, r' \in \mathbb{F}_2$, $(r, r') \neq (0,0)$, such that $\iota : r\gamma + r'\gamma'$ is *constant* on $\mathfrak{l} \setminus \mathfrak{p}$, and $\mathbb{P}^2 \setminus \mathfrak{l}$, i.e., ι is a flag map on $\mathbb{P}^2$. This last property fails for $p = 2$, as noted in Example 6. Nevertheless, one can extract the following general fact:

- if γ, γ' satisfy $(*)$ then there exists a nontrivial R-linear combination $\iota := r\gamma + r'\gamma'$ such that

  $$\iota : \mathbb{P}_k(K) \to R$$

  is a flag map, i.e., $\iota \in \mathrm{I}^a_\nu(R)$, for some $\nu \in \mathcal{V}_K$.

The proof is based on a lemma from projective geometry:

Lemma 8. *Let $\mathbb{P}^2(k) = S_1 \sqcup_{j \in J} S_j$ be a set-theoretic decomposition into at least three nonempty subsets such that for every $\mathbb{P}^1(k) \subset \mathbb{P}^2(k)$ we have*

$$\mathbb{P}^1(k) \subseteq S_1 \sqcup S_j, \quad \textit{for some} \;\; j \in J, \qquad \textit{or} \;\; \mathbb{P}^1 \subseteq \sqcup_{j \in J} S_j$$

Then one of the S_j is a flag subset of $\mathbb{P}^2(k)$.

These considerations lead to the following characterization of multiplicative groups of valuation rings, one of the main results of [6]:

Proposition 9. *Let $\mathfrak{o}^\times \subset K^\times/k^\times$ be a subgroup such that its intersection with any projective plane $\mathbb{P}^2 \subset \mathbb{P}_k(K) = K^\times/k^\times$ is a flag subset, i.e., its set-theoretic characteristic function is a flag function. Then there exists a $\nu \in \mathcal{V}_K$ such that $\mathfrak{o}^\times = \mathfrak{o}^\times_\nu/k^\times$.*

We now turn to more classical objects, namely Galois groups of fields. Let G_K be the absolute Galois group of a field K, G^a_K its abelianization and $G^c_K = G_K/[G_K, [G_K, G]]$ its canonical central extension. Let K be a function field of transcendence degree ≥ 2 over $k = \bar{\mathbb{F}}_p$. Fix a prime $\ell \neq p$ and replace G^a_K and G^c_K by their maximal pro-ℓ-quotients

$$\mathcal{G}^a_K = \mathrm{Hom}(K^\times/k^\times, \mathbb{Z}_\ell) = \mathrm{W}^a_K(\mathbb{Z}_\ell), \quad \text{and} \quad \mathcal{G}^c_K.$$

Note that $\mathcal{G}^a_K$ is a torsion-free $\mathbb{Z}_\ell$-module of infinite rank.

A *commuting pair* is a pair of nonproportional $\gamma, \gamma' \in \mathcal{G}^a_K$ which lift to commuting elements in $\mathcal{G}^c_K$ (this property does not depend on the choice of the lift). The main result of the theory of commuting pairs in [6] says that if

(**) $\gamma, \gamma' \in \mathcal{G}_K^a$ form a commuting pair

then the $\mathbb{Z}_\ell$-linear span of $\gamma, \gamma' \in \mathcal{G}_K^a$ contains an inertia element of some valuation ν of K.

A key observation is that Property $(**)$ implies $(*)$, for each $\mathbb{P}_k^2 \subset \mathbb{P}_k(K)$, which leads to a flag structure on $\mathbb{P}_k(K)$, which in turn gives rise to a valuation.

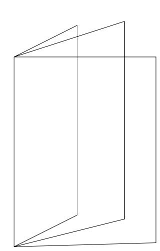
Commuting pairs are part of an intricate *fan* Σ_K on $\mathcal{G}_K^a$, which is defined as the set of all topologically noncyclic subgroups $\sigma \subset \mathcal{G}_K^a$ such that the preimage of σ, under the natural projection $\mathcal{G}_K^c \to \mathcal{G}_K^a$, is abelian. We have:

- $\mathrm{rk}_{\mathbb{Z}_\ell}(\sigma) \leq \mathrm{tr}\deg_k(K)$, for all $\sigma \in \Sigma_K$;
- every $\sigma \in \Sigma_K$ contains a subgroup of corank one which is the inertia subgroup of some valuation of K.

Intersections of subgroups in Σ_K reflect subtle dependencies among valuations of K. Let $\mathcal{I}_\nu^a := \mathrm{I}_\nu^a(\mathbb{Z}_\ell) \subset \mathcal{G}_K^a$ be the subgroup of inertia elements with respect to $\nu \in \mathcal{V}_K$ and $\mathcal{D}_\nu^a = \mathrm{D}_\nu^a(\mathbb{Z}_\ell)$ the corresponding *decomposition group*. In our context, $\mathcal{D}_\nu^a$ is also the group of all elements $\gamma \in \mathcal{G}_K^a$ forming a commuting pair with every $\iota \in \mathcal{I}_\nu^a$. As noted above, the definitions of inertia and decomposition groups given here are equivalent to the classical definitions in Galois theory. We have

$$\mathcal{G}_{\boldsymbol{K}_\nu}^a = \mathcal{D}_\nu^a / \mathcal{I}_\nu^a,$$

the Galois group of the residue field $\boldsymbol{K}_\nu$ of ν, and $\Sigma_{\boldsymbol{K}_\nu}$ is the set of projections of subgroups of $\sigma \in \Sigma_K$ which are contained in $\mathcal{D}_\nu^a$. When X is a surface, $K = k(X)$, and ν a *divisorial* valuation of K, we have $\mathcal{I}_\nu^a \simeq \mathbb{Z}_\ell$ and $\mathcal{D}_\nu^a$ is a large group spanned by subgroups σ of rank two consisting of all element commuting with the topological generator δ_ν of $\mathcal{I}_\nu^a$. The picture is:



To summarize, the Galois group $\mathcal{G}^c_K$ encodes information about affine and projective structures on $\mathcal{G}^a_K$, in close parallel to what we saw in the context of K-theory in Section 2. These structures are so individual that they allow to recover the field, via the reconstruction of the abstract projective structure on $\mathbb{P}_k(K)$ (see [14] for the axiomatic foundations of projective geometry):

Theorem 10. [7], [9] *Let K and L be function fields of transcendence degree ≥ 2 over $k = \bar{\mathbb{F}}_p$ and $\ell \neq p$ a prime. Let*

$$\psi^* : \mathcal{G}^a_L \to \mathcal{G}^a_K$$

be an isomorphism of abelian pro-ℓ-groups inducing a bijection of sets

$$\Sigma_L = \Sigma_K.$$

Then there exists an isomorphism of perfect closures

$$\bar{\psi} : \bar{K} \to \bar{L},$$

unique modulo rescaling by a constant in $\mathbb{Z}_\ell^\times$.

