UNRAMIFIED COHOMOLOGY OF FINITE GROUPS OF LIE TYPE

by

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ABSTRACT. — We prove vanishing results for unramified stable cohomology of finite groups of Lie type.

Contents

1. Introduction

Let k be an algebraically closed field, G a finite group and V a faithful representation of G over k. In this note we compute cohomological obstructions to stable rationality of quotients of V by G introduced by Saltman [12] and [4] and studied in [7], [10], [5].

KEY WORDS AND PHRASES. — Rationality, finite simple groups, unramified cohomology.

Let $K = k(V)^G$ be the function field of the quotient variety and

$$s : \mathcal{G}_K \to G.$$

the natural homomorphism from the absolute Galois group of K to G. We have an induced map on cohomology with coefficients in the torsion group $\mathbb{Z}/\ell$, with trivial G-action,

$$s_i^* : H^i(G, \mathbb{Z}/\ell) \to H^i(\mathcal{G}_K, \mathbb{Z}/\ell).$$

Note that s_i^* depends on the ground field k, but not on the choice of the faithful representation V over that field. The groups

$$H^i_{k,s}(G, \mathbb{Z}/\ell) := H^i(G, \mathbb{Z}/\ell)/\operatorname{Ker}(s_i^*),$$

are called *stable cohomology groups* over k. They form a finite ring. We may consider them as subgroups of $H^i(\mathcal{G}_K, \mathbb{Z}/\ell)$. Every divisorial valuation $\nu \in \mathrm{Val}_K$ of K defines a residue map

$$\partial_\nu : H^i(\mathcal{G}_K, \mathbb{Z}/\ell) \to H^{i-1}(\mathcal{G}_{\boldsymbol{K}_\nu}, \mathbb{Z}/\ell),$$

where $\boldsymbol{K}_\nu$ is the residue field of ν. The groups

$$H^i_{k,un}(G, \mathbb{Z}/\ell) := \bigcap_{\nu \in \mathrm{Val}_K} \operatorname{Ker}(\partial_\nu \circ s_i^*) \subset H^i_{k,s}(G, \mathbb{Z}/\ell)$$

form a subring of $H^*_{k,s}(G, \mathbb{Z}/\ell)$. A basic fact is that if there exists a faithful representation V of G over k and a unirational parametrization of the quotient V/G whose degree is prime to ℓ then

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0, \quad \text{for all } i > 0.$$

In particular, these cohomology groups vanish if this quotient is stably rational.

For example, the rings of invariants of finite groups generated by pseudo-reflections are polynomial, the corresponding quotient varieties rational, and the cohomological invariants trivial. In particular, all Weyl groups $\mathcal{W}$ of semi-simple Lie groups have

$$H^i_{k,un}(\mathcal{W}, \mathbb{Z}/\ell) = 0, \quad \text{for all } i > 0, \quad \text{and all } \ k.$$

Conjecture 1.1. — Let G be a finite simple group. Then

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0, \quad \text{for all } \ i > 0, \quad \text{all } \ k \ \text{ and all } \ \ell.$$

The $i = 2$ case of this conjecture was proved for $G = \mathsf{PSL}_n(\mathbb{F}_q)$ and $k = \mathbb{C}$ in [5] and for simple and quasi-simple groups of Lie type in [9]. Examples of functions fields with vanishing second and nonvanishing third unramified cohomology were given in [11]. Here we prove that many of these cohomology groups vanish for finite groups of Lie type, for $k = \bar{\mathbb{F}}_q$. In fact, we prove stable rationality of many associated quotients spaces. Our main theorem is:

Theorem 1.2. — *Let G be one of the following groups*

$$\mathsf{SL}_n(\mathbb{F}_q), \quad \mathsf{Sp}_{2n}(\mathbb{F}_q), \ \textit{or } {}^2\mathsf{SL}_n(\mathbb{F}_q).$$

Let V be a faithful representation of G over $k = \bar{\mathbb{F}}_p$. Then the quotient of V by G is stably rational over k.

In particular, Conjecture 1.1 holds in these cases, for $\ell \nmid q$. Our main tool is a theorem of Lang which proves rationality of certain quotient spaces over $\bar{\mathbb{F}}_p$.

Theorem 1.3. — *Let G be a semi-simple simply-connected Lie group defined over a finite field $\mathbb{F}_q$. Then the image*

$$H^i(\mathsf{G}(\mathbb{F}_q), \mathbb{Z}/\ell) \to H^i_{k,s}(\mathsf{G}/\mathsf{G}(\mathbb{F}_q), \mathbb{Z}/\ell)$$

is zero, for $k = \bar{\mathbb{F}}_q$, all $i > 0$ and $\ell \nmid q$.

