CO-FIBERED PRODUCTS OF ALGEBRAIC CURVES

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Abstract. We give examples of failure of the existence of co-fibered products in the category of algebraic curves.

Contents

1. Introduction

Let C_1, C_2 be smooth complex projective curves. Assume that one has a diagram

$$\begin{array}{ccc} C & \xrightarrow{f_1} & C_1 \\ {\scriptstyle f_2}\big\downarrow & & \\ C_2 & & \end{array}$$

with f_1, f_2 étale surjective morphisms. A *co-fibered product* is the universal diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{f_1} & C_1 \\ {\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\ C_2 & \xrightarrow[g_2]{} & C' \end{array}$$

with C' a curve, g_1, g_2 surjective finite morphisms and

$$g_1 f_1 = g_2 f_2.$$

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The starting point for this note was the following question of J. Kollár: are there obstructions to the existence of co-fibered products for unramified covers in the category of Riemann surfaces? In the language of function fields, the condition is equivalent to the triviality of the intersection of the function fields $k(C_1) \cap k(C_2) \subset k(C)$.

More generally, let X be an algebraic variety over an algebraically closed field k of dimension n. Let $K = k(X)$ be its function field. Consider subfields $k(Y_1), k(Y_2) \subset k(X)$, with $\dim(Y_1) = \dim(Y_2) = \dim(X)$. We show that under mild conditions on k and the varieties Y_1, Y_2, one has indeed

$$k(Y_1) \cap k(Y_2) = k.$$

This can be achieved as soon as k^* has an element of infinite order. We will also show that both field extensions $k(X)/k(Y_1)$ and $k(X)/k(Y_2)$ can be unramified, thus satisfying the condition that f_1 and f_2 above be étale. Using a theorem of Margulis we show that co-fibered products for unramified covers exist unless the curves in question are Shimura curves. Thus our construction provides *all* counterexamples. In case of curves over the complex numbers, we will give examples with small covering degrees, e.g., $\deg(g_1) = \deg(g_2) = 3$.

In positive characteristic, related questions on intersections of some specific function fields have been considered in [Ber73], [BM78], [Wat04], [BWZ07], [ZM08]. A sample result from [Ber73] is: If k is perfect field of characteristic p then

$$k(x^{pn} + x^{pn-1}) \cap k(x^n) \neq k$$

if and only if $\gcd(p, n) = 1$. A more precise description of $k(f) \cap k(g)$ is in [BM78]. However, the question remains whether co-fibered products exist for unramified covers of curves over $\bar{\mathbb{F}}_p$.

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2. Elementary examples

It is easy to construct examples of *ramified* covers: Let $K := k(\mathbb{P}^1)$ and assume that k^* contains an element a of infinite order. Let us take two involutions

$$\sigma : x \to 1/x, \quad \text{and} \quad \sigma_a : x \to a/x.$$

They generate a dihedral group $\mathfrak{D}_a$ with commutator

$$[\sigma, \sigma_2] \,:\, x \mapsto x/a^2$$

of *infinite* order. The fields of invariants $k(x)^{\sigma}$ and $k(x)^{\sigma_a}$ have index 2 in $k(x)$, but the intersection consists of elements which are invariant under $\mathfrak{D}_a$ and hence only of constants.

More generally, let $G \in \mathrm{PGL}_2(k)$ be an infinite subgroup generated by two elements of finite order. The subfields of invariants of $k(\mathbb{P}^1)$ have the required property. Note that such groups G do not exist for k an algebraic closure of a finite field. Indeed, any finite set of elements in $\mathrm{PGL}_2(\bar{\mathbb{F}}_p)$ is contained in a subgroup $\mathrm{PGL}_2(\mathbb{F}_q)$, for some $q = p^n$, so that this approach fails to produce nonintersecting subfields.

3. Shimura varieties

Natural examples of curve covers arise in the theory of arithmetic groups. Let G be a semi-simple algebraic group defined over a number field F. Fix a model $\mathcal{G}$ of G over the ring of integers $\mathfrak{o}_F$. For every real embedding $\iota \,:\, F \to \mathbb{R}$ we have a complex symmetric space

$$\mathbb{D}_\iota := \mathcal{G}_\iota(\mathbb{R})/\mathbf{K}_\iota, \quad \mathbb{D} := \prod_\iota \mathbb{D}_\iota$$

where $\mathbf{K}_\iota$ is a maximal compact subgroup. We have a homomorphism

$$\phi \,:\, \mathcal{G}(\mathfrak{o}_F) \to \prod_\iota \mathcal{G}_\iota(\mathbb{R}).$$