4. $\mathbb{Z}$-version of the Galois group

In this section, we introduce a functorial version of the *reconstruction / recognition* theories presented in Sections 2 and 3. This version allows to recover not only field isomorphisms from data of K-theoretic or Galois-theoretic type, but also *sections*, i.e., rational points on varieties over higher-dimensional function fields.

We work in the following setup: let X be an algebraic variety over $k = \bar{\mathbb{F}}_p$, with function field $K = k(X)$. We use the notation:

- for $x, y \in K^\times$ we let $\mathfrak{l}(x, y) \subset \mathbb{P}_k(K)$ be the projective line through the images of x and y in $K^\times/k^\times$;
- for planes $\mathbb{P}^2(1, x, y) \subset \mathbb{P}_k(K)$ (the span of lines $\mathfrak{l}(1, x)$, $\mathfrak{l}(1, y)$) we set $\mathbb{P}^2(1, x, y)^\circ := \mathbb{P}^2(1, x, y) \setminus \{1\}$.

We will say that $\bar{x}, \bar{y} \in K^\times/k^\times$ are algebraically dependent, and write $\bar{x} \approx \bar{y}$, if this holds for any of their lifts x, y to $K^\times$.

Define the *abelian Weil group*

$$\mathrm{W}^a_K := \mathrm{Hom}(K^\times/k^\times, \mathbb{Z}) = \mathrm{Hom}(K^\times, \mathbb{Z}),$$

with discrete topology. Since $K^\times/k^\times$ is a free abelian group, W^a_K is a torsion-free, infinite rank $\mathbb{Z}$-module. The functor

$$K \mapsto \mathrm{W}^a_K$$

is contravariant; it is a $\mathbb{Z}$-version of $\mathcal{G}^a_K$, since it can be regarded as a $\mathbb{Z}$-sublattice of

$$\mathcal{G}^a_K = \mathrm{Hom}(K^\times/k^\times, \mathbb{Z}_\ell), \quad \ell \neq p,$$

via the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell$.

We proceed to explore a functorial version of Theorem 10 for $(\mathrm{W}^a_K, \Sigma_K)$, where Σ_K is the corresponding *fan*, i.e., the set of c-subgroups $\sigma \subset \mathrm{W}^a_K$. We work with the following diagram

$$K \xrightarrow{\psi} L$$

$$K^\times/k^\times \xrightarrow{\psi_1} L^\times/l^\times$$

$$\mathrm{W}^a_K \xleftarrow{\psi^*} \mathrm{W}^a_L$$

where K and L are function fields over algebraically closed ground fields k and l. We are interested in situations when ψ_1 maps subgroups of the form $E^\times/k^\times$, where $E \subset K$ is a subfield with $\mathrm{tr}\deg_k(E) = 1$, into similar subgroups in $L^\times/l^\times$. The *dual* homomorphisms

$$\psi^* : \mathrm{W}^a_L \to \mathrm{W}^a_K \tag{4.1}$$

to such ψ_1 *respect the fans*, in the following sense: for all $\sigma \in \Sigma_L$ either $\psi^*(\sigma) \in \Sigma_K$ or $\psi^*(\sigma)$ is cyclic.

Example 11. The following examples of homomorphisms ψ^* as in Equation 4.1 arise from geometry:

(1) If $X \to Y$ is a dominant map of varieties over k then $k(Y) = L \subset K = k(X)$ and the induced homomorphism

$$\psi^* : \mathrm{W}^a_K \to \mathrm{W}^a_L$$

respects the fans.

(2) Let $\pi : X \to Y$ be a dominant map of varieties over k and $s : Y \to X$ a section of π. There exists a valuation $\nu \in \mathcal{V}_K$ with center the generic point of $s(Y)$ such that

$$L^\times \subset \mathfrak{o}^\times_\nu \subset K^\times$$

and the natural projection

$$\rho_\nu : \mathfrak{o}^\times_\nu \to \boldsymbol{K}^\times_\nu$$

onto the multiplicative group of the residue field induces an isomorphism

$$L^\times \simeq \boldsymbol{K}_\nu^\times$$

which extends to an isomorphism of fields

$$L \simeq \boldsymbol{K}_\nu.$$

Let $\tilde{\rho}_\nu : K^\times \to L^\times$ be any multiplicative extension of ρ_ν to $K^\times$. Such $\tilde{\rho}_\nu$ map multiplicative subgroups of the form $k(x)^\times$ to similar subgroups of $L^\times$, i.e., the dual map

$$\mathrm{W}_L^a \to \mathrm{W}_K^a$$

preserves the fans.

(3) More generally, let $\nu \in \mathcal{V}_K$ be a valuation, with valuation ring $\mathfrak{o}_\nu$, maximal ideal $\mathfrak{m}_\nu$ and residue field $\boldsymbol{K}_\nu$. Combining the exact sequences

$$\begin{array}{ccccccccc} & & 1 & \longrightarrow & \mathfrak{o}_\nu^\times & \longrightarrow & K^\times & \xrightarrow{\nu} & \Gamma_\nu \longrightarrow 1 \\ & & & & \| & & & & \\ 1 & \longrightarrow & (1+\mathfrak{m}_\nu)^\times & \longrightarrow & \mathfrak{o}_\nu^\times & \xrightarrow{\rho_\nu} & \boldsymbol{K}_\nu^\times & \longrightarrow & 1 \end{array}$$

we have an exact sequence

$$1 \to \boldsymbol{K}_\nu^\times \to (1+\mathfrak{m}_\nu)^\times \backslash K^\times \to \Gamma_\nu \to 1.$$

Let $\boldsymbol{K}_\nu = k(Y)$, for some algebraic variety Y over k, and let L be any function field containing $\boldsymbol{K}_\nu = k(Y)$. Assume that there is a diagram

$$\begin{array}{ccc} \mathfrak{o}_\nu^\times/k^\times & \hookrightarrow & K^\times/k^\times \\ \downarrow{\scriptstyle \rho_\nu} & & \downarrow{\scriptstyle \psi_1} \\ \boldsymbol{K}_\nu^\times/k^\times & \xrightarrow[\phi]{} & L^\times/k^\times \end{array}$$

where ϕ is injective and ψ_1 is an extension of ϕ. Such extensions exist provided we are given a splitting of $\nu : K^\times \to \Gamma_\nu$. In this case, we will say that ψ_1 is *defined by a valuation*. The dual map to such ψ_1 respects the fans. Indeed, this can be checked on multiplicative groups E^*/k^* of one-dimensional subfields $E \subset K$: either E^*/k^* embedds into $\boldsymbol{K}_\nu^*/k^*$ or the restriction of ν to E^*/k^*

is nontrivial and hence the image of E^*/k^* is cyclic. Theorem 12 shows a converse result: *any* ψ_1 respecting one-dimensional subfields can be obtained via this construction.

