MULTIPLE MIXING FOR ADELE GROUPS AND RATIONAL POINTS

ALEXANDER GORODNIK, RAMIN TAKLOO-BIGHASH, AND YURI TSCHINKEL

Abstract. We prove an asymptotic formula for the number of rational points of bounded height on projective equivariant compactifications of $H\backslash G$, where H is a connected simple algebraic group embedded diagonally into $G := H^n$.

Introduction

Let $X \subset \mathbf{P}^n$ be a smooth projective variety over a number field F. Fix a height function

$$\mathsf{H}\colon \mathbf{P}^n(F) \to \mathbb{R}_{>0} \tag{1}$$

and consider the counting function

$$\mathsf{N}(X, T) := \{x \in X(F) \,|\, \mathsf{H}(x) \le T\}.$$

Manin's conjecture [8] and its refinements by Batyrev–Manin [1], Peyre [17], and Batyrev–Tschinkel [3] predict precise asymptotic formulas for $\mathsf{N}(X^\circ, T)$ as $T \to \infty$, where $X^\circ \subset X$ is an appropriate Zariski open subset of an algebraic variety with sufficiently positive anticanonical class. These formulas involve geometric invariants of X:

- the Picard group $\mathrm{Pic}(X)$ of X;
- the anticanonical class $-K_X \in \mathrm{Pic}(X)$;
- the cone of pseudo-effective divisors $\Lambda_{\mathrm{eff}}(X)_{\mathbb{R}} \subset \mathrm{Pic}(X)_{\mathbb{R}}$,

and they depend on an *adelic metrization* $\mathcal{L} = (L, \|\cdot\|_v)$ of the polarization L giving rise to the embedding $X \subset \mathbf{P}^n$, i.e., on a choice of the height function in (1). Given these, one introduces the invariants:

$$a(L), b(L), \text{ and } c(\mathcal{L})$$

so that the number of F-rational points on X° of $\mathcal{L}$-height bounded by T is, conjecturally, given by

$$\mathsf{N}(X^\circ, \mathcal{L}, T) = \frac{c(\mathcal{L})}{a(L)(b(L)-1)!} T^{a(L)} \log(T)^{b(L)-1}(1 + o(1)), \quad T \to \infty, \tag{2}$$

see, e.g., Definitions 2.2.4, 2.3.11, and Section 3 of [3], for precise definitions of the constants.

Key words and phrases. Rational points, heights, mixing, counting.

These conjectures have stimulated intense research; see [20], [16], [4], [5] for surveys of the current state of this subject. Of particular importance are equivariant compactifications of algebraic groups and their homogeneous spaces. In all equivariant cases considered previously, it was essential that X admits an action, with a dense orbit, of a solvable algebraic group. For example, the paper [19] proves Manin's conjecture for equivariant compactifications of the symmetric space $G\backslash(G \times G)$, a spherical variety. In this paper, we establish these conjectures for a new class of varieties, which includes *nonspherical* varieties, under a technical assumption on automorphic characters (Assumption 5.1):

Theorem 1. *Let H be a connected simple algebraic group defined over a number field F, $G := H^n$ its n-fold product. Let X be a smooth projective G-equivariant compactification of $X^\circ := H\backslash G$, where H acts on the left diagonally, with boundary $X \setminus X^\circ$ a divisor with strict normal crossings. Assume 5.1. Then X satisfies Manin's conjecture and its refinements, i.e.,* (2) *holds for $L = -K_X$.*

This generalizes the case $n = 2$ treated in [19] and [10] to arbitrary n. The proof presented here also works, with minor modifications, for semi-simple groups H. Compactifications of the homogeneous space $H\backslash H^n$ have played an important role in work of L. Lafforgue on the Langlands' conjecture over function fields of curves over finite fields (see, e.g., Chapter 3 in [13]). The geometry of these compactifications is surprisingly rich.

Our proof combines ergodic-theoretic methods developed in [11] with geometric integration techniques developed in [6] and [7]; in particular, it uses neither the theory of height zeta functions nor spectral theory on adelic spaces. On the other hand, it does not allow to establish effective error terms as in the $n = 2$ case in [19].

Organization of the paper. In Sections 1 and 2 we discuss geometric and analytic background and, in particular, establish meromorphic continuation of Igusa-type integrals (Theorem 2.3) that implies an asymptotic formula for volumes of height balls. In Section 3, we give a classification of intermediate subgroups M with $H \subset M \subset H^n$. This result is used in Section 4 where we establish the multiple mixing property for the adelic spaces using measure-rigidity techniques. Finally, our main result is deduced from multiple mixing in Section 5.

Acknowledgments. We are grateful to Antoine Chambert-Loir, Amos Nevo, and Brendan Hassett for useful comments and suggestions. We are especially grateful to the referee who carefully read the paper and made a number of valuable suggestions; they are incorporated below. The first author was supported by EPSRC, ERC, and RCUK. The second authors was supported by NSF grant 0701753. The third author was partially supported by NSF grants DMS-0739380, 0901777, and 1160859.

1. Geometric background

Let F be an algebraically closed field of characteristic zero, G a connected semi-simple algebraic group defined over F and $H \subset G$ a connected closed subgroup. Let X be a projective equivariant compactification of $X^\circ := H\backslash G$. Throughout, we assume that X is smooth and that the boundary

$$\cup_{\alpha\in\mathcal{A}} D_\alpha = X\backslash X^\circ$$

is a divisor with normal crossings, with irreducible components D_α. We will identify line bundles and divisors with their classes in the Picard group $\mathrm{Pic}(X)$. If H is a parabolic subgroup, then there is no boundary, i.e., $\mathcal{A}$ is empty, and $H\backslash G$ is a generalized flag variety. Distribution of rational points of bounded height on flag varieties was studied in [8]. We will assume that

- $\mathcal{A}$ is not empty,
- X° is affine (this holds, e.g., when H is reductive),
- the groups of algebraic characters of G and H are trivial.

Recall that a 1-parameter subgroup of G is a homomorphism $\xi : \mathbb{G}_m \to G$.

Proposition 1.1. *Let G be a connected reductive group, $H \subset G$ a closed connected reductive subgroup, and X a smooth projective G-equivariant compactification of $X^\circ = H\backslash G$. Assume that G and H have no nontrivial algebraic characters. Then*

(1) the classes of irreducible boundary components D_α span the Picard group $\mathrm{Pic}(X)_{\mathbb{Q}}$ and the pseudo-effective cone $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$;

(2) the class of the anticanonical line bundle is given by

$$-K_X = \sum_{\alpha\in\mathcal{A}} \kappa_\alpha D_\alpha,$$

where all $\kappa_\alpha \geq 1$.

Proof. Fix a polarization L of X and let $X \subset \mathbf{P}^n$ be the corresponding projective embedding. After taking a suitable multiple, we may assume that L is G-linearized, i.e., the action of G on X extends to an action on the ambient $\mathbf{P}^n$ (by [14, Corollary 1.6]). Let D be an effective divisor such that the generic point of D is in $H\backslash G$. There exists a 1-parameter subgroup moving the generic point of D. After specializing D, at least one of the irreducible components of the limit is supported in the boundary. We can now apply induction on the L-degree of the remaining components, if any, to conclude that D is equivalent to an effective divisor with support in the boundary.

On the other hand, the only invertible functions on $H\backslash G$ are constants, by assumption and Rosenlicht's theorem. It follows that there are no relations between classes of the boundary components.