Let $\Gamma \subset \phi(\mathcal{G}(\mathfrak{O}_K))$ be a subgroup of finite index and

$$X_\Gamma := \Gamma\backslash\mathbb{D}.$$

The quotient X_Γ is a complex algebraic variety defined over some finite extension of $\mathbb{Q}$. For $h \in \phi(\mathcal{G}(F))$ let $\Gamma_h := h\Gamma h^{-1}$. Then $\Lambda_h := \Gamma_h \cap \Gamma$ is a subgroup of finite index in Γ and Γ_h. Thus there are surjective maps

$$\begin{array}{ccc} X := \mathbb{D}/\Lambda_h & \xrightarrow{f_1} & \mathbb{D}/\Gamma = X_\Gamma \\ \Big\downarrow{\scriptstyle f_2} & & \\ \mathbb{D}/\Gamma_h = X_{\Gamma_h} & & \end{array}$$

Both maps are defined over some number field F'. Thus we have two field embeddings

$$f_1^*(F'(X_\Gamma)) \subset F'(X), \quad f_2^*(F'(X_{\Gamma_h})) \subset F'(X).$$

Lemma 1. *If h is of infinite order in $G(F)$ (modulo the center) then the intersection*

$$f_1^*(F'(X_\Gamma)) \cap f_2^*(F'(X_{\Gamma_h})) \subset K'(X)$$

is a subfield of transcendence degree strictly smaller then $\dim(X)$*. If h and Γ generate a subgroup of $G(F)$ which acts densely on $\mathbb{D}/\Gamma$ then the intersection*

$$f_1^*(F'(X_{\Gamma_h})) \cap f_2^*(F'(X_\Gamma)) \subset F'(X)$$

consists only of constants.

The same results hold for arbitrary extensions of F', in particular for complex numbers.

Proof. The field $\mathbb{C}(X)$ consists of meromorphic functions on $\mathbb{D} := \prod_\iota D_\iota$ which are invariant under the action of $\Gamma_h \cap \Gamma$, and the subfields

$$f_1^*(\mathbb{C}(X_\Gamma)), f_2^*(\mathbb{C}(X_{\Gamma_h}))$$

of meromorphic functions invariant under Γ, Γ_h respectively. The intersection $f_1^*(\mathbb{C}(X_\Gamma)) \cap g_2^*(\mathbb{C}(X_{\Gamma_h}))$ consists of functions invariant under both Γ, Γ_h. If h has infinite order and if its power is not a central element in $G(F)$ then $\Gamma \cap \Gamma_h$ has infinite index in the group generated by Γ, Γ_h. This is equivalent to $\mathbb{C}(X_h)$ having an infinite degree over the intersection $f_1^*(\mathbb{C}(X_\Gamma)) \cap f_2^*(\mathbb{C}(X_{\Gamma_h}))$. Since both fields are algebraic subfields the intersection is also algebraic, i.e., a finite extension of $\mathbb{C}(y_1, \dots, y_k)$ for some set $y_1, \dots, y_k$. This implies that $k < \dim(X_h)$.

If Γ_h, Γ generate a subgroup which acts on $\mathbb{D}$ with a dense orbit then there are no invariant meromorphic functions on $\mathbb{D}$. Thus

$$f_1^*(\mathbb{C}(X_\Gamma)) \cap f_2^*(\mathbb{C}(X_{\Gamma_h})) = \mathbb{C}.$$

Since the maps f_i are defined over F' the same holds for arbitrary intermediate subfields $\tilde{F} \subset \mathbb{C}, F' \subset \tilde{F}$. □

If the action of Γ on $\mathbb{D}$ is cocompact then for a subgroup of finite index the stabilizers become trivial. Then both maps f_1, f_2 are finite unramified covers.

4. Curve covers

Consider $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_h := h\,\mathrm{SL}_2(\mathbb{Z})h^{-1}$, where $h \in \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{Z})$. The intersection $\Gamma \cap \Gamma_h$ has finite index in both groups. However, the action of the group generated by Γ and Γ_h on the upper-half plane $\mathfrak{H}$ is not discrete. In this case we have cusps, i.e., the corresponding maps are ramified. A similar argument applies to any arithmetic group acting on $\mathfrak{H}$.