We proceed to describe the restriction of ψ_1 to projective subspaces of $\mathbb{P}_k(K)$, when ψ_1 is obtained from the geometric construction (3); in particular, ψ_1 is not injective. In this case, for any $E = k(x) \subset K$ the restriction of ψ_1 to $E^\times/k^\times$ is either

- injective, or
- its restriction to $\mathfrak{l}(1,x)$ is constant on the complement of one point, i.e., it factors through a valuation homomorphism.

On planes $\mathbb{P}^2(1,x,y)$ there are more possibilities: an inertia element

$$\iota = \iota_\nu \in \mathrm{W}_K^a = \mathrm{Hom}(K^\times/k^\times, \mathbb{Z})$$

restricts to any $\mathbb{P}^2 = \mathbb{P}_k^2(1,x,y)$ as a flag map, in particular, it takes at most three values. This leads to the following cases:

(a) ι is constant on $\mathbb{P}^2(1,x,y)$; then $\mathbb{P}^2(1,x,y) \subset \mathfrak{o}_\nu^\times/k^\times$ since the corresponding linear space does not intersect the maximal ideal $\mathfrak{m}_\nu$ and contains 1. The projection

$$\rho_\nu : \mathbb{P}^2(1,x,y) \to \boldsymbol{K}_\nu^\times/k^\times$$

is injective. If x,y are algebraically independent the values of ψ_1 on $\mathbb{P}^2(1,x,y)^\circ$ are algebraically independent in $\boldsymbol{K}_\nu^\times$.

(b) ι takes two values and (after an appropriate multiplicative shift) is constant on $\mathbb{P}^2(1,x,y) \setminus \{x\}$; in this case ψ_1 on $\mathbb{P}^2(1,x,y) \setminus \{x\}$ is a composition of a projection from x onto $\mathfrak{l}(1,y)$ and an embedding of $\mathfrak{l}(1,y) \hookrightarrow \boldsymbol{K}_\nu^\times/k^\times$ with the map $x \to \iota(x)$.

(c) ι takes two values and is constant on the complement of a projective line, say $\mathfrak{l}(x,y)$; then $\psi_1 \equiv 1$ on the complement to $\mathfrak{l}(x,y)$. Note that $1/x \cdot \mathfrak{l}(x,y) \subset \mathfrak{o}_\nu^\times$ and hence embeds into $\boldsymbol{K}_\nu^\times/k^\times$.

(d) ι takes three values.

The proof of Theorem 12 below relies on a reconstruction of these and similar flag structures from fan data, more precisely, it involves a construction of a special multiplicative subset $\mathfrak{o}^\times \subset K^\times/k^\times$ which will be equal to $\mathfrak{o}_\nu^\times/k^\times$, for some valuation $\nu \in \mathcal{V}_K$.

Theorem 12. *Assume that K, L are function fields over algebraic closures of finite fields k, l, respectively. Assume that*

(a)

$$\psi_1 : K^\times/k^\times \to L^\times/l^\times$$

is a homomorphism such that for any one-dimensional subfield $E \subset K$, there exists a one-dimensional subfield $F \subset L$ with

$$\psi_1(E^\times/k^\times) \subseteq F^\times/l^\times,$$

(b) $\psi_1(K^\times/k^\times)$ *contains at least two algebraically independent elements of* $L^\times/l^\times$.

If ψ_1 is not injective then

(1) *there is a $\nu \in \mathcal{V}_K$ such that ψ_1 is trivial on $(1+\mathfrak{m}_\nu)^\times/k^\times \subset \mathfrak{o}_\nu^\times/k^\times$;*

(2) *the restriction of ψ_1 to*

$$\boldsymbol{K}_\nu^\times/k^\times = \mathfrak{o}_\nu^\times/k^\times(1+\mathfrak{m}_\nu)^\times \to L^\times/l^\times \tag{4.2}$$

is injective and satisfies (a).

If ψ_1 is injective, then there exists a subfield $F \subset L$, a field isomorphism

$$\phi : K \xrightarrow{\sim} F \subset L,$$

and an integer $m \in \mathbb{Z}$, coprime to p, such that ψ_1 coincides with the homomorphism induced by ϕ^m.

The case of injective ψ_1 has been treated in [8]. The remainder of this section is devoted to the proof of Theorem 12 in the case when ψ_1 is not injective.

An immediate application of Condition (a) is: if $f \approx f'$, i.e., f, f' are algebraically dependent then $\psi_1(f) \approx \psi_1(f')$. The converse does not hold, in general, and we are lead to introduce the following decomposition:

$$\mathbb{P}_k(K) = S_1 \sqcup_f S_f, \tag{4.3}$$

where

$$S_1 := \psi_1^{-1}(1)$$

and, for $f \notin S_1$,

$$S_f := \{f' \in \mathbb{P}_k(K) \setminus S_1 \,|\, \psi_1(f') \approx \psi_1(f)\} \subset \mathbb{P}_k(K)$$

are equivalence classes of elements whose images are algebraically dependent. We record some properties of this decomposition:

Lemma 13.

(1) *For all f, the set $S_1 \sqcup S_f$ is closed under multiplication,*

$$f', f'' \in S_1 \sqcup S_f \Rightarrow f' \cdot f'' \in S_1 \sqcup S_f.$$

(2) *Every projective line $\mathfrak{l}(1, f) \subset \mathbb{P}_k(K)$, with $\psi_1(f) \neq 1$, is contained in $S_1 \sqcup S_f$.*

(3) *Assume that f, g are such that $\psi_1(f), \psi_1(g)$ are nonconstant and distinct. If $\psi_1(f) \approx \psi_1(g)$ then $\mathfrak{l}(f, g) \in S_1 \sqcup S_f$. Otherwise, $\mathfrak{l}(f, g) \cap S_1 = \emptyset$.*

(4) *Let $\Pi \subset \mathbb{P}_k(K)$ be a projective subspace such that there exist $x, y, z \in \Pi$ with distinct images and such that $\psi_1(x/z) \not\approx \psi_1(y/z)$.*
Then, for any $h \in K^\times/k^\times$, the projective subspace $\Pi' := h \cdot \Pi$ satisfies the same property.