For the second claim, see, e.g., [12, Section 6]. □

Let L be a big line bundle on X. We define

$$a(L) := \inf\{t \in \mathbb{Q} : t[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\},$$

$$b(L) := \text{ the maximal codimension of the face containing } a(L)L + K_X.$$

These invariants depend on the chosen compactification X. By Proposition 1.1, we have

$$L = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha D_\alpha, \quad \lambda_\alpha \in \mathbb{Q}_{>0},$$

so that the corresponding invariants are given by

$$a(L) := \max_\alpha \frac{\kappa_\alpha}{\lambda_\alpha} \tag{3}$$

and

$$b(L) := \#\{\alpha \in \mathcal{A} \,|\, a(L) = \frac{\kappa_\alpha}{\lambda_\alpha}\}. \tag{4}$$

Remark 1.2. The invariants $a(L)$ and $b(L)$ may be computed even if X is not smooth. Consider an equivariant resolution of singularities $\tilde{X} \to X$, and let $\tilde{L}$ be the pullback of L to $\tilde{X}$. Put

$$a(L) := a(\tilde{L}), \qquad b(L) := b(\tilde{L}).$$

A basic result is that this does not depend on the chosen resolution (see, e.g., [12, Section 2]).

The following proposition, used in the proof of Theorem 1 in conjunction with results in Section 5, has been established in [12]:

Proposition 1.3. *Let $M \subsetneq G$ be a closed connected subgroup containing H and let Y be the closure of $H\backslash M$ in X. Then*

$$(a(-K_X|_Y), b(-K_X|_Y)) < (a(-K_X), b(-K_X)),$$

in the lexicographic ordering.

Remark 1.4. This fails in the non-equivariant context, see [2] for a counterexample and [3] for a discussion of this "saturation" phenomenon.

2. Heights and height integrals

Let F be a number field, $\mathbb{A}$ its ring of adeles, and $\mathbb{A}_f$ the subring of finite adeles. Let v be a place of F and F_v the corresponding completion; for nonarchimedian v we let $\mathfrak{o}_v$ denote the ring of v-integers and $\mathfrak{m}_v$ its maximal ideal.

Let X be a projective variety over F, $U \subset X$ a Zariski open subset with boundary

$$\cup_{\alpha\in\mathcal{A}} D_\alpha = X \setminus U$$

being a normal crossings divisor. Here D_α are F-irreducible components, which could be reducible over an algebraic closure $\bar{F}$ of F. For each α one can endow the line bundle $\mathcal{O}(D_\alpha)$ with an adelic metric which allows to define local and global heights (see, e.g., Section 2.3 of [7]):

$$\mathsf{H}_{D_{\alpha,v}} : U(F_v) \to \mathbb{R}_{>0}, \qquad \mathsf{H}_{D_\alpha} := \prod_v \mathsf{H}_{D_{\alpha,v}}. \tag{5}$$

The heights in (5) give rise to an *adelic height system*

$$\begin{array}{ccc} \oplus_\alpha \mathbb{C}^{\mathcal{A}} \times U(\mathbb{A}) & \xrightarrow{\mathsf{H}} & \mathbb{C} \\ (\sum s_\alpha D_\alpha, (u_v)) & \mapsto & \prod_\alpha \prod_v \mathsf{H}_{D_{\alpha,v}}(u_v)^{s_\alpha} \end{array}$$

which restricts to a Weil height, for $u \in U(F)$ and $(s_\alpha) \in \mathbb{Z}^{\mathcal{A}}$ (see Section 2 of [7] for more details).

We will apply this theory in the setup of Section 1. Let G be a connected semi-simple algebraic group over F with trivial characters, H a closed connected reductive subgroup, and X a smooth projective G-equivariant compactification of the affine variety $X^\circ := H\backslash G$ with boundary

$$\cup_{\alpha \in \mathcal{A}} D_\alpha = X \setminus G,$$

which we assume to be a divisor with strict normal crossings. The divisors D_α can be equipped with an adelic metrization which defines local and global heights on $X^\circ(\mathbb{A})$. Furthermore, G-equivariance implies that for all but finitely many v, the local height functions H_v are right-invariant under $G(\mathfrak{o}_v)$ (see, e.g., Section 3 in [6]). The local and global measures $\mathrm{d}\tau_v$ and $\mathrm{d}\tau$ coincide with suitably normalized Haar measures $\mathrm{d}x_v$ and $\mathrm{d}x$ on $X^\circ(\mathbb{A}) = (H\backslash G)(\mathbb{A})$.

Lemma 2.1. *Let G be a connected semi-simple algebraic group defined over a field F and H a closed subgroup. Let $X^\circ = H\backslash G$ and assume that the map of sets*

$$\mathrm{H}^1(F, H) \to \mathrm{H}^1(F, G) \tag{6}$$

is injective. Then

$$X^\circ(F) = H(F)\backslash G(F).$$

Proof. See, e.g., [18, Chapter 1, Section 5.4]. □

Corollary 2.2. *Let H be a connected simple algebraic group defined over a field F, acting diagonally on $G := H^n$ Then*

$$X^\circ(F) = H(F)\backslash G(F).$$

In particular, if F is a number field, and F_v its completion with respect to a place v, then

$$X^\circ(F_v) = H(F_v)\backslash G(F_v) \quad \text{and} \quad X^\circ(\mathbb{A}) = H(\mathbb{A})\backslash G(\mathbb{A}) \tag{7}$$

Proof. The map

$$\mathrm{H}^1(F, H) \to \mathrm{H}^1(F, G)$$

is injective, since $G = H^n$ and the map is the diagonal one. □

The geometric framework developed in Section 4 of [7] allows to establish analytic properties of local and global integrals of the form

$$\int_{U(F_v)} \mathsf{H}_v(\mathbf{s}, u_v)^{-1}\,\mathrm{d}\tau_v, \quad \int_{U(\mathbb{A})} \mathsf{H}(\mathbf{s}, u)^{-1}\,\mathrm{d}\tau, \tag{8}$$

where τ_v and τ are certain Tamagawa measures defined in Section 2 of [7]. Proposition 4.1.2 and Proposition 4.3.5 of [7] provide meromorphic continuations for integrals in (8) (see also Theorem 7.1 of [19]):

Theorem 2.3. *Let G be a connected semi-simple algebraic group and $H \subset G$ a closed subgroup, defined over a number field F, satisfying the vanishing condition (6) for F and all of its completions. Let X be a smooth projective equivariant compactification of $X^\circ = H\backslash G$ with normal crossing boundary $\cup_{\alpha\in\mathcal{A}} D_\alpha$ and*

$$\mathsf{H} : \mathbb{C}^{\mathcal{A}} \times X^\circ(\mathbb{A}) \to \mathbb{C}$$

an adelic height system. Then there exist a function Φ, holomorphic and bounded in vertical strips for $\Re(s_\alpha) > \kappa_\alpha - \epsilon$, for some $\epsilon > 0$, such that for $\mathbf{s} = (s_\alpha)$ in this domain one has

$$\int_{X^\circ(\mathbb{A})} \mathsf{H}(\mathbf{s}, x)^{-1}\,\mathrm{d}x = \prod_{\alpha\in\mathcal{A}} \zeta_F(s_\alpha - \kappa_\alpha + 1)\cdot\Phi(\mathbf{s}),$$

where ζ_F is the Dedekind zeta function.