Combining this with results of Tits [14] we obtain the following:

Theorem 1.4. — *Let G be a finite quasi-simple group of Lie type over a finite field of characteristic p. Put*

$$d(G) := \begin{cases} \emptyset & \textit{if } G \textit{ is } \mathsf{SL}_n(\mathbb{F}_q), {}^2\mathsf{SL}_n(\mathbb{F}_q), \mathsf{Sp}_n(\mathbb{F}_q), \mathsf{SO}_n(\mathbb{F}_q) \textit{ or } G_2(\mathbb{F}_q); \\ \{2,3\} & \textit{if } G \textit{ is of type } F_4, E_6, E_7; \\ \{2,3,5\} & \textit{if } G \textit{ is of type } E_8. \end{cases}$$

Then, for $k = \bar{\mathbb{F}}_p$, one has

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0, \quad \textit{for all } \ i > 0, \quad \textit{and all } \ \ell \notin d(G).$$

Here is the roadmap of the paper. In Section 2 we study the birational type of quotients $G \backslash \mathsf{G}/H$, where G is an algebraic group over an algebraically closed field k and $G, H \subset \mathsf{G}(k)$ are finite subgroups, acting on G by translations on the left, resp. on the right. In Section 3 we study the classical groups. In Section 4 we introduce stable and unramified cohomology over arbitrary algebraically closed fields and prove their basic properties. In Section 7 we establish general vanishing results, applying theorems of Lang and Tits.

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2. Equivariant birational geometry

We work over an algebraically closed field k. We say that k-varieties X and Y are stably birational, and write $X \sim Y$, if $X \times \mathbb{A}^n$ is birational to $Y \times \mathbb{A}^m$, for some $n, m \in \mathbb{N}$.

Let G be an algebraic group and X an algebraic variety over k, with a G-action

$$\lambda : G \times X \to X.$$

We will sometimes consider different actions of the same group. To emphasize the action we will write $\lambda(G)\backslash X$ for the quotient of X by the λ-action of G; we write $G\backslash X$, when the action is clear from the context.

We say that the action of G is *almost free* if there exists a Zariski open subset $X^\circ \subset X$ on which the action is free. In particular, the quotient map $X \to G\backslash X$ is separable.

Example 2.1. — Let V be a faithful complex representation of G. Then G acts almost freely on V.

Lemma 2.2. — *Let G be a finite group and V a faithful representation of G over an algebraically closed field k. Let Y be an affine variety over k, with a free G-action, and $y \in Y(k)$ a point.*

For every Zariski open $U \subset V$ there exist a G-equivariant k-morphism $\phi_U : Y \to V$ and a Zariski open G-invariant subset $Y^\circ \subset Y$ such that

- *$y \in Y^\circ(k)$;*
- *$\phi_U(Y^\circ) \subset U$.*

Proof. — It suffices to consider $V := \mathbb{A}_k^{|G|}$, the affine k-space, with the induced faithful G-action. For any divisor $D \subset V$ there exists a Zariski open subset $U \subset V$ such that for every point $v \in U(k)$ its G-orbit $G \cdot v \notin D$. For every Zariski open $U \subset V$ there exists a G-equivariant k-morphism $\phi_U : Y \to V$

such that $\phi_U(y) \in U(k)$ (functions separate points). This implies the existence of a Zariski open G-invariant subset $Y^\circ \subset Y$ with the claimed properties. □

A G-variety X is called *G-affine*, and the corresponding action *affine*, if there exists a G-equivariant birational isomorphism between X and a faithful representation of G. Let $\mathcal{V}$ and X be affine G-varieties. A G-morphism $\pi : \mathcal{V} \to X$ is called an *affine G-bundle* if it is an affine bundle over some open subset $X^\circ \subset X$ and the G-action is compatible with this structure of an affine bundle.

By Hilbert 90, an affine G-bundle $\mathcal{V} \to X$ is G-birational to a finite dimensional G-representation over the function field of $K = k(X)$, compatible with the given G-action on K. A morphism $\rho : X \to B$ of G-varieties will be called a G-ruling (and X - G-ruled) over B if there exists a finite set of affine G-varieties

$$X_n = X, B_{n-1}, X_{n-1}, B_{n-2}, X_{n-2}, \ldots, X_1, B_0 = B$$

such that $X_i \to B_{i-1}$ is an affine G-bundle and $B_i \subset X_i$ a G-stable Zariski open subset, for $i = 1, \ldots, n$.