Proof. The first property is evident. Condition (a) of Theorem 12 implies that for every $f \in K^\times/k^\times$ we have

$$\psi_1(\mathfrak{l}(1, f)) \subset \mathbb{P}_l(F), \quad F = l(\psi_1(f)), \tag{4.4}$$

this implies the second property.

Considering the shift $\mathfrak{l}(f, g) = f \cdot \mathfrak{l}(1, g/f)$ we see that

$$\psi_1(\mathfrak{l}(f, g) \subseteq \psi_1(f) \cdot \psi_1(\mathfrak{l}(1, g/f)) \subset \psi_1(f) \cdot \mathbb{P}_l(F),$$

where $F = l(\psi_1(g/f))$. If $\psi_1(f) \approx \psi_1(g/f)$ then $\psi_1(\mathfrak{l}(f, g)) \subset S_1 \sqcup S_f$. Otherwise, $\psi_1(\mathfrak{l}(f, g))$ is disjoint from 1.

To prove the last property is suffices to remark that $\psi_1(\Pi')$ contains $\psi_1(hx), \psi_1(hy), \psi_1(hz)$, and $\psi_1(hx/hz) \not\approx \psi_1(hy/hz)$. □

Lemma 14. *Let $\Pi = \mathbb{P}^2 \subset \mathbb{P}_k(K)$ be a projective plane satisfying condition (4) of Lemma 13 and such that the restriction of ψ_1 to Π is not injective. Then*

(1) *there exists a line $\mathfrak{l} \subset \Pi$ such that ψ_1 is constant on $\Pi \setminus \mathfrak{l}$ or*
(2) *there exists a $g \in \Pi$ such that*
   - *ψ_1 is constant on every punctured line $\mathfrak{l}(g, f) \setminus g$;*
   - *$\psi_1(g) \not\approx \psi_1(f)$, for every $f \neq g$;*
   - *$\psi_1(f) \approx \psi_1(f')$ for all $f, f' \notin g$.*

Proof. After an appropriate shift and relabeling, and using Lemma 13, we may assume that $\Pi = \mathbb{P}^2(1, x, y)$ and that $\psi_1^{-1}(1)$ contains a nontrivial element $z \in \mathbb{P}^2(1, x, y)$, i.e., $z \in S_1$. Let

$$\mathbb{P}^2(1, x, y) = S_1 \sqcup_{f \in \mathcal{F}} S_f$$

be the decomposition induced by (4.3).

Step 1. Neither of the sets S_1 nor S_f contains a line. Indeed, assume that there is a projective line $\mathfrak{l} \subseteq S_1$ and let $g \in \mathbb{P}^2(1, x, y) \setminus S_1$. Every $f' \in \mathbb{P}^2(1, x, y)$ lies on a line through g and $\mathfrak{l}(f', g)$ which intersects $\mathfrak{l}$, and thus S_1, i.e., all f' lie in $S_1 \sqcup S_g$, by Lemma 13. Assume that $\mathfrak{l} \subset S_f$. Every $g \in \mathbb{P}^2(1, x, y) \setminus S_1$ lies on a line of the form $\mathfrak{l}(1, g)$, which intersects $\mathfrak{l} \subseteq S_f$. It follows that $g \in S_f$, contradicting our assumption that $\psi_1(\mathbb{P}^2(1, x, y))$ contains at least two algebraically independent elements.

Step 2. Split $\mathcal{F} = \mathcal{F}' \sqcup \mathcal{F}''$ into nonempty subsets, arbitrarily, and let

$$\mathbb{P}^2(1, x, y) = S_1 \sqcup S' \sqcup S'', \quad S' := \sqcup_{f' \in \mathcal{F}'} S_{f'}, \quad S' := \sqcup_{f'' \in \mathcal{F}'} S_{f''}$$

be the induced decomposition. By Lemma 13, every $\mathfrak{l}$ is in either

$$S_1 \sqcup S', \ S_1 \sqcup S'', \ \text{ or } \ S' \sqcup S''.$$

By Lemma 8, one of these subsets is a flag subset of $\mathbb{P}^2(1, x, y)$.

Step 3. Assume that S_1 is a flag subset. Since it contains at least two elements and does not contain a projective line, by Step 1, we have:

- $S_1 = \mathbb{P}^2 \setminus \mathfrak{l}$, for some line $\mathfrak{l}$, and we are in Case (1), or
- $S_1 = \mathfrak{l}(1, z)^\circ = \mathfrak{l}(1, z) \setminus g$, for some $g \in S'$ (up to relabeling).
  Choose a $g'' \in S''$, so that $\psi_1(g'') \not\approx \psi_1(g)$, and let $\mathfrak{l}$ be a line through g'', $\mathfrak{l} \neq \mathfrak{l}(g, g'')$. Then $\mathfrak{l}$ intersects $\mathfrak{l}(1, z)^\circ = S_1$, which implies that the complement $\mathfrak{l} \setminus (\mathfrak{l} \cap \mathfrak{l}(1, z)^\circ) \subseteq S''$. It follows that S'' contains the complement of $\mathfrak{l}(1, z) \cap \mathfrak{l}(g, g'')$. Considering projective lines through 1 and elements in S'' and applying Lemma 13 we find that $S'' \supseteq \mathbb{P}^2(1, x, y) \setminus \mathfrak{l}(1, z)$. Since $g \notin S''$, we have equality. Thus all elements in $\mathbb{P}^2(1, x, y) \setminus g$ have algebraically dependent images. If ψ_1 were not constant on a line $\mathfrak{l}$ through g, with $\mathfrak{l} \neq \mathfrak{l}(1, z)$, let $f_1, f_2 \in \mathfrak{l}$ be elements with $\psi_1(f_1) \neq \psi_1(f_2)$. Lemma 13 implies that $g \in S_{f_1}$, contradicting our assumption that $\psi_1(g) \not\approx \psi_1(f_1)$. Thus we are in Case (2).

Step 4. Assume that S' is a flag subset and S_1 is not. We have the following cases:

- $S' = \{g\}$. Then $S' = S_g$ and $\mathfrak{l}(1, g)^\circ := \mathfrak{l}(1, g) \setminus g \subseteq S_1$. Assume that there exist $f_1, f_2 \in \mathbb{P}^2(1, x, y) \setminus \mathfrak{l}(1, g)$ with $\psi_1(f_1) \neq \psi_1(f_2)$. If $f_2 \notin \mathfrak{l}(g, f_1)$ then $\psi_1(f_1) \approx \psi_1(f_2)$, as $\mathfrak{l}(f_1, f_2)$ intersects $\mathfrak{l}(1, g)^\circ$. If there exists at least one $f_2 \notin \mathfrak{l}(g, f_1)$ with nonconstant $\psi_1(f_2)$, then by the argument above, all elements on the complement to

$\mathfrak{l}(1,g)$ are algebraically dependent, on every line $\mathfrak{l}$ through g, ψ_1 is constant on $\mathfrak{l}\setminus g$, and we are in Case (2). If ψ_1 is identically 1 on $\mathbb{P}^2(1,x,y)\setminus\mathfrak{l}(g,f_1)$ then $\psi_1\neq 1$ on $\mathfrak{l}(g,f_1)$ (otherwise S_1 would contain a projective line). Thus S_1 is a flag subset, contradiction.