Let D be a division algebra of dimension 4 over $\mathbb{Q}$ which embeds into the 2×2-matrices $M_2(\mathbb{R})$. The spitting field of D is a real-quadratic field $\mathbb{Q}(\sqrt{d})$. Let $\Gamma \subset D$ be a subgroup of finite index in the group of integral quaternions with norm one which does not contain torsion elements. It acts discretely on $\mathfrak{H}$ with a compact quotient, the complex points of a projective algebraic curve C. Let $h \in D$ be an element with a nontrivial denominator and $\Gamma_h := h\Gamma h^{-1}$. Write

$$\Lambda_h := \Gamma_h \cap \Gamma.$$

As in Section 3, we have covers

$$\begin{array}{ccc} X := \mathbb{D}/\Lambda_h & \xrightarrow{f_1} & \mathbb{D}/\Gamma = X_\Gamma \\ \downarrow{\scriptstyle f_2} & & \\ \mathbb{D}/\Gamma_h = X_{\Gamma_h} & & \end{array}$$

On the other hand, the group generated by h and Γ acts nondiscretely on $\mathfrak{H}$. Thus there are no h, Γ-invariant elements in the function field $\mathbb{C}(C')$ and hence no nontrivial common quotient. The groups Γ, Γ_h and Λ_h contain no elements of finite order. Hence they act freely on $\mathbb{D}$ and the covers f_1 and f_2 are unramified.

We now present an series of examples with small covering degrees. Let D be a quaternion algebra over $\mathbb{Q}$ with splitting field $\mathbb{Q}(\sqrt{d})$, for $d > 0$. Denote by $\Gamma \subset D^1$ the subgroup of integer elements in D of norm 1. Assume that D has the following properties:

(1) d is a square in $\mathbb{Q}_2$ and hence $D \times \mathbb{Q}_2 = M_2(\mathbb{Q}_2)$,
(2) Γ does not contain elements of finite order,
(3) the completion of Γ surjects onto $\mathrm{SL}_2(\mathbb{Z}_2)$.

These conditions are easily satisfied. Note that D is dense in $\mathrm{GL}_2(\mathbb{Q}_2)$. Let $h \in D$ be an element which modulo 4 is equal to

$$\begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix},$$

and put $\Gamma_h := h\Gamma h^{-1}$. The intersection

$$\Lambda_h := \Gamma_h \cap \Gamma$$

contains a subgroup $x = 1 \mod 2$ and a subgroup modulo 4 generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is a subgroup of index 3 in Γ and also in Γ_h. The group Λ_h is the preimage of a congruence subgroup in $\mathrm{SL}_2(\mathbb{Z}_2)$. Since $\mathrm{SL}_2(\mathbb{Z}/2) = \mathfrak{S}_3$ and the unipotent subgroup is $\mathbb{Z}_2$ we obtain that Λ_h has index 3.

The construction shows that it suffices to assume that Γ is a subgroup of finite index in the group of integral elements in D of norm 1, which has no torsion and whose completion surjects onto $\mathrm{SL}_2(\mathbb{Z}_2)$. For example, we can insist that $g \in \Gamma$ satisfies $g = 1 \mod p$, for some prime $p \neq 2$.

Remark 2. We cannot achieve that both g_1, g_2 are of degree 2 and unramified. Indeed, in this case the corresponding extensions would be Galois, and the actions of both $\mathbb{Z}/2$ could be realized inside an action of a finite group H on C. Thus $k(C') \cap k(C'')$ contains $k(C)^H$, a nontrivial field.

Let G be a semi-simple algebraic group over $\mathbb{Q}$ and

$$\mathrm{Comm}_G(\Gamma) := \left\{ g \in G(\mathbb{R}) \,|\, [\Gamma : \left(g\Gamma g^{-1} \cap \Gamma\right)] < \infty \right\}.$$

This is a well-defined subgroup of $G(\mathbb{R})$ containing Γ.

Assume that $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete cocompact subgroup without torsion elements. Then X admits maps f_1, f_2 as above, if and only if Γ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$. Note that

$$[\mathrm{Comm}_{\mathrm{SL}_2(\mathbb{R})}(\Gamma) : \Gamma] < \infty$$

unless Γ is arithmetic. Indeed, we have the following

Theorem 3. [Mar91, Theorem (B), p. 298] *Let G be a semi-simple group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{R})$ an irreducible lattice. Assume that Γ*

(i) *is of infinite index in* $\mathrm{Comm}_G(\Gamma)$*;*
(ii) *is finitely generated;*
(iii) *satisfies property QD.*

Then Γ is arithmetic.

In our applications, Γ automatically satisfies properties (ii) and (iii) (see [Mar91, Chapter IX] for definitions and results).

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