Lemma 2.3. — *Assume that $\rho : X \to B$ is a G-ruling over B and that the action of G on X is almost free. Then X is G-affine.*

Proof. — Follows from Hilbert 90. □

Let X, Y be smooth varieties with an almost free action of G. We write $X \overset{G}{\rightsquigarrow} Y$ if there exist a G-representation V, a Zariski open G-stable subset $X^\circ \subset X$ and a G-morphism (not necessarily dominant) $\beta : X^\circ \times V \to Y$. We write $X \overset{G}{\leftrightsquigarrow} Y$, and say that the G-actions are equivalent, if $X \overset{G}{\rightsquigarrow} Y$ and $Y \overset{G}{\rightsquigarrow} X$.

Lemma 2.4. — *If $X \overset{G}{\rightsquigarrow} Y$ then the morphism*

$$\beta : G\backslash(X \times Y) \to G\backslash X$$

has a rational section.

Proof. — Consider the morphism

$$\beta' : G\backslash(X \times V \times Y) \to G\backslash X,$$

where G acts diagonally. The graph of the map $X \to (Y \times V)$ is G-stable and gives a section of β'. The projection of this section to $G\backslash(X \times Y)$ is a section of β. □

LEMMA 2.5. — *Let* G *be a Lie group over an algebraically closed field* k*. Let* $G \subset \mathsf{G}(k)$ *be a finite subgroup. Let* X *be an algebraic variety over* k *with an almost free action of* G*. Assume that* $X \stackrel{G}{\rightsquigarrow} \mathsf{G}$*, where* G *is considered as a* G*-variety, with a left action. Then*

$$G\backslash(X \times \mathsf{G}) \sim G\backslash X.$$

Proof. — By Lemma 2.4, there is a Zariski open subset G-stable subset $X^\circ \subset X$ so that the G-morphism (projection to the first factor)

$$\beta : G\backslash(X \times \mathsf{G}) \to (G\backslash X)$$

has a section. We also have a *right* action of G, which preserves the fibration structure given by β. Thus it is a principal homogeneous space over $G\backslash X^\circ$, for some G-stable Zariski open $X^\circ \subset G$, with a section. Hence it is birational to $(G\backslash X) \times \mathsf{G}$. It suffices to recall that G is rational over k. □

Let G be a connected algebraic group and $F \in \mathrm{Aut}_k(\mathsf{G})$ a k-automorphism of G. Let $G \subset \mathsf{G}(k)$ be finite subgroup, with a natural left action

$$\begin{array}{rccc} \lambda : & G \times \mathsf{G} & \to & \mathsf{G} \\ & (\gamma, g) & \mapsto & \gamma \cdot g \end{array}$$

We also have an F-twisted right action

$$\begin{array}{rccc} \rho^F : & G \times \mathsf{G} & \to & \mathsf{G} \\ & (\gamma, g) & \mapsto & g \cdot F(\gamma^{-1}). \end{array}$$

and an F-conjugation

$$\begin{array}{rccc} \kappa^F : & G \times \mathsf{G} & \to & \mathsf{G} \\ & (\gamma, g) & \mapsto & \gamma \cdot g \cdot F(\gamma^{-1}). \end{array}$$

LEMMA 2.6. — *Assume that* $G \subset \mathsf{G}(k)$ *has the following properties:*

(1) *there exists a faithful* G*-representation* V *such that* $V \stackrel{G}{\rightsquigarrow} \mathsf{G}$*, where* G *acts on* G *via* λ*;*
(2) *the twisted action* ρ^F *on* G *is almost free.*

Then the quotient of G *by the* F*-twisted conjugation* κ^F *of* G *is stably birational to the quotient of* G *by* λ*.*

Proof. — Consider the diagonal action of G on $\mathsf{G}\times\mathsf{G}$:

$$\begin{array}{ccc} G\times\mathsf{G}\times\mathsf{G} & \xrightarrow{(\lambda,\rho^F)} & \mathsf{G}\times\mathsf{G} \\ (\gamma,g,g') & \mapsto & (\gamma\cdot g, g'\cdot F(\gamma^{-1})). \end{array}$$

Let $\Delta^F := \{(g, F^{-1}(g))\}\subset \mathsf{G}\times\mathsf{G}$ be the F-twisted (anti)diagonal. Then Δ^F is preserved under the (λ,ρ^F)-action of G and descends to a section of the principal (right) G-fibration

$$(\lambda,\rho^F)(G)\backslash(\mathsf{G}\times\mathsf{G})\to\rho^F(G)\backslash\mathsf{G},$$

projection to the second factor. It follows that

$$\rho^F(G)\backslash\mathsf{G}\sim(\lambda,\rho^F)(G)\backslash(\mathsf{G}\times\mathsf{G}).$$

Observe that

$$\rho^F(G)\backslash\mathsf{G}\sim\lambda(G)\backslash\mathsf{G}.$$

Now we show that

$$\kappa^F(G)\backslash\mathsf{G}\sim(\lambda,\rho^F)(G)\backslash(\mathsf{G}\times\mathsf{G}). \tag{2.1}$$