- $S'=\mathfrak{l}^\circ=\mathfrak{l}\setminus g$, for some line $\mathfrak{l}$ and $g\in\mathfrak{l}$. Since S_1 has at least two elements, there is a $z'\in(\mathbb{P}^2\setminus\mathfrak{l})\cap S_1$. Lemma 13 implies that ψ_1 equals 1 on $\mathfrak{l}(z',g')\setminus g'$, for all $g'\in S'=\mathfrak{l}^\circ$. Similarly, ψ_1 equals 1 on $\mathfrak{l}(g,z')^\circ:=\mathfrak{l}(g,z')\setminus g$, as every point on this punctured line lies on a line passing through S_1 and intersecting S'. Thus $S_1\supseteq\mathbb{P}^2(1,x,y)\setminus\mathfrak{l}$ and since S_1 does not contain a line, these sets must be equal. It follows that we are in Case (1).
- Assume that $S'=\mathbb{P}^2(1,x,y)\setminus\mathfrak{l}$ and we are not in the previous case. Then $\mathfrak{l}$ contains at least two points in S_1 and the complement $S''=\mathfrak{l}\setminus(\mathfrak{l}\cap S_1)$ also has at least two points, f_1'',f_2''. Thus $S''=S_{f''}$ for some f'', by Lemma 13. Since we were choosing the splitting $\mathcal{F}=\mathcal{F}'\sqcup\mathcal{F}''$ arbitrarily, we conclude that $S'=S_{f'}$ for some f'.

  The same argument as in Step 3. implies that ψ_1 is constant on $\mathfrak{l}(g',f_i'')\setminus f_i''$ for any $g'\in S'$. Hence ψ_1 is constant on $\mathbb{P}^2(1,x,y)\setminus\mathfrak{l}$ and we are in Case (1).

□

Lemma 15. *Let*

$$\mathfrak{u}:=\cup\,\mathfrak{l}(1,x)\subseteq K^\times/k^\times=\mathbb{P}_k(K)$$

be union over all lines such that ψ_1 is injective on $\mathfrak{l}(1,x)$. Assume that there exist nonconstant $x,y\in\mathfrak{u}$ such that $\psi_1(x)\not\approx\psi_1(y)$. Then

$$\mathfrak{o}^\times:=\mathfrak{u}\cdot\mathfrak{u}\subset\mathbb{P}_k(K)$$

is a multiplicative subset.

Proof. First of all, if $x\in\mathfrak{u}$ then $x^{-1}\in\mathfrak{u}$, since $\mathfrak{l}(1,x)=x\cdot\mathfrak{l}(1,x^{-1})$.

It suffices to show that $\mathfrak{u}\cdot\mathfrak{o}^\times=\mathfrak{o}^\times$, i.e., for all $x,y,z\in\mathfrak{u}$ one has $xyz=tw$, for some $t,w\in\mathfrak{u}$.

Assume first that $\psi_1(x)\not\approx\psi_1(y)$ and consider $\mathbb{P}^2(1,x,y^{-1})$. A shift of this plane contains the line $\mathfrak{l}(1,xy)$. If $xy\notin\mathfrak{u}$ then ψ_1 is not injective on this plane and we may apply Lemma 14. We are not in Case (1) and not in Case (2), since ψ_1 injects $\mathfrak{l}(1,x)$ and $\mathfrak{l}(1,y^{-1})$, contradiction. Thus ψ_1 is injective on $\mathbb{P}^2(1,x,y^{-1})$ and $xy=t$, for some $t\in\mathfrak{u}$, which proves the claim.

Now assume that $\psi_1(x), \psi_1(y), \psi_1(z) \in F^\times/l^\times$, for some 1-dimensional $F \subset L$. By assumption, there exists a $w \in K^\times/k^\times$ such that $\psi_1(w)$ is algebraically independent of $F^\times/l^\times$ and such that $\mathfrak{l}(1,w)$ injects. By the previous argument, ψ_1 is injective on the lines $\mathfrak{l}(1,xw)$ and $\mathfrak{l}(1,w^{-1}y)$, so that $xw, w^{-1}y \in \mathfrak{u}$. By our assumptions, ψ_1 is injective on $\mathfrak{l}(1,z)$. Now we repeat the previous argument for xw and z: there is a $t \in \mathfrak{u}$ such that $xw \cdot z = t$. □

Proposition 16. *Let*

$$\psi_1 : K^\times/k^\times \to L^\times/l^\times$$

be a homomorphism satisfying the assumptions of Theorem 12 and such that $\mathfrak{u} \subset \mathbb{P}_k(K)$ *contains* x, y *with* $\psi_1(x) \not\approx \psi_1(y)$. *Let* $\mathfrak{o}^\times = \mathfrak{u}\cdot\mathfrak{u} \subset K^\times/k^\times$ *be the subset defined above. Then*

$$\nu : K^\times/k^\times \to K^\times/\mathfrak{o}^\times$$

is a valuation homomorphism, i.e., there exists an ordered abelian group Γ_ν *with* $K^\times/\mathfrak{o}^\times \simeq \Gamma_\nu$ *and* ν *is the valuation map.*

Proof. By Lemma 15, we know that $\mathfrak{o}^\times$ is multiplicative. We claim that the restriction of ν to every $\mathbb{P}^1 \subset \mathbb{P}_k(K)$ is a flag map. Indeed, we have $\mathbb{P}^1 = x \cdot \mathfrak{l}(1,y)$, for some $x, y \in K^\times/k^\times$. On every line $\mathfrak{l}(1,y) \subset \mathfrak{u}$, the value of ν is 1, and on every line $\mathfrak{l}(1,x) \not\subset \mathfrak{u}$, the value of ν is constant on the complement to one point.

By Proposition 5, this implies that ν is a flag map, i.e., defines a valuation on $K^\times/k^\times$ with values on $\Gamma_\nu := K^\times/\mathfrak{o}^\times$. □

To complete the proof of Theorem 12 we need to treat homomorphisms ψ_1 such that for most one-dimensional $E \subset K$, the image $\psi_1(E^\times/k^\times)$ is cyclic, or even trivial. We have two cases:

(A) there exists a line $\mathfrak{l} = (1,x)$ such that ψ_1 is not a flag map on this line.