Applying Tauberian theorems, we obtain the following:

Lemma 2.4 (Well-roundedness of adelic height balls)**.** *Let L be a class in the interior of the cone of effective divisors and H the associated height. Then the corresponding height balls*

$$B_T = \{x \in X^\circ(\mathbb{A}) \,:\, \mathsf{H}(x) < T\}$$

are well-rounded, i.e.,

$$\lim_{\kappa\to 1^+} \limsup_{T\to\infty} \frac{\mathrm{vol}(B_{\kappa T}) - \mathrm{vol}(B_{\kappa^{-1}T})}{\mathrm{vol}(B_T)} = 0.$$

Proof. This is a corollary of the main theorem of [7, Theorem 1.3], which establishes the claimed asymptotic expansion for B_T and well-roundedness in the smooth case. The general case follows from the smooth case: for any constant $c > 1$ there exists a smooth metrization such that the corresponding height function H' satisfies

$$c^{-1}\mathsf{H}' < \mathsf{H} < c\mathsf{H}'.$$

Thus for any $T > 0$, we have

$$B'_{c^{-1}T} \subseteq B_T \subseteq B'_{cT}$$

so that

$$\limsup_{T} \frac{\mathrm{vol}(B_{\kappa T}) - \mathrm{vol}(B_{\kappa^{-1}T})}{\mathrm{vol}(B_T)} \le \limsup_{T} \frac{\mathrm{vol}(B'_{c\kappa T}) - \mathrm{vol}(B'_{c^{-1}\kappa^{-1}T})}{\mathrm{vol}(B'_{c^{-1}T})},$$

which can be made arbitrarily small by taking $c\kappa$ close enough to 1. This implies that the height balls are well-rounded. $\square$

3. Intermediate subgroups

Let H be a connected almost simple algebraic group defined over an algebraically closed field of characteristic zero and $Z(H)$ its center. For $n \in \mathbb{N}$, let H^n be the n-fold product of H and $\Delta_n = H \hookrightarrow H^n$ the diagonal. The symmetric group $\mathfrak{S}_n$ acts on H^n by permution of the coordinates. We call subgroups M, N of H^n *permutation equal* if there is a $\sigma \in \mathfrak{S}_n$ such that $M = \sigma(N)$. The following proposition is used in the proof of the multiple mixing property in Section 4.

Proposition 3.1. *Let H be a connected simple algebraic group and M a connected algebraic group such that*

$$\Delta_n \subseteq M \subseteq H^n.$$

Then there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $\sum_{i=1}^k n_i = n$ and M is permutation equal to

$$\Delta_{n_1} \times \cdots \times \Delta_{n_k}.$$

The remainder of this section is devoted to a proof of Proposition 3.1. The main step is the following version of Goursat's lemma:

Lemma 3.2. *Let $\underline{x}_r = (x_1, \dots, x_r) \in H^r$ be such that for all i and $j \neq i$, we have $x_i, x_i x_j^{-1} \notin Z(H)$. Let $L_r \subseteq H^r$ be the smallest subgroup containing*

$$\Gamma_r := \{(\delta x_1 \delta^{-1}, \dots, \delta x_r \delta^{-1}) \mid \delta \in H\}.$$

Then $L_r = H^r$.

Proof. We assume that $Z(H) = 1$ and proceed by induction on r. Note that Γ_1 is nontrivial and that it is closed under conjugation so that the closed subgroup of H generated by Γ_1 is normal. Since H is simple, $L_1 = H$.

For $r > 1$. Let L_r be the subgroup corresponding to $\underline{x}_r := (x_1, \dots, x_r)$, we assume that $L_r = H^r$. Clearly, L_r is the projection of L_{r+1} onto the first r entries. Applying the case $r = 1$, we deduce that the projection of L_{r+1} onto the last entry is equal to H. Suppose that there is an element $h \in H^r$ such that for two distinct elements $u, v \in H$, we have $(h, u) \in L_{r+1}$ and $(h, v) \in L_{r+1}$. Then $(e_r, uv^{-1}) \in L_{r+1}$, where e_r denotes the vector in H^r consisting of identity elements in every entry. Again by the case when $r = 1$, we see that $\{e_r\} \times H \subset L_{r+1}$. Since the projection onto the first r coordinates is surjective, $L_{r+1} = H^r \times H$, as required. It remains to rule out the case when for every $h \in H^r$ there is a unique $u := u(h)$ such that $(h, u(h)) \in L_{r+1}$. It follows from the uniqueness that the map $\varphi : h \mapsto u(h)$ is a homomorphism $H^r \to H$, and

$$L_{r+1} = \{(h, \varphi(h)) \mid h \in H^r\}.$$

Moreover, φ is surjective. By construction, if $(h, \varphi(h)) \in L_{r+1}$, then for any $\delta \in H$, we have

$$(\delta_r h \delta_r^{-1}, \delta\varphi(h)\delta^{-1}) \in L_{r+1},$$

where δ_r denotes the vector in H^r with δ in every entry. It follows from uniqueness that

$$\varphi(\delta_r h \delta_r^{-1}) = \delta\varphi(h)\delta^{-1}.$$

Hence, $\delta^{-1}\varphi(\delta_r)$ commutes with $\varphi(h)$ for every $h \in H^r$. Since φ is surjective, we see that $\varphi(\delta_r) = \delta$ for every $\delta \in H$.

Set $y = (x_1 x_{r+1}^{-1}, \dots) = (y', e)$, by definition of φ, one has $x_{r+1} = \varphi(x_1, \dots, x_r)$, hence $\varphi(y') = e$. Consequently, $(y', e) \in L_{r+1}$. Moreover, $y' \in H^r$ satisfies the condition of the lemma, so that L_{r+1} contains $L_r \times \{e\}$. Since the last projection is surjective, this implies $L_{r+1} = H^{r+1}$.

□

Definition 3.3. Let $r \leq n$ be integers. An *admissible embedding* of H^r in H^n is a morphism $\varphi : H^r \to H^n$ of the form

$$\varphi(h_1, \dots, h_r) = (h_{i_1}, \dots, h_{i_n}),$$

for some integers $i_1, \dots, i_n \in \{1, \dots, r\}$. Up to permutation of coordinates on H^n, it is of the form

$$\begin{array}{rcl} H^r & \to & \Delta_{n_1} \times \cdots \times \Delta_{n_r} \subset H^n \\ (h_1, \dots, h_r) & \mapsto & (\underbrace{h_1, \dots, h_1}_{n_1}, \underbrace{h_2, \dots, h_2}_{n_2}, \dots, \underbrace{h_r, \dots, h_r}_{n_r}), \end{array}$$

with $\sum_i n_i = n$. An *admissible subgroup* of H^n is the image of an admissible embedding.

Definition 3.4. Given $r \leq n$, we say an element $\underline{x} \in H^n$ is of rank $\leq r$, if $\underline{x} \in \iota(H^r)$ for some admissible embedding ι. We say $\underline{x}$ is of rank r_0, written $r(\underline{x}) = r_0$, if r_0 is the smallest number r such that $\underline{x}$ is of rank $\leq r$.