Let V be a faithful representation of G as in (1) and $V^\circ\subset V$ a G-stable Zariski subset admitting a G-map into G, considered with the λ-action of G. We know that there exists a G-morphism $\xi:\mathsf{G}\to V$, where G is considered with the κ^F-action of G, such that $\xi(\mathsf{G})\cap V^\circ\neq\emptyset$ (see Lemma 2.2). It follows that $\mathsf{G}\overset{G}{\rightsquigarrow}\mathsf{G}$, where the source carries the κ^F-action of G and the image the λ-action of G. Equation (2.1) now follows from Lemma 2.4. □

Corollary 2.7. — *Let $G\subset\mathsf{G}(k)$ by a finite subgroup satisfying Assumption (1) of Lemma 2.6. Let V be a faithful representation of G over k. Then*

$$G\backslash\mathsf{G}\sim G\backslash V$$

More generally, for any X with an almost free action of G we have

$$G\backslash(X\times V)\sim G\backslash X.$$

Proof. — Note that $G\backslash(\mathsf{G}\times V)$ is a vector bundle over $G\backslash\mathsf{G}$, and hence stably birational to it. On the other hand, it is a right G-fibration over $G\backslash V$ with section defined by the G-equivariant map $V\to\mathsf{G}$.

To prove the statement for X it suffices to notice that a dense Zariski open G-stable subset $X^\circ\subset X$ admits a nontrivial G-morphism to V. □

3. Equivariant birational geometry of classical groups

In this section, k is an algebraically closed field, of any characteristic.

LEMMA 3.1. — *The conjugation action* $\kappa : \mathsf{SL}_2 \to \mathsf{SL}_2$ *is equivalent to a linear action.*

Proof. — Realize SL_2 as a nonsingular quadric in $\mathbb{A}^4 = \mathsf{M}_2$. The conjugation action is linear on M_2 and has a fixed point corresponding to the identity. The projection of SL_2 from the identity to the locus of trace zero matrices is equivariant and has degree 1. Hence the conjugation action is rationally equivalent to the action on trace zero matrices. □

LEMMA 3.2. — *Let* $G = (\mathbb{Z}/2)^n$ *and* X *be a* G*-affine variety over* k*. Then* $G\backslash X$ *is rational.*

LEMMA 3.3. — *Let* $G \subset \mathsf{PGL}_2(k) \simeq \mathsf{SO}_3(k) \subset \mathsf{M}_2(k)$ *be a finite subgroup. Then the action of* G *on* PGL_2 *by conjugation is equivalent to the left (linear) action of* G *on trace zero matrices* $\mathsf{M}_2^0 \subset \mathsf{M}_2$*. In particular,* $G\backslash\mathsf{PGL}_2$ *is stably rational.*

LEMMA 3.4. — *The left action of* $(\mathbb{Z}/2)^2 \subset \mathsf{SO}_3 \hookrightarrow \mathsf{SO}_4$ *is linear.*

Proof. — Consider the subgroup $(\mathbb{Z}/2)^3 \subset \mathsf{SO}_4$. It contains a central subgroup $\mathbb{Z}/2$ and a complementary subgroup $H_2 = (\mathbb{Z}/2)^2 \subset \mathsf{SO}_3 = \mathsf{PGL}_2$. By Lemma 3.1, the conjugation action $H_2 \subset \mathsf{PGL}_2$ is linear.

This is a subgroup of the diagonal subgroup $\mathsf{SO}_3 = \mathsf{PGL}_2 \subset \mathsf{SL}_2 \times \mathsf{SL}_2/(\mathbb{Z}/2)$. Note that SO_4 is a product of $\mathsf{SO}_3 = (h, h)$ and $\mathsf{Spin}_3 = (g, 1)$, and that conjugation by elements in $(\mathbb{Z}/2)^2$ respects this decomposition. Thus the action is a vector bundle over $\mathsf{SL}_2/(\mathsf{Q}_8)_{conj} = \mathsf{SL}_2/(\mathbb{Z}/2)^2$ (where Q_8 are the quaternions).