(B) ψ_1 is a flag map on *every line* $\mathfrak{l} \subset \mathbb{P}_k(K)$.

In Case (A), let $\mathfrak{u}'$ be the union of lines $\mathfrak{l}(1,x)$ such that ψ_1 is not a flag map on $\mathfrak{l}(1,x)$, i.e., is not constant on the line minus a point. By the arguments in Lemma 14, Lemma 15, and Proposition 16, there exists a unique 1-dimensional normally closed subfield $F \subset L$ such that $\psi_1(\mathfrak{u}') \subseteq F^\times/l^\times$. Define

$$\mathfrak{o}^\times := \psi_1^{-1}(F^\times/l^\times) \subset K^\times/k^\times.$$

By assumption on ψ_1, $\mathfrak{o}^\times$ is a proper multiplicative subset of $K^\times/k^\times$. The induced homomorphism

$$\nu : K^\times/k^\times \to (K^\times/k^\times)/\mathfrak{o}^\times =: \Gamma_\nu$$

is a valuation map, since it is a flag map on every line $\mathfrak{l} \subset \mathbb{P}_k(K)$. Indeed, it is constant on all $\mathfrak{l}(1,x) \subseteq \mathfrak{u}$ and a valuation map on all $\mathfrak{l}(1,y) \not\subset \mathfrak{o}^\times$. Hence the same holds for any projective line in $\mathbb{P}_k(K)$ which implies the result.

In Case (B), we conclude that ψ_1 is a flag map on $\mathbb{P}_k(K)$ (see, e.g., Lemma 4.16 of [7]), i.e., there exists a valuation $\nu \in \mathcal{V}_K$ with value group $\Gamma_\nu = \psi_1(K^\times/k^\times)$ such that the valuation homomorphism $\nu = \psi_1$.

Remark 17. The main steps of the proof (Lemmas 13, 14, and 15) are valid for more general fields: the splitting of $\mathbb{P}^2$ into subsets satisfying conditions (1) and (2) of Lemma 13 is related to a valuation, independently of the ground field (see [6]). Here we restricted to $k = \bar{\mathbb{F}}_p$ since in this case the proof avoids some technical details which appear in the theory of general valuations. Furthermore, the condition that K, L are function fields is also not essential. The only essential property is the absence of an infinite tower of roots for the elements of K, L which are not contained in the ground field.

5. Galois cohomology

By duality, the main result of Section 4 confirms the general concept that birational properties of algebraic varieties are functorially encoded in the structure of the Galois group G^c_K. On the other hand, it follows from the proof of the Bloch–Kato conjecture that G^c_K determines the full cohomology of G_K. Here and in Section 5 we discuss group-theoretic properties of G_K and its Sylow subgroups which we believe are ultimately responsible for the validity of the Bloch–Kato conjecture.

Let G be a profinite group, acting continuously on a topological G-module M, and let $\mathrm{H}^i(G,M)$ be the (continuous) i-cohomology group. These groups are contravariant with respect to G and covariant with respect to M; in most of our applications M either $\mathbb{Z}/\ell$ or $\mathbb{Q}/\mathbb{Z}$, with trivial G-action. We recall some basic properties:

- $\mathrm{H}^0(G,M) = M^G$, the submodule of G-invariants;
- $\mathrm{H}^1(G,M) = \mathrm{Hom}(G,M)$, provided M has trivial G-action;

- $\mathrm{H}^2(G, M)$ classifies extensions
$$1 \to M \to \tilde{G} \to G \to 1,$$
up to homotopy.
- if G is abelian and M finite with trivial G-action then
$$\mathrm{H}^n(G, M) = \wedge^n(\mathrm{H}^1(G, M)), \quad \text{for all} \quad n \geq 1,$$
- if $M = \mathbb{Z}/\ell^m$ then
$$\mathrm{H}^n(G, M) \hookrightarrow \mathrm{H}^n(\mathfrak{G}_\ell, M), \quad \text{for all} \quad n \geq 0,$$
where $\mathfrak{G}_\ell$ is the ℓ-Sylow subgroup of G.

See [1] for further background on group cohomology and [28], [23] for background on Galois cohomology. Let
$$G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$$
the n-th term of its *derived* series, $G^{(1)} = [G, G]$. We will write
$$G^a = G/[G, G], \quad \text{and} \quad G^c = G/[[G, G], G]$$
for the abelianization, respectively, the second *lower central series* quotient of G. Consider the diagram, connecting the first terms in the derived series of G with those in the lower central series:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^{(1)} & \longrightarrow & G & \longrightarrow & G^a & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a & \longrightarrow & 1 \end{array}$$

We have a homomorphism between E_2-terms of the spectral sequences computing $\mathrm{H}^n(G, \mathbb{Z}/\ell^m)$ and $\mathrm{H}^n(G^c, \mathbb{Z}/\ell^m)$, respectively. Suppressing the coefficients, we have

$$\begin{array}{ccccccc} \mathrm{H}^p(G^a, \mathrm{H}^q(Z)) & = & \mathrm{E}_2^{p,q}(G^c) & \Longrightarrow & \mathrm{E}^{p+q}(G^c) & = & \mathrm{H}^{p+q}(G^c) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^p(G^a, \mathrm{H}^q(G^{(1)})) & = & \mathrm{E}_2^{p,q}(G) & \Longrightarrow & \mathrm{E}^{p+q}(G) & = & \mathrm{H}^{p+q}(G) \end{array}$$

We have $\mathrm{H}^0(G, \mathbb{Z}/\ell^m) = \mathbb{Z}/\ell^m$, for all $m \in \mathbb{N}$, and
$$\mathrm{H}^0(G^a, \mathrm{H}^1(Z)) = \mathrm{H}^1(Z).$$
The diagram of corresponding five term exact sequences takes the form:

$$\begin{array}{ccccccc}
\mathrm{H}^1(G^a) = \mathrm{H}^1(G^c) & \longrightarrow & \mathrm{H}^1(Z) & \xrightarrow{d_2} & \mathrm{H}^2(G^a) & \longrightarrow & \mathrm{H}^2(G^c) \\
 & & \simeq\downarrow & & \| & & \downarrow \\
\mathrm{H}^1(G^a) = \mathrm{H}^1(G) & \longrightarrow & \mathrm{H}^0(G^a, \mathrm{H}^1(G^{(1)})) & \xrightarrow{d_2'} & \mathrm{H}^2(G^a) & \longrightarrow & \mathrm{H}^2(G)
\end{array}$$

where the left arrows map $\mathrm{H}^1(G) = \mathrm{H}^1(G^a)$ to zero and the isomorphism in the middle follows from the description of G^c as a maximal quotient of G that is a central extension of G^a.