It is clear that for every $\underline{x} \in H^n$, $r(\underline{x}) \leq n$. Note that if $\underline{x} \in H^n$ and $\underline{\delta} \in \Delta_n$ then

$$r(\underline{x} \cdot \underline{\delta}) = r(\underline{x}), \quad \text{for } \underline{x} \in H^n.$$

Proof of Proposition 3.1. A reformulation of the statement of the proposition is that if M is a connected subgroup of H^n satisfying

$$\Delta_n \subset M \subset H^n,$$

then M is admissible. Since the isogeny $\pi : H^r \to \bar{H}^r$, where $\bar{H} = H/Z(H)$, defines a bijection between a closed connected subgroup of H^r and $\bar{H}^r$, it is sufficient to prove the claim assuming that $Z(H) = 1$.

Let $r = \max_{\underline{x}\in M} r(\underline{x})$, and let $\underline{x}$ be an element of M which realizes this maximum. As $\Delta_n \subset M$, we may assume that no entry of $\underline{x}$ is equal to identity. After rearranging the coordinates, if necessary, we may assume that

$$\underline{x} = (x_1, \dots, x_1, x_2, \dots, x_2, \cdots, x_r, \dots, x_r) \in \Delta_{n_1} \times \cdots \times \Delta_{n_r}$$

where $x_i x_j^{-1} \neq e$ for $i \neq j$. Then since $\Delta_n \subset M$, it follows from Lemma 3.2 that

$$N := \Delta_{n_1} \times \cdots \times \Delta_{n_r} \subseteq M.$$

It suffices to establish that $N = M$. If M were larger than N, multiplying a generic element of N by an element of $M \setminus N$ we would get an element $\underline{x}'$ with $r(\underline{x}') > r(\underline{x})$, a contradiction. □

4. Multiple mixing

Let H be a connected semi-simple algebraic group defined over a number field F. The aim of this section is to prove the multiple mixing property for the adelic homogeneous space $Y := H(F)\backslash H(\mathbb{A})$. However, when the group H is not simply connected, $\mathsf{L}^2(H(F)\backslash H(\mathbb{A}))$ contains nontrivial one-dimensional representations, and the multiple mixing property holds only on a subset $Y_W \subset Y$, which we now introduce. Let $\pi : \tilde{H} \to H$ be the universal cover of H and W a compact subgroup of $H(\mathbb{A})$ such that $W \cap H(\mathbb{A}_f)$ is open in $H(\mathbb{A}_f)$. We set

$$H_W := H(F)\pi(\tilde{H}(\mathbb{A}))W. \tag{9}$$

By [11], Corollary 4.10, H_W is a normal closed co-abelian subgroup of finite index in $H(\mathbb{A})$. We consider the homogeneous space

$$Y_W := H(F)\backslash H_W,$$

equipped with the normalized Haar measure $\mathrm{d}y$. Let $\mathsf{C}_c(Y_W)^W$ denote the space of continuous compactly supported and W-invariant functions on Y_W.

The following theorem is an adelic version of the multiple mixing of S. Mozes [15].

Theorem 4.1 (multiple mixing)**.** *Let H be a connected simple group over F and*

$$\{(b_1^{(n)}, \dots, b_r^{(n)})\}_{n\in\mathbb{N}} \subset H_W[r] = H_W \times \cdots \times H_W$$

a sequence such that for all $i \neq j$,

$$\lim_{n\to\infty} (b_i^{(n)})^{-1} b_j^{(n)} = \infty \quad \text{in } H_W.$$

Then for all $f_1, \dots, f_r \in \mathsf{C}_c(Y_W)^W$, we have

$$\lim_{n\to\infty} \int_{Y_W} f_1(yb_1^{(n)}) \cdots f_r(yb_r^{(n)})\,\mathrm{d}y = \left(\int_{Y_W} f_1\,\mathrm{d}y\right) \cdots \left(\int_{Y_W} f_r\,\mathrm{d}y\right). \tag{10}$$

The proof of Theorem 4.1 is based on an interpretation of integrals in (10) as a sequence of probability measures supported on $Y_W \times \cdots \times Y_W$ and on an analysis of their limit behaviour using the theory of unipotent flows on adelic spaces developed in [11]. The main technical tools are a partial case of Theorem 1.7 of [11] combined with the description of intermediate subgroups from Section 3.

For $g \in G(\mathbb{A})$ and a measure ν on $G(F)\backslash G(\mathbb{A})$, let $g \cdot \nu$ be the push-forward of ν via the right multiplication by g.

Theorem 4.2 ([11], Theorem 1.7)**.** *Let G be a connected semi-simple algebraic group defined over a number field F, H a connected semi-simple subgroup defined over F, and V a compact subgroup of $G(\mathbb{A})$ such that $V \cap G(\mathbb{A}_f)$ is open in $G(\mathbb{A}_f)$. Let ν_L be the unique $\tilde{L}(\mathbb{A})$-invariant probability measure supported on $G(F)\pi(\tilde{L}(\mathbb{A})) \subset G(F)\backslash G_V$, and let $g^{(n)}$ be a sequence in $G(F)\pi(\tilde{G}(\mathbb{A})) \subset G_V$. Then*

1. *If the centralizer of L in G is anisotropic over F, then the sequence of measures $\{g^{(n)} \cdot \nu_L\}$ is precompact in the weak* topology.*
2. *Suppose that a probability measure μ on $G(F)\backslash G_V$ is a limit of the sequence $\{g^{(n)} \cdot \nu_L\}$ in the weak* topology. Then there exist a connected algebraic subgroup M of G over F and sequences $\{\delta^{(n)}\} \subset G(F)$, $\{l^{(n)}\} \subset \pi(\tilde{L}(\mathbb{A}))$ such that*
   - $\delta^{(n)} L (\delta^{(n)})^{-1} \subset M$,
   - $\delta^{(n)} l^{(n)} g^{(n)} \to g \in \pi(\tilde{G}(\mathbb{A}))$,

   and the limit measure μ can be described as follows: there is a normal subgroup $M_0 \subset M(\mathbb{A})$ of finite index, containing $M(F)\pi(\tilde{M}(\mathbb{A}))$, such that for all $f \in \mathsf{C}_c(G(F)\backslash G_V)^V$,
   $$\int_{G(F)\backslash G_V} f \,\mathrm{d}\mu = \int_{G(F)\backslash G_V} f \,\mathrm{d}(g \cdot \nu_{M_0}),$$
   where ν_{M_0} denotes the unique invariant probability measure supported on $G(F)M_0 \subset G(F)\backslash G_V$.

Proof of Theorem 4.1. We apply Theorem 4.2 to the groups

$$\begin{aligned} G &= H^r = H \times \cdots \times H, \\ L &= \Delta_r = \{(h, \dots, h) \,|\, h \in H\}, \\ V &= W \times \cdots \times W. \end{aligned}$$

Since $H(F)\pi(\tilde{H}(\mathbb{A}))$ is a normal subgroup of H_W (see [11], Section 4) and W is compact, the normalized Haar measure on Y_W can be written as

$$\int_{Y_W} f \,\mathrm{d}y = \int_{Y_W \times W} f(uw) \,\mathrm{d}\nu_H(u) \mathrm{d}w, \quad f \in \mathsf{C}_c(Y_W), \tag{11}$$

where ν_H is the unique $\tilde{H}(\mathbb{A})$-invariant probability measure on $H(F)\pi(\tilde{H}(\mathbb{A})) \subset Y_W$, and $\mathrm{d}w$ is the probability invariant measure on W. Therefore,

$$\int_{Y_W} f_1(xb_1^{(n)})\cdots f_r(xb_r^{(n)})\,\mathrm{d}x = \int_{Y_W\times W} f_1(uwb_1^{(n)})\cdots f_r(uwb_r^{(n)})\,\mathrm{d}\nu_H(u)\mathrm{d}w.$$

If we show that for every fixed $w \in W$, we have

$$\lim_{n\to\infty}\int_{Y_W} f_1(uwb_1^{(n)})\cdots f_r(uwb_r^{(n)})\,\mathrm{d}\nu_H(u) = \left(\int_{Y_W} f_1\,\mathrm{d}y\right)\cdots\left(\int_{Y_W} f_r\,\mathrm{d}y\right),$$

then the theorem would follow from the Lebesgue dominated convergence theorem.