Hence the action on SO_4 is linear and the automorphism F is the identity on the diagonal SO_3. The same holds for the twisted F-action. □

LEMMA 3.5. — *Let* G *be an algebraic group over* k*. Assume that* G *admits an affine action on itself, e.g.,* $\mathsf{G} = \mathsf{GL}_n, \mathsf{SL}_n, \mathsf{Sp}_n$*. Let* $G \subset \mathsf{G}(k)$ *be a finite subgroup which has trivial intersection with the center of* G*.*

Then the conjugation action of G *on* G *is stably birationally equivalent to a linear action.*

Proof. — By Corollary 2.7, the diagonal left translation action of G on $\mathsf{G} \times \mathsf{G}$ is equivalent to the action on a principal G-bundle over G, with G acting on the base by conjugation. This proves the equivalence. □

Corollary 3.6. — *Let $G \subset \mathsf{G}(k)$ be a finite subgroup as in Lemma 3.5. Assume that the action of G on G by left translations is affine. Then both the left and the conjugation action of G on G are stably birational to a linear action.*

Proposition 3.7. — *Let G be a classical simply-connected Lie group of type A or C, i.e., $\mathsf{G} = \mathsf{SL}_n$ or $\mathsf{G} = \mathsf{Sp}_{2n}$ over k. Let $G \subset \mathsf{G}(k)$ be a finite subgroup. Then G is a G-affine variety, for the standard left action of G.*

Proof. — Note first that any finite subgroup $G \subset \mathsf{GL}_n(k)$ induces a G-affine structure on GL_n. Indeed, $\mathsf{GL}_n \subset \mathsf{M}_{n\times n} = \oplus_{i=1}^n V^{(i)}$, a direct sum of n copies of the standard representation V of GL_n. Put $B_0 = 0$, a point, and $X_1 := V^{(1)} = V$. We have a canonical projection $X_1 \to B_0$. Define $X_j \subset \oplus_{i=1}^j V^{(i)}$ as the set of those vectors, whose projections to $\oplus_{i=1}^{j-1} V^{(i)}$ are linearly independent, and $B_j \subset X_j$ as the subset of vectors in X_j, which are linearly independent in $\oplus_{i=1}^j V^{(i)}$, under the standards identification $V = V^{(i)}$. For any $G \subset \mathsf{GL}_n$ this defines the structure of a G-ruling on GL_n over a point.

For $\mathsf{G} = \mathsf{SL}_n$, and $G \subset \mathsf{G}$ we have a similar G-ruling: For $j = 1, \ldots, n-1$ it is the same as above. For $j = n$, put $X_n := \mathsf{SL}_n$. The map $X_n \to B_{n-1}$ is the restriction of the map above. Explicitly, it is the projection to the first $(n-1)$-vectors $(v_1, \ldots, v_{n-1})$, with fiber an affine subspace $F_{(v_1,\ldots,v_{n-1})} \subset V^{(n)} = V$, given by the affine equation in the coordinates of the last vector $\det(v_1, \ldots, v_n) = 1$. Now we can apply Lemma 2.3 to conclude that SL_n is G-affine.

Note that by Tsen's theorem, the morphism $\mathsf{GL}_n \to \mathsf{GL}_n/\mathsf{SL}_n$ has a section. This gives a G-equivariant birational isomorphism $\mathsf{SL}_n \times \mathbb{G}_m \to \mathsf{GL}_n$.

The group $\mathsf{G} = \mathsf{Sp}_{2n}$ has a canonical embedding into $\mathsf{M}_{2n\times 2n}$, defined by the equations

$$\omega(v_i, v_{i'}) = \delta_{i',n+i}, \quad \text{for} \quad i < i', \tag{3.1}$$

(ω is the standard bilinear form and δ is the delta function). The system of projections is induced from the one above:

$$X_j = \{(v_1, \ldots, v_j)\} \subset \oplus_{i=1}^j V^{(i)},$$

satisfying equations (3.1), for indices $1 \le i < i' \le j$, and the property that the vectors $v_1, \ldots, v_{j-1}$ are linearly independent in V, under the identifications $V^{(i)} = V$. The subvariety $B_j \subset X_j$ is given as the locus where $v_1, \ldots, v_j$ are linearly independent. Each map $X_j \to B_{j-1}$ is an affine G-bundle, for any finite subgroup $G \subset \mathsf{G}(k)$ - its fibers are given by a system of *linear* equations on the coordinates of v_j. □

PROPOSITION 3.8. — *Let $\mathsf{G} = \mathsf{SO}_n$ and $G \subset \mathsf{G}(k)$ be a finite subgroup. Then there exist a G-ruling X, a variety Y with trivial G-action and a G-equivariant finite morphism*

$$\pi : \mathsf{G} \times Y \to X.$$

Moreover, $\deg(\pi) \mid 2^{n-1}$.

Proof. — Keep the notations in the proof of Proposition 3.7: $\mathsf{G} \subset \mathsf{M}_n$.