Let K be *any* field containing ℓ^m-th roots of 1, for all $m \in \mathbb{N}$, and let G_K be its absolute Galois group. We apply the cohomological considerations above to G_K. By *Kummer theory*,

$$\mathrm{H}^1(G_K, \mathbb{Z}/\ell^m) = \mathrm{H}^1(G_K^a, \mathbb{Z}/\ell^m) = \mathrm{K}_1(K)/\ell^m \tag{5.1}$$

and we obtain a diagram

$$\begin{array}{ccc}
\mathrm{H}^1(G_K^a, \mathbb{Z}/\ell^m) \otimes \mathrm{H}^1(G_K^a, \mathbb{Z}/\ell^m) & \xrightarrow{\mathfrak{s}_K} & \wedge^2(\mathrm{H}^1(G_K^a, \mathbb{Z}/\ell^m)) = \mathrm{H}^2(G_K^a, \mathbb{Z}/\ell^m) \\
\| & & \\
\mathrm{K}_1(K)/\ell^m \otimes \mathrm{K}_1(K)/\ell^m & \xrightarrow{\sigma_K} & \mathrm{K}_2(K)/\ell^m
\end{array}$$

where $\mathfrak{s}_K$ is the skew-symmetrization homomorphism (cup product) and σ_K is the symbol map, and $\mathrm{Ker}(\sigma_K)$ is generated by symbols of the form $f \otimes (1-f)$. The Steinberg relations imply that

$$\mathrm{Ker}(\mathfrak{s}) \subseteq \mathrm{Ker}(\sigma_K)$$

(see, e.g., [18, Section 11]). We obtain a diagram

$$\begin{array}{ccccc}
\mathrm{H}^1(Z_K, \mathbb{Z}/\ell^m) & \stackrel{d_2}{\hookrightarrow} & \mathrm{H}^2(G_K^a, \mathbb{Z}/\ell^m) & \xrightarrow{\pi_a^*} & \mathrm{H}^2(G_K^c, \mathbb{Z}/\ell^m) \\
 & & \| & & \uparrow h_K \\
I_K(2)/\ell^m & \hookrightarrow & \wedge^2(\mathrm{H}^1(G_K^a, \mathbb{Z}/\ell^m)) & \twoheadrightarrow & \mathrm{K}_2(K)/\ell^m
\end{array}$$

where h_K is the *Galois symbol* (cf. [23, Theorem 6.4.2]) and $I_K(2)$ is defined by the exact sequence

$$1 \to I_K(2) \to \wedge^2(K^\times) \to \mathrm{K}_2(K) \to 1.$$

A theorem of Merkurjev–Suslin [17] states that h_K is an isomorphism

$$\mathrm{H}^2(G_K, \mathbb{Z}/\ell^m) = \mathrm{K}_2(K)/\ell^m. \tag{5.2}$$

This is equivalent to:

- $\pi_a^* : \mathrm{H}^2(G_K^a, \mathbb{Z}/\ell^m) \to \mathrm{H}^2(G_K)$ is surjective and
- $\mathrm{H}^1(Z_K, \mathbb{Z}/\ell^m) = I_K(2)/\ell^m$.

The Bloch–Kato conjecture, proved by Voevodsky, Rost, and Weibel, generalizes (5.1) and (5.2) to all n. This theorem is of enormous general interest, with far-reaching applications to algebraic and arithmetic geometry. It states that for any field K and any prime ℓ, one has an isomorphism between Galois cohomology and the mod ℓ Milnor K-theory:

$$\mathrm{H}^n(G_K, \mu_\ell^{\otimes n}) = \mathrm{K}_n^M(K)/\ell. \tag{5.3}$$

It substantially advanced our understanding of relations between fields and their Galois groups, in particular, their Galois cohomology. Below we will focus on Galois-theoretic consequences of (5.3).

We have canonical central extensions

$$\begin{array}{ccccccccc}
 & & & & G_K & & & & \\
 & & & & \downarrow & \searrow & & & \\
1 & \longrightarrow & Z_K & \longrightarrow & G_K^c & \longrightarrow & G_K^a & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \xrightarrow{\pi_a^c} & \mathcal{G}_K^a & \longrightarrow & 1
\end{array}$$

and the diagram

$$\begin{array}{ccc}
 & G_K & \\
{}^{\pi_c}\swarrow & & \searrow^{\pi_a} \\
\mathcal{G}_K^c & \xrightarrow[\pi_a^c]{} & \mathcal{G}_K^a.
\end{array}$$

The following theorem relates the Bloch–Kato conjecture to statements in Galois-cohomology (see also [12], [11], [27]).

Theorem 18. [10, Theorem 11] *Let $k = \bar{\mathbb{F}}_p$, $p \neq \ell$, and $K = k(X)$ be the function field of an algebraic variety of dimension ≥ 2. The Bloch–Kato conjecture for K is equivalent to:*

(1) *The map*

$$\pi_a^* \colon \mathrm{H}^*(\mathcal{G}_K^a, \mathbb{Z}/\ell^n) \to \mathrm{H}^*(G_K, \mathbb{Z}/\ell^n)$$

is surjective and

(2) $\mathrm{Ker}(\pi_a^c \circ \pi_c)^* = \mathrm{Ker}(\pi_a^*)$.

This implies that the Galois cohomology of the pro-ℓ- quotient $\mathcal{G}_K$ of the absolute Galois group G_K encodes important birational information of X. For example, in the case above, $\mathcal{G}_K^c$, and hence K, modulo purely-inseparable extensions, can be recovered from the cup-products

$$\mathrm{H}^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \times \mathrm{H}^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \to \mathrm{H}^2(\mathcal{G}_K, \mathbb{Z}/\ell^n), \quad n \in \mathbb{N}.$$

6. Freeness

Let K be a function field over an arbitrary ground field k and G_K the absolute Galois group of K. The pro-ℓ-quotient $\mathcal{G}_K$ of K is highly individual: for $k = \bar{\mathbb{F}}_p$ it determines K up to purely-inseparable extensions. On the other hand, let $\mathfrak{G}_\ell(G_K)$ be an ℓ-Sylow subgroup of G_K. This group is *universal*, in the following sense:

Proposition 19. [5] *Assume that X has dimension n and that X contains a smooth k-rational point. Then*

$$\mathfrak{G}_\ell(G_K) = \mathfrak{G}_\ell(G_{k(\mathbb{P}^n)}).$$

In particular, when k is algebraically closed, the ℓ-Sylow subgroups depend only on the dimension of X. This universal group will be denoted by $\mathfrak{G}_\ell$. The following *Freeness* conjecture captures an aspect of this universality. It implies the (proved) Bloch–Kato conjecture; but more importantly, it provides a structural explanation for its truth.