We write $wb_i^{(n)} = s_i^{(n)}w_i^{(n)}$ for $s_i^{(n)} \in H(F)\pi(\tilde{H}(\mathbb{A}))$ and $w_i^{(n)} \in W$. Since the functions f_i are assumed to be W-invariant,

$$\int_{Y_W} f_1(uwb_1^{(n)})\cdots f_r(uwb_r^{(n)})\,\mathrm{d}\nu_H(u) = \int_{Y_W} f_1(us_1^{(n)})\cdots f_r(us_r^{(n)})\,\mathrm{d}\nu_H(u).$$

Since W is compact, we have

$$(s_i^{(n)})^{-1}s_j^{(n)} = w_j^{(n)}\cdot(b_i^{(n)})^{-1}b_j^{(n)}\cdot(w_j^{(n)})^{-1} \to \infty \tag{12}$$

for all $i \neq j$. We set

$$s^{(n)} = (s_1^{(n)},\ldots,s_r^{(n)}) \in G(F)\pi(\tilde{G}(\mathbb{A})).$$

Then

$$\int_{Y_W} f_1(us_1^{(n)})\cdots f_r(us_r^{(n)})\,\mathrm{d}\nu_H(u) = \int_{G(F)\backslash G_V} (f_1\otimes\cdots\otimes f_l)\,\mathrm{d}(s^{(n)}\cdot\nu_L).$$

Now it remains to determine the limit points of the sequence of measures $s^{(n)}\cdot\nu_L$ in the weak* topology. We first note that the centraliser of L in G is equal to $Z(H)\times\cdots\times Z(H)$. Hence, by Theorem 4.2(1), the sequence of measures $s^{(n)}\cdot\nu_L$ is precompact. Let μ be a probability measure on $G(F)\backslash G_V$ which is a limit point of this sequence. The measure μ is described by in Theorem 4.2(2). In particular, we obtain that there exist a connected algebraic subgroup M of G and a sequence $\delta^{(n)} \in G(F)$ such that

$$L \subseteq (\delta^{(n)})^{-1}M\delta^{(n)} \subseteq G$$

From the classification of intermediate subgroups in Proposition 3.1, we deduce that

$$M = \delta^{(n)}N_n(\delta^{(n)})^{-1},$$

where N_n is an admissible subgroup (in the sense of Definition 3.3).

We claim that $M = G$. Indeed, suppose that $M \subsetneq G$. Since the number of admissible subgroups is finite, we may assume, after passing to a subsequence, that $N_n = N \subsetneq G$ is independent of n. Then there exist indices $i \neq j$ such that for the corresponding projection map $\pi_{ij} : G \to H\times H$, we have $\pi_{ij}(N) = \Delta$, where Δ denotes the diagonal subgroup in $H\times H$. Let $\delta = \delta^{(1)}$ and $\sigma^{(n)} = \delta^{-1}\delta^{(n)}$. Since

$$\delta^{(1)}N(\delta^{(1)})^{-1} = \delta^{(n)}N(\delta^{(n)})^{-1},$$

we obtain

$$\pi_{ij}(\sigma^{(n)})\,\Delta\,\pi_{ij}(\sigma^{(n)})^{-1} = \Delta,$$

and

$$(1, (\sigma_i^{(n)})^{-1}\sigma_j^{(n)})\,\Delta\,(1, (\sigma_i^{(n)})^{-1}\sigma_j^{(n)}) = \Delta.$$

This implies that

$$z_n := (\sigma_i^{(n)})^{-1}\sigma_j^{(n)} \in Z(H).$$

By Theorem 4.2(2), we also know that there exist $l^{(n)} \in \pi(\tilde{L}(\mathbb{A}))$ such that the sequence $\delta^{(n)} l^{(n)} s^{(n)}$ converges. Then the sequence $\sigma^{(n)} l^{(n)} s^{(n)}$ converges too, and in particular,

$$(\sigma_i^{(n)} l_i^{(n)} s_i^{(n)})^{-1}(\sigma_j^{(n)} l_j^{(n)} s_j^{(n)})$$

converges. Since $l_i^{(n)} = l_j^{(n)}$ and $z_n \in Z(H)$, we obtain

$$\begin{aligned}(\sigma_i^{(n)} l_i^{(n)} s_i^{(n)})^{-1}(\sigma_j^{(n)} l_j^{(n)} s_j^{(n)}) &= (s_i^{(n)})^{-1}(l_i^{(n)})^{-1}(\sigma_i^{(n)})^{-1}\sigma_j^{(n)} l_j^{(n)} s_j^{(n)} \\ &= (s_i^{(n)})^{-1}(l_i^{(n)})^{-1} z_n l_j^{(n)} s_j^{(n)} \\ &= z_n^{-1}(s_i^{(n)})^{-1} s_j^{(n)}.\end{aligned}$$

Since z_n runs over the finite set $Z(H)$, it follows that $(s_i^{(n)})^{-1}s_j^{(n)}$ converges, which is a contradiction. This proves that $M = G$.

By the last statement of Theorem 4.2, there is a finite index subgroup $M_0 \subseteq M(\mathbb{A}) = G(\mathbb{A})$, containing $G(F)\pi(\tilde{G}(\mathbb{A}))$, and $g \in \pi(\tilde{G}(\mathbb{A}))$ such that for all $f \in \mathsf{C}_c(G(F)\backslash G_V)^V$,

$$\int_{G(F)\backslash G_V} f\,\mathrm{d}\mu = \int_{G(F)\backslash G_V} f\,\mathrm{d}(g\cdot\nu_{M_0}),$$

Since $G(F)\pi(\tilde{G}(\mathbb{A}))$ is a normal coabelian subgroup of G_V (see [11], Section 4), M_0 is also normal coabelian. As in (11), the normalized Haar measure $\mathrm{d}z$ on $G(F)\backslash G_V$ is given by

$$\int_{G(F)\backslash G_V} f\,\mathrm{d}z = \int_{G(F)\backslash G_V\times V} f(uv)\,\mathrm{d}\nu_{M_0}(u)\mathrm{d}v, \quad f \in \mathsf{C}_c(G(F)\backslash G_V),$$

where $\mathrm{d}v$ is the normalized Haar measure on V. For $f \in \mathsf{C}_c(G(F)\backslash G_V)^V$, using that M_0 is coabelian, we obtain

$$\begin{aligned}\int_{G(F)\backslash G_V} f\,\mathrm{d}z &= \int_{G(F)\backslash M_0\times V} f(uvg)\,\mathrm{d}\nu_{M_0}(u)\mathrm{d}v \\ &= \int_{G(F)\backslash M_0\times V} f(ugv)\,\mathrm{d}\nu_{M_0}(u)\mathrm{d}v \\ &= \int_{G(F)\backslash G_V} f\,\mathrm{d}(g\cdot\nu_{M_0}).\end{aligned}$$

This proves that every limit point of the sequence $g^{(n)} \cdot \nu_L$ is a probability measure which is equal to $\mathrm{d}z$ on $\mathsf{C}_c(G(F)\backslash G_V)^V$ which completes the proof of Theorem 4.1. $\square$

5. Counting rational points

Let H be a connected simple algebraic group defined over a number field F, $G = H^r$, and X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$, where H is embedded diagonally. Let L be a line bundle on X such that its class is in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$. By Proposition 1.1, we can write

$$L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha, \quad \lambda_\alpha \in \mathbb{Q}_{>0}.$$

Let

$$\mathsf{H} = \mathsf{H}_{\mathcal{L}} : X^\circ(F) \to \mathbb{R}_{>0}$$

be a height corresponding to a smooth metrization of L as in Section 2 (or in Section 2.1 of [7]).