Assume that $\mathrm{char}(k) \neq 2$. Every quadratic form can be diagonalized over k. Let $X_n \subset \mathsf{M}_n$ be the subvariety given by:

$$(v_i, v_{i'}) = \delta_{i',n+i}, \quad \text{for} \quad i < i'. \tag{3.2}$$

The system of projections is the same as above: $X_j \subset \oplus_{i=1}^{j} V^{(i)}$ is the subset of vectors satisfying equations (3.2), for indices $1 \leq i < i' \leq j$, and the condition that $v_1, \ldots, v_{j-1}$ are linearly independent. The subvariety $B_j \subset X_j$ corresponds to j-tuples $(v_1, \ldots, v_j)$ which are linearly independent (as vectors in $V = V^{(i)}$). Each map $X_j \to B_{j-1}$ is a G-equivariant vector bundle, for any finite subgroup $G \subset \mathsf{G}(k)$. Each X_j carries the action of a j-dimensional torus $\mathbb{G}_m^j$, over k, commuting with the action of G. The action of $\mathsf{G} \times \mathbb{G}_m^n$ on X_n is transitive, and the stabilizer of a general k-point has order 2^{n-1}. The claim follows, for $Y := \mathbb{G}_m^n$.

Assume that $\mathrm{char}(k) = 2$. In this case, $\mathsf{SO}_{2n+1} \simeq \mathsf{Sp}_{2n}$ and we can apply Proposition 3.7. We also have $\mathsf{SO}_{2n} \subset \mathsf{Sp}_{2n}$, where $\mathsf{Sp}_{2n}(k)$ is the set of elements of $\mathsf{GL}_{2n}(k)$ which preserve a symplectic bilinear form ω, and $\mathsf{SO}_{2n}(k)$ the set of those elements which in addition preserve a quadratic form f. The forms are related by the condition

$$f(x+y) = f(x) + f(y) + \omega(x,y).$$

We may identify a general element $\gamma \in \mathsf{Sp}_{2n}(k)$ with a choice of an orthogonal basis $\{v_1, \ldots, v_{2n}\}$. Observe that the map

$$\begin{array}{ccc} \mathsf{Sp}_{2n} & \to & \mathsf{Sp}_{2n}/\mathsf{SO}_{2n} \sim \mathbb{A}^{2n} \\ \{v_1, \ldots, v_{2n}\} & \mapsto & (f(v_1), \ldots, f(v_{2n})). \end{array}$$

Indeed, $\gamma \in \mathsf{SO}_{2n}(k)$ iff and only if $f(\gamma x) = f(x)$, for all $x \in V$. We have

$$\begin{aligned} f(\sum_{i=1}^{2n} a_i v_i) &= \sum_{i=1}^{2n} a_i^2 f(v_i) + \sum_{i \neq j} a_i a_j (v_i, v_j) \\ &= \sum_{i=1}^{2n} a_i^2 f(\gamma v_i) + \sum_{i \neq j} a_i a_j \omega(\gamma v_i, \gamma v_j) \\ &= f(\gamma(\sum_{i=1}^{2n} a_i v_i)), \end{aligned}$$

since $f(\gamma v_i) = f(v_i)$ and γ preserves ω.

We claim that the bundle $\mathsf{Sp}_{2n} \to \mathsf{Sp}_{2n}/\mathsf{SO}_{2n} \sim \mathbb{A}^{2n}$ admits a multisection of degree 2^{2n}. Explicitly, it can be constructed as follows: fix an orthogonal basis $\{v_1, \ldots, v_{2n}\}$ such that $f(v_i) \neq 0$, for $i = 1, \ldots, 2n$. We have an action of the affine group $\mathsf{B} = \mathbb{G}_m \rtimes \mathbb{G}_a \subset \mathsf{SL}_2$ given by

$$(x_i, x_{n+i}) \mapsto (\lambda x_i, \mu x_i + \lambda^{-1} x_{n+i}), \quad \text{for} \quad i = 1, \ldots, n.$$

We claim that this gives a generically surjective map:

$$\mathsf{SO}_{2n} \times \mathsf{B}^n \to \mathsf{Sp}_{2n}$$

of degree 2^{2n}. The image of $\mathsf{B}^n \cdot \{f(v_1), \ldots, f(v_{2n})\}$ is dense in $\mathbb{A}^{2n}$. Consider the intersection $\mathsf{B}^n \cap \mathsf{SO}_{2n}$:

$$f(v_i) = \lambda^2 f(v_i), \quad \text{and} \quad f(v_{i+n}) = \mu^2 f(v_i) + \lambda\mu\omega(v_i, v_{i+n}) + \lambda^{-2} f(v_{i+n}).$$

These equations can be solved in k, for each $i = 1, \ldots, n$, and we have a dominant map $\mathsf{SO}_{2n} \times \mathsf{B} \to \mathbb{A}_i^2$ of degree 4, for each i. This concludes the proof. □

4. Stable cohomology

In this section we collect background material on stable cohomology of finite groups, developing the theory over arbitrary algebraically closed fields k. We will omit k from the notation when the field is clear from from the context.