Conjecture 20 (Bogomolov)**.** Let k be an algebraically closed field of characteristic $\neq \ell$, X an algebraic variety over k of dimension ≥ 2, $K = k(X)$, and write

$$\mathfrak{G}_\ell^{(1)} := [\mathfrak{G}_\ell, \mathfrak{G}_\ell]$$

for the commutator of an ℓ-Sylow subgroup of G_K. Then

$$\mathrm{H}^i(\mathfrak{G}_\ell^{(1)}, \mathbb{Z}/\ell^m) = 0, \quad \text{for all} \quad i \geq 2, m \in \mathbb{N}. \tag{6.1}$$

Remark 21. For profinite ℓ-groups, the vanishing in Equation (6.1) for $i = 2$ implies the vanishing for *all* $i \geq 2$ (see [16]).

We now return to the cohomological considerations in Section 5. The standard spectral sequence associated with

$$1 \to G^{(1)} \to G \to G^a \to 1,$$

gives

$$\mathrm{H}^p(G^a, \mathrm{H}^q(G^{(1)}, \mathbb{Z}/\ell^m)) \Rightarrow \mathrm{H}^n(G, \mathbb{Z}/\ell^m).$$

We apply this to $G = \mathfrak{G}_\ell$; suppressing the coefficients we obtain:

$$\begin{array}{cccc}
0 & 0 & 0 & \cdots \\
\mathrm{H}^0(G^a, \mathrm{H}^2(G^{(1)})) & 0 & 0 & \cdots \\
\mathrm{H}^0(G^a, \mathrm{H}^1(G^{(1)})) & \mathrm{H}^1(G^a, \mathrm{H}^1(G^{(1)})) & \mathrm{H}^2(G^a, \mathrm{H}^1(G^{(1)})) & \cdots \\
 & \searrow^{d_2} & & \\
\mathrm{H}^0(G^a, \mathrm{H}^0(G^{(1)})) & \mathrm{H}^1(G^a, \mathrm{H}^0(G^{(1)})) & \mathrm{H}^2(G^a, \mathrm{H}^0(G^{(1)})) & \cdots
\end{array}$$

Conjecture 20 would imply that, for $G = \mathfrak{G}_\ell$, and also for $G = G_K$, $\mathrm{H}^2(G^{(1)})$ and consequently *all* entries above the second line vanish. In this case, we have a long exact sequence (see, e.g., [23, Lemma 2.1.3])

$$0 \to \mathrm{H}^1(G^a) \to \mathrm{H}^1(G) \to \mathrm{H}^0(G^a, \mathrm{H}^1(G^{(1)})) \to \mathrm{H}^2(G^a) \to \cdots$$

$$\cdots \to \mathrm{H}^n(G^a, \mathrm{H}^1(G^{(1)})) \xrightarrow{d_2} \mathrm{H}^{n+2}(G^a) \to \mathrm{H}^{n+2}(G) \to \cdots$$

In Section 5 we saw that for $n = 0$ the homomorphism d_2' in the sequence

$$\mathrm{H}^n(G_K^a, \mathrm{H}^1(G_K^{(1)})) \xrightarrow{d_2'} \mathrm{H}^{n+2}(G_K^a) \to \mathrm{H}^{n+2}(G_K) = \mathrm{K}_{n+2}(K)/\ell^m$$

can be interpreted as the embedding of the *skew-symmetric* relations:

$$\mathrm{H}^0(G^a, \mathrm{H}^1(Z)) = I_K(2)/\ell^m \xrightarrow{d_2'} \wedge^2(\mathrm{H}^1(G_K^a)) = \wedge^2(K^\times/\ell^m) \to \mathrm{K}_2(K)/\ell^m.$$

This relied on Kummer theory and the Merkurjev–Suslin theorem. We proceed to interpret the differential d_2 for higher n.

We work with $G = \mathfrak{G}_\ell$, an ℓ-Sylow subgroup of the absolute Galois group of a function field K over an algebraically closed field. We have an exact sequence of continuous G^a-modules

$$1 \to [G, G^{(1)}] \to G^{(1)}/G^{(2)} \to Z \to 1.$$

Note that G^a acts trivially on Z, and $\mathrm{H}^1(Z)$, and via $x \mapsto gxg^{-1} - x$ on $G^{(1)}/G^{(2)}$. Dually we have a sequence of G^a-modules:

$$\mathrm{Hom}([G, G^{(1)}], \mathbb{Z}/\ell^m) \leftarrow \mathrm{Hom}(G^{(1)}/G^{(2)}, \mathbb{Z}/\ell^m) \leftarrow \mathrm{Hom}(Z, \mathbb{Z}/\ell^m) \leftarrow 1.$$

Define

$$M := \mathrm{Hom}(G^{(1)}/G^{(2)}, \mathbb{Z}/\ell^m),$$

then

$$M^{G^a} = \mathrm{Hom}(Z, \mathbb{Z}/\ell^m) = \mathrm{H}^1(Z) = I_K(2)/\ell^m.$$

We have a homomorphism

$$\mathrm{H}^n(M^{G^a}) \to \mathrm{H}^n(M) \tag{6.2}$$

via the natural embedding. Since $\mathrm{H}^0(M^{G^a})$ embeds into $\mathrm{H}^2(G^a)$ via d_2 we obtain a natural homomorphism

$$t_n : \mathrm{H}^n(M^{G^a}) \to \mathrm{H}^{n+2}(G^a)$$

and the differential d_2 on the image of $\mathrm{H}^n(M^{G^a}) \to \mathrm{H}^n(M)$ coincides with t_n. Thus the fact that the kernel of $\mathrm{H}^n(G^a) \to \mathrm{H}^n(G)$ is generated by trivial symbols will follow from the surjectivity of the homomorphism in (6.2). This, in turn, would follow if the projection $M \to M/M^{G^a}$ defined a trivial map on cohomology. Thus we can formulate the following conjecture which complements the Freeness conjecture 20:

Conjecture 22. The projection $M \to M/M^{G^a}$ can be factored as

$$M \hookrightarrow D \twoheadrightarrow M/M^{G^a},$$

where D is a cohomologically trivial G^a-module.

We hope that a construction of a natural module D can be achived via algebraic geometry.

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Courant Institute of Mathematical Sciences, N.Y.U., 251 Mercer str., New York, NY 10012, U.S.A.
E-mail address: bogomolo@cims.nyu.edu

Courant Institute of Mathematical Sciences, N.Y.U., 251 Mercer str., New York, NY 10012, U.S.A.
E-mail address: tschinkel@cims.nyu.edu