The adelic height function H is invariant under a compact open subgroup V of $G(\mathbb{A}_f)$, and we may assume that $V = W \times \cdots \times W$ for a compact open subgroup W of $H(\mathbb{A}_f)$. From now on, we make the following

Assumption 5.1. *There are no nontrivial automorphic characters of $G(\mathbb{A})$ invariant under V.*

This assumption simplifies the analysis of volume asymptotics of adelic height balls, via Tauberian theorems applied to Igusa integrals. It holds, e.g., in the following cases (see Proposition 2.1 and Remark 2.2 of [19]):

- If G is a connected semi-simple simply connected group.
- If G is split and of adjoint type over a number field of class number 1, and H is invariant under $\prod_{v \nmid \infty} G(\mathfrak{o}_v)$.

Example 5.2. Consider $G = PGL_2$ over a field F of class number 2. Let E/F be the Hilbert class field and $\omega_{E/F}$ the corresponding quadratic character. Then

$$\omega_{E/F} \circ \det : PGL_2(\mathbb{A}_F) \to \pm 1$$

is an unramified nontrivial automorphic character.

We define the subgroups $G_V \subset G(\mathbb{A})$ and $H_W \subset H(\mathbb{A})$ as in (9). The homogeneous space

$$X_V := H_W \backslash G_V$$

naturally embeds into $X^\circ(\mathbb{A})$ as an open subset, and by Corollary 2.2,

$$X^\circ(\mathbb{A}) \simeq H(\mathbb{A}) \backslash G(\mathbb{A}).$$

We equip $X^\circ(\mathbb{A})$ with the Tamagawa measures $\mathrm{d}x$, defined as in Section 2 of [7]. The regularization of the measure as in [7] requires vanishing of cohomology $\mathrm{H}^i(X, \mathcal{O}_X)$, for $i = 1, 2$, which follows in our case by general vanishing arguments, as $-K_X$ is big (by [9, Thm 1.2]); it is also evident from the explicit volume computation in Lemma 2.4.

Define the height balls in X_V by

$$B_T = B_{T,\mathcal{L}} = \{x \in X_V \mid \mathsf{H}_{\mathcal{L}}(x) < T\}.$$

Lemma 5.3. *Assume that the line bundle L is in the interior of the effective cone. Then*

$$\mathrm{vol}(B_T) = c(\mathcal{L}) \cdot T^{a(L)} \log(T)^{b(L)-1}(1 + o(1)) \quad \text{as } T \to \infty,$$

with $c(\mathcal{L}) > 0$ and $a(L), b(L)$ as in (2).

Proof. Using a standard Tauberian argument (see, for instance, [7]), it suffices to show that

$$\mathsf{Z}(s) = \int_{X_V} \mathsf{H}(x)^{-s}\,\mathrm{d}x$$

has an isolated pole at $a(L)$ of order $b(L)$ and that it admits a meromorphic continuation to $\Re(s) > a(L) - \epsilon$, for some $\epsilon > 0$. We recall (see [11], Section 4) that, in general, G_V is a normal closed coabelian subgroup of $G(\mathbb{A})$. Under the Assumption 5.1, it coincides with $G(\mathbb{A})$. It remains to invoke Theorem 2.3. ∎

Definition 5.4. Let X be an equivariant compactification of $X^\circ = H\backslash G$ and $H' \subset G$ any closed proper subgroup containing the diagonal, i.e., $H \subset H'$. Let $X' \subsetneq X$ be the induced equivariant compactification of H'. A line bundle L on X is called *balanced with respect to H'* if

$$(a(L|_{X'}), b(L|_{X'})) < (a(L), b(L)),$$

in the lexicographic ordering. It is called if this property holds for every such $H' \subsetneq G$.

If X' is not smooth, the invariants are computed on an equivariant resolution of singularities (see Remark 1.2).

Remark 5.5. This property fails in simple examples: $X = \mathbf{P}^3 \times \mathbf{P}^3$ considered as an equivariant compactification of $\mathbb{G}_m^6$ or $\mathbb{G}_a^6$, or $\mathrm{PGL}_2 \times \mathrm{PGL}_2$, with $L = (\lambda_1, \lambda_2)$ and $\lambda_1 \neq \lambda_2$.

Lemma 5.6. *Assume that the line bundle L is balanced. Then, for every smooth adelic metrization of L, every compact subset K of H_W and $i \neq j$, one has*

$$\frac{\mathrm{vol}(B_T \cap \{(x_1, \ldots, x_n) \in H\backslash G \mid x_i^{-1} x_j \in K\})}{\mathrm{vol}(B_T)} \to 0 \quad \text{as } T \to \infty. \tag{13}$$

Proof. Let $M \subset G = H^n$ be the subgroup defined by $x_i = x_j$. Lemma 2.4 implies that, for $T \to \infty$, one has

$$\begin{array}{rcl} \mathrm{vol}(B_T) & = & c\, T^{a(X,L)} \log(T)^{b(X,L)-1}(1+o(1)) \\ \mathrm{vol}(B_T \cap \{x_i^{-1}x_j = 1\}) & = & c'\, T^{a(Y,L|_Y)} \log(T)^{b(Y,L|_Y)-1}(1+o(1)), \end{array}$$

where Y is the induced equivariant compactification of

$$Y^\circ := (H\backslash M) \subset (H\backslash G) = X^\circ \subset X$$

and

$$a(L), b(L), \quad \text{resp.} \quad a(L|_Y), b(L|_Y)$$

are the geometric invariants defined in Section 1. When L is balanced, Equation 13 follows, by definition.

Let $K \subset G(\mathbb{A})$ be a compact subset. Consider translates M_k of M by $k \in K$. The asymptotic of

$$\mathrm{vol}(B_T \cap \{(x_1, \ldots, x_n) \in H\backslash G \mid x_i^{-1}x_j = k\})$$

is determined by analytic properties of the height integral

$$I(\mathbf{s}, k) := \int_{Y^\circ(\mathbb{A})} \mathsf{H}(\mathbf{s}, yk)^{-1}\, \mathrm{d}y = \prod_v \int_{Y^\circ(F_v)} \mathsf{H}_v(\mathbf{s}, y_v k_v)^{-1}\, \mathrm{d}y_v$$

where $Y^\circ = H\backslash M$ and $\mathrm{d}y$, $\mathrm{d}y_v$ are suitably normalized Haar measures. Note that the adelic function

$$k \mapsto \mathsf{H}(\mathbf{s}, yk),$$

is continuous, with $\mathsf{H}_v(\mathbf{s}, y_v k_v) = \mathsf{H}_v(\mathbf{s}, y_v)$ for all but finitely many v. Specialize the integral $I(\mathbf{s}, k)$ to $\mathbf{s} = sL$. We know that each local integral

$$\int_{Y^\circ(F_v)} \mathsf{H}_v(sL, y_v k_v)^{-1}\, \mathrm{d}y_v$$

is holomorphic for $\Re(s) > a(L|_Y) - \epsilon$, for some $\epsilon > 0$, and that the Euler product $I(sL, k)$ has an isolated pole at $s = a := a(L|_Y)$ of order $b := b(L|_Y)$. When L is balanced, Equation 13 holds for translates M_k.