For every finite group G and a G-module M we have the notion of *group cohomology*, as the derived functor $M \mapsto M^G$, the G-invariants, or, topologically, as the cohomology of the classifying space $BG = X/G$, where X is a contractible space with a fixed point free action of G.

Passing to algebraic geometry, let X be an algebraic variety over k, with an almost free action of a G. Let $X^\circ \subset X$ be the locus where the action is free. Let M be a finite G-module. It defines a sheaf on $\tilde{X} := X^\circ/G$. This gives a homomorphism from group cohomology of G to étale cohomology of $\tilde{X}$:

$$H^i(G, M) \to H^i_{et}(\tilde{X}, M).$$

Composing with restriction to the generic point we get a homomorphism

$$\sigma_i^* : H^i(G, M) \to H^i(\mathcal{G}_K, M),$$

where $\mathcal{G}_K$ is the absolute Galois group of the function field $K = k(\tilde{X})$. There are canonical isomorphisms

$$H^i(\mathcal{G}_K, M) = \varinjlim_{D} H^i(X \setminus D, M),$$

where the limit is taken over divisors of D. We can interpret elements in the kernel of σ^* as classes vanishing on some Zariski open subvariety $\tilde{U} \subset \tilde{X}$.

Remark 4.1. — Note that for fixed G and M, the groups $\sigma_i^*(H^i(G, M)) = 0$, for all $i > \dim(X)$, while the usual group cohomology need not vanish.

Proposition 4.2. — *There exist a finite group $\tilde{G}$ and a sequence*

$$\mathcal{G}_K \xrightarrow{\tilde{\sigma}} \tilde{G} \xrightarrow{\rho} G$$

of homomorphisms $\tilde{\sigma}$ and ρ such that for all $0 \le i \le \dim(X)$ one has

$$\mathrm{Ker}(\sigma_i^*) \subset \mathrm{Ker}(\rho_i^*),$$

where $\rho_i^* : H^i(\tilde{G}, M) \to H^i(G, M)$ is the induced map on group cohomology.

Proof. — The cohomology classes in $H^i(\mathcal{G}_K, \mathbb{Z}/\ell)$ are represented by continuous cocycles (in the natural topology on $\mathcal{G}_K$). Any element is induced from a finite group H. If it vanishes it also vanishes on a finite quotient $\tilde{G}$ of $\mathcal{G}_K$ and the maps $\mathcal{G}_K \to \tilde{G} \to G$ are continuous. Since the initial group $H^i(G, \mathbb{Z}/\ell)$ is finite there exists a $\tilde{G}$ where all elements from $H^i(G, \mathbb{Z}/\ell)$, which vanish on $\mathcal{G}_K$ are killed. □

A special case of the above construction arises as follows: let $\varrho : G \to V$ be a faithful representation over an algebraically closed field k and let $K = k(V)^G$ be the function field of the quotient. We have induced maps

$$s_i^* : H^i(G, M) \to H^i(\mathcal{G}_K, M)$$

and we can define the *stable cohomology groups* over k:

$$H^i_{k,s}(G, M) := H^i(G, M)/\operatorname{Ker}(s_i^*),$$

which we will often identify with their image in $H^i(\mathcal{G}_K, M)$.

Proposition 4.3. — *The cohomology groups $H^i_{k,s}(G, M)$*

1. *do not depend on the representation;*
2. *are functorial in G;*
3. *are universal for G-actions: for any G-variety X over k the homomorphism $H^i(G, M) \to H^i(\mathcal{G}_{k(X)}, M)$ factors through $H^i_{k,s}(G, M)$;*
4. *if M is an ℓ-torsion module, then*

$$H^i_{k,s}(G, M) = H^i_{k,s}(\mathrm{Syl}_\ell, M)^{\mathfrak{N}_\ell}$$

where $\mathrm{Syl}_\ell = \mathrm{Syl}_\ell(G)$ is an ℓ-Sylow subgroup of G and $\mathfrak{N}_\ell = \mathfrak{N}_\ell(G)$ its normalizer in G.

Proof. — We apply Lemma 2.2. Choosing an appropriate Zariski open G-invariant subvariety $X^\circ \subset X$ we can reduce to the affine case, with free G-action. Let $V^\circ \subset V$ be a Zariski open subset where the action of G is free. Put $\tilde{X} := G\backslash X^\circ$ and $\tilde{V} := G\backslash V^\circ$. We need to show that a class $\alpha \in H^i(G, M)$ whose image in $H^i(\mathcal{G}_{k(\tilde{V})}, M)$ is zero also vanishes in $H^i(\mathcal{G}_{k(\tilde{X})}, M)$. Such a class vanishes in $H^i_{et}(\tilde{U}, M)$, where $\tilde{U} \subset \tilde{V}$ is an affine Zariski open subset. The preimage U of $\tilde{U}$ in V is a nonempty G-invariant affine Zariski open subset. Thus there exist an affine nonempty G-invariant Zariski open subset $U_X \subset X^\circ$ and a G-morphism $\phi_U : X^\circ \to V$ such that $\phi_U(U_X) \subset U$. This descends to a morphism $\tilde{X} \supset \tilde{U}_X \to \tilde{U} \subset \tilde{V}$. The image of α under the composition

$$H^i(G, M) \to H^i_{et}(\tilde{U}, M) \to H^i_{et}(\tilde{U}_X, M) \to H^i_{et}(\mathcal{G}_{k(\tilde{X})}, M)$$

is zero. This proves (3). Applying this to $X = V'$, for another faithful representation, we get (1).

Property (2) is proved as follows: First, let $H \subset G$ be a subgroup and V a faithful G-representation. Consider the morphism $H\backslash V \to G\backslash V$. A class vanishing on a Zariski open subset of $G\backslash V$ also vanishes on a Zariski open subset of $H\backslash V$. Next, let $G \to H$ be a surjective homomorphism and V_G, resp. V_H, a faithful representation of G, resp. H. Then $W_G := V_H \oplus V_G$ is a faithful representation of G and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{k(G\backslash W_G)} & \to & \mathcal{G}_{k(H\backslash V_H)} \\ \downarrow & & \downarrow \\ G & \to & H, \end{array}$$

giving natural maps on cohomology.

We proceed with the proof of Property (4). Since ℓ and the cardinality of G/Syl_ℓ are coprime, the map $\mathrm{Syl}_\ell\backslash V \to G\backslash V$ induces an invertible map on cohomology of open subvarieties of $G\backslash V$. The group $\mathfrak{N}_\ell(G)/\mathrm{Syl}_\ell(G)$ has order prime to ℓ. The action of $\mathfrak{N}_\ell(G)/\mathrm{Syl}_\ell(G)$ on M decomposes the module into a direct sum; so that

$$H^i(\mathrm{Syl}_\ell(G), M) = H^i(\mathrm{Syl}_\ell(G), M)^{\mathfrak{N}_\ell(G)} \oplus R,$$

so that the restriction of the trace map is zero on the module R.

We have

$$H^i(G, M) \xrightarrow{\sim} H^i(\mathrm{Syl}_\ell(G), M) \subset H^i(\mathrm{Syl}_\ell(G), M)^{\mathfrak{N}(\mathrm{Syl}_\ell(G))}.$$

Consider the image of $r \in R$ in $H^*_{k,s}(\mathrm{Syl}_\ell(G), M)$. We get a direct decomposition $H^i_s(\mathrm{Syl}_\ell(G), M)^{\mathfrak{N}(\mathrm{Syl}_\ell(G))} \oplus R_s$, with $\mathrm{Tr}(r) = 0$. Thus $H^i_{k,s}(G, M)$ surjects onto $H^i(\mathrm{Syl}_\ell(G), M)^{\mathfrak{N}(\mathrm{Syl}_\ell(G))}$, and the map is an isomorphism.

□

Lemma 4.4. — *Let V be a representation space for a faithful representation of group G over an algebraically closed field k. Assume that $G\backslash V$ is isomorphic to affine space. Then any nontrivial element $\alpha \in H_{k,s}(G, \mathbb{Z}/\ell)$ has nontrivial restriction to the stable cohomology of a centralizer of a quasi-reflection in G.*

Proof. — If $\alpha \in H^*_{k,s}(k(\mathbb{A}^n), \mathbb{Z}/\ell)$ is nontrivial then the residue of α is nontrivial on some irreducible divisor $D \subset \mathbb{A}^n = G\backslash V$ (see [6]). The preimage of D in V is a union of irreducible divisors $D_1, \ldots, D_r$. For each i, there exists a nontrivial $\gamma_i \in G$ acting trivially on all points of D_i. Thus each D_i is a hyperplane in V. Hence γ_i is a quasi-reflection. □

Corollary 4.5. — *Let $\mathcal{W}$ be a Weyl group. Then*

$$H^i_{k,s}(\mathcal{W}, \mathbb{Z}/\ell) = 0, \quad \text{for all } i > 0 \text{ and all } \ell \neq 2.$$

We have

$$H^i_{k,s}(\mathcal{W}, \mathbb{Z}/2) \hookrightarrow \