Moreover, the function

$$k \mapsto (s-a)^b \cdot I(sL, k)$$

uniformly continuous and nonvanishing, for $\Re(s) > a - \epsilon$, since only finitely many v are affected and the local integrals vary uniformly continuously with k. We conclude that

$$s \mapsto \int_K I(sL, k)\, \mathrm{d}k$$

has an isolated pole at $s = a$ of order b. It follows that, for $T \to \infty$,

$$\mathrm{vol}(B_T \cap \{x_i^{-1}x_j \in K\}) = \int_K \mathrm{vol}(B_T \cap \{x_i^{-1}x_j = k\})\, \mathrm{d}k = c\, T^a \log(T)^{b-1}(1+o(1)),$$

with some constant $c = c(\mathcal{L}) > 0$. □

Remark 5.7. If the height function is not balanced, the proper subvariety defined by

$$\{x_i^{-1}x_j = \text{constant}\}$$

contributes a positive proportion of rational points to the asymptotic. This is an example of the saturation phenomenon observed in [3], cf. Remark 1.4.

As a corollary of Theorem 4.1 we obtain an equidistribution on the space $Z_V = G(F)\backslash G_V$. We denote by $\mathrm{d}y$ and $\mathrm{d}z$ the normalized Haar measures supported on $Y_W = H(F)\backslash H_W$ and $Z_V = G(F)\backslash G_V$ respectively. Let $\mathrm{d}x$ denote the restriction of the Tamagawa measure on X_V. We consider Y_W as a subspace of Z_V embedded in Z_V diagonally.

Corollary 5.8. *If the line bundle L is balanced, then for every $f \in \mathsf{C}_c(Z_V)$,*

$$\lim_{T\to\infty} \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} \left(\int_{Y_W} f(yx)\,\mathrm{d}x \right) \mathrm{d}y = \int_{Z_V} f\,\mathrm{d}z.$$

Proof. By the Stone-Weierstrass theorem, it suffices to consider functions of the form $f = f_1 \otimes \cdots \otimes f_n$ with $f_i \in \mathsf{C}_c(Y_W)$. In this case,

$$I(x) := \int_{Y_W} f(yx)\,\mathrm{d}y = \int_{Y_W} f_1(yx_1)\cdots f_r(yx_r)\,\mathrm{d}y.$$

Since B_T is invariant under $V = W \times \cdots \times W$,

$$\int_{B_T} I(x)\,\mathrm{d}x = \int_{B_T}\int_{Y_W} \bar{f}_1(yx_1)\cdots \bar{f}_r(yx_r)\,\mathrm{d}y\,\mathrm{d}x,$$

where $\bar{f}_i(y) = \int_W f_i(yw)\,\mathrm{d}w$, where $\mathrm{d}w$ denotes the normalized Haar measure on W. Hence, we may assume that f_i's are W-invariant.

Given a compact subset K of H_W, we set

$$B_{T,K} = \{x \in B_T \mid x_i^{-1}x_j \notin K, \quad \forall i \neq j\}.$$

By Theorem 4.1, for every $\epsilon > 0$, there exists a compact subset K of H_W such that for all $x = (x_1, \ldots, x_r) \in B_{T,K}$, we have

$$\left| I(x) - \left(\int_{Y_W} f_1\,\mathrm{d}y \right) \cdots \left(\int_{Y_W} f_r\,\mathrm{d}y \right) \right| < \epsilon,$$

and

$$\int_{B_{T,K}} I(x)\,\mathrm{d}x = \mathrm{vol}(B_{T,K}) \left(\int_{Y_W} f_1\,\mathrm{d}y \right) \cdots \left(\int_{Y_W} f_r\,\mathrm{d}y \right) + O(\epsilon\,\mathrm{vol}(B_{T,K})). \tag{14}$$

Also,

$$\int_{B_T \setminus B_{T,K}} I(x)\,\mathrm{d}x = O(\mathrm{vol}(B_T \setminus B_{T,K})). \tag{15}$$

Since the line bundle is balanced, it follows from Lemma 5.6 that

$$\frac{\mathrm{vol}(B_T \setminus B_{T,K})}{\mathrm{vol}(B_T)} \to 0 \quad \text{as } T \to \infty.$$

Hence, combining (14) and (15), we deduce that

$$\limsup_{T\to\infty} \left| \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} I(x)\,\mathrm{d}x - \left(\int_{Y_W} f_1\,\mathrm{d}y\right) \cdots \left(\int_{Y_W} f_r\,\mathrm{d}y\right) \right| = O(\epsilon)$$

for every $\epsilon > 0$, which proves the corollary. □

From Corollary 5.8, we deduce:

Theorem 5.9. *Let H be a connected simple algebraic group over F, $G = H^n$, and X a G-equivariant compactification of $X^\circ = H\backslash G$. Let $H_W \subseteq H(\mathbb{A})$ be the normal subgroup of finite index defined in* (9)*. Let L be a balanced line bundle on X. Then*

$$\begin{aligned} |X^\circ(F) \cap B_T| &= \mathrm{vol}(H(F)\backslash H_W)^{1-r} \cdot \mathrm{vol}(B_T)(1 + o(1)) \\ &= c\,\mathrm{vol}(H(F)\backslash H_W)^{1-r} \cdot T^{a(L)}(\log T)^{b(L)-1}(1 + o(1)), \quad \textit{as } T \to \infty, \end{aligned}$$

where $c = c(\mathcal{L})$ is as in Lemma 5.3.

Proof. Let $\mathrm{d}h$ be the Tamagawa measure on $H(\mathbb{A})$ restricted to H_W. Then the Haar measure $\mathrm{d}g$ on G_V can be written as

$$\int_{G_V} \tilde{f}\,\mathrm{d}g = \int_{X_V} \left(\int_{H_W} \tilde{f}(hx)\,\mathrm{d}h\right) \mathrm{d}x, \quad \tilde{f} \in \mathsf{C}_c(G_V).$$

Take $\kappa > 1$, and let U be a symmetric neighborhood of identity in G_V such that

$$B_T \cdot U \subset B_{\kappa T} \quad \text{for all } T. \tag{16}$$

Let $\tilde{f} \in \mathsf{C}_c(G_V)$ be a nonnegative function with $\mathrm{supp}(\tilde{f}) \subset U$ and $\int_{G_V} \tilde{f}\,\mathrm{d}g = 1$. Put

$$f(g) := \sum_{\gamma \in G(F)} \tilde{f}(\gamma^{-1} g).$$

Then, for every $x \in X_V$,

$$\int_{Y_W} f(yx)\,\mathrm{d}y = \frac{1}{\mathrm{vol}(H(F)\backslash H_W)} \sum_{\gamma \in H(F)\backslash G(F)} \int_{H_W} \tilde{f}(\gamma^{-1} hx)\,\mathrm{d}h. \tag{17}$$

If $x \in B_{\kappa^{-1}T}$ and $\gamma^{-1}hx \in U$, then $\gamma \in hxU$; using (16) we have $\gamma \in B_T$, since B_T consists of cosets of H_W and $h \in H_W$. Hence, (17) implies that

$$\begin{aligned}
(18)\qquad \mathrm{vol}(H(F)\backslash H_W) \int_{B_{\kappa^{-1}T}} \left(\int_{Y_W} f(yx)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \sum_{\gamma \in H(F)\backslash G(F)\cap B_T} \int_{H_W \times B_{\kappa^{-1}T}} \tilde{f}(\gamma^{-1}hx)\,\mathrm{d}h\mathrm{d}x \\
&\leq \sum_{\gamma \in H(F)\backslash G(F)\cap B_T} \int_{G_V} \tilde{f}(\gamma^{-1}g)\,\mathrm{d}g \\
&= |H(F)\backslash G(F) \cap B_T|
\end{aligned}$$

If $\gamma \in B_T$ and $\gamma^{-1}hx \in U$, then $x \in h^{-1}\gamma U$; using (16) we have $x \in B_{\kappa T}$. Now (17) implies that

$$\begin{aligned}
(19)\qquad |H(F)\backslash G(F) \cap B_T| &= \sum_{\gamma \in H(F)\backslash G(F)\cap B_T} \int_{G_V} \tilde{f}(\gamma^{-1}g)\,\mathrm{d}g \\
&= \sum_{\gamma \in H(F)\backslash G(F)\cap B_T} \int_{H_W \times X_V} \tilde{f}(\gamma^{-1}hx)\,\mathrm{d}h\mathrm{d}x \\
&\leq \sum_{\gamma \in H(F)\backslash G(F)} \int_{H_W \times B_{\kappa T}} \tilde{f}(\gamma^{-1}hx)\,\mathrm{d}h\mathrm{d}x \\
&= \mathrm{vol}(H(F)\backslash H_W) \int_{B_{\kappa T}} \left(\int_{Y_W} f(yx)\,\mathrm{d}y \right) \mathrm{d}x.
\end{aligned}$$

By Lemma 5.3,

$$\lim_{T\to\infty} \frac{\mathrm{vol}(B_{\kappa T})}{\mathrm{vol}(B_T)} = \kappa^{a(L)}.$$

Combining (18) with Corollary 5.8, we obtain

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{|H(F)\backslash G(F) \cap B_T|}{\mathrm{vol}(B_T)} \\
\geq &\left(\lim_{T\to\infty} \frac{\mathrm{vol}(B_{\kappa^{-1}T})}{\mathrm{vol}(B_T)} \right) \left(\lim_{T\to\infty} \frac{\mathrm{vol}(H(F)\backslash H_W)}{\mathrm{vol}(B_{\kappa^{-1}T})} \int_{B_{\kappa^{-1}T}} \left(\int_{Y_W} f(yx)\,\mathrm{d}y \right) \mathrm{d}x \right) \\
= &\kappa^{-a(L)} \mathrm{vol}(H(F)\backslash H_W) \int_{Z_V} f\,\mathrm{d}z \\
= &\kappa^{-a(L)} \frac{\mathrm{vol}(H(F)\backslash H_W)}{\mathrm{vol}(G(F)\backslash G_V)} = \kappa^{-a(L)}\, \mathrm{vol}(H(F)\backslash H_W)^{1-r}.
\end{aligned}$$

Similarly, it follows from (19) that

$$\limsup_{T\to\infty} \frac{|H(F)\backslash G(F) \cap B_T|}{\mathrm{vol}(B_T)} \le \kappa^{a(L)} \mathrm{vol}(H(F)\backslash H_W)^{1-r}.$$

Since these estimates hold for all $\kappa > 1$, we conclude that

$$|H(F)\backslash G(F) \cap B_T| = \mathrm{vol}(H(F)\backslash H_W)^{1-r}\mathrm{vol}(B_T)(1 + o(1))$$

as $T \to \infty$. Since $X^\circ(F) = H(F)\backslash G(F)$ by Corollary 2.2, this proves the first part of the theorem. The second part follows from Lemma 5.3. □

Theorem 1 follows by applying Proposition 1.3, which insures that the anticanonical line bundle $-K_X$ is balanced.

References

[1] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.

[2] V. V. Batyrev and Y. Tschinkel. Rational points on some Fano cubic bundles. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(1):41–46, 1996.

[3] V. V. Batyrev and Y. Tschinkel. Tamagawa numbers of polarized algebraic varieties. *Astérisque*, (251):299–340, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[4] T. D. Browning. *Quantitative arithmetic of projective varieties*, volume 277 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2009.

[5] A. Chambert-Loir. Lectures on height zeta functions: at the confluence of algebraic geometry, algebraic number theory, and analysis. In *Algebraic and analytic aspects of zeta functions and L-functions*, volume 21 of *MSJ Mem.*, pages 17–49. Math. Soc. Japan, Tokyo, 2010.

[6] A. Chambert-Loir and Y. Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.

[7] A. Chambert-Loir and Y. Tschinkel. Igusa integrals and volume asymptotics in analytic and adelic geometry. *Confluentes Math.*, 2(3):351–429, 2010.

[8] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[9] B. Fu and D.-Q. Zhang. A characterization of compact complex tori via automorphism groups. *Math. Ann.* 357(3):961-968, 2013.

[10] A. Gorodnik, F. Maucourant, and H. Oh. Manin's and Peyre's conjectures on rational points and adelic mixing. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3):383–435, 2008.

[11] A. Gorodnik and H. Oh. Rational points on homogeneous varieties and equidistribution of adelic periods. *GAFA*, 21(2):319–392, 2011.

[12] B. Hassett, S. Tanimoto, and Y. Tschinkel. Balanced line bundles and equivariant compactifications of homogeneous spaces. `arXiv:1307.5467`, to appear in *IMRN*

[13] L. Lafforgue. *Chirurgie des grassmanniennes*, volume 19 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2003.

[14] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[15] S. Mozes, *Mixing of all orders of Lie groups actions. Invent. Math.* 107, no. 2, 235–241, 1992.

[16] H. Oh. Orbital counting via mixing and unipotent flows. In *Homogeneous flows, moduli spaces and arithmetic*, volume 10 of *Clay Math. Proc.*, pages 339–375. Amer. Math. Soc., Providence, RI, 2010.

[17] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.

[18] J.-P. Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.

[19] J. Shalika, R. Takloo-Bighash, and Y. Tschinkel. Rational points on compactifications of semi-simple groups. *J. Amer. Math. Soc.*, 20(4):1135–1186 (electronic), 2007.

[20] Y. Tschinkel. Algebraic varieties with many rational points. In *Arithmetic geometry*, volume 8 of *Clay Math. Proc.*, pages 243–334. Amer. Math. Soc., Providence, RI, 2009.

School of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.
E-mail address: a.gorodnik@bristol.ac.uk

University of Illinois at Chicago, 851 S. Morgan str. (M/C 249), Chicago, IL 60607, U.S.A.
E-mail address: rtakloo@math.uic.edu

Courant Institute of Mathematical Sciences, N.Y.U., 251 Mercer str., New York, NY 10012, U.S.A.
E-mail address: tschinkel@cims.nyu.edu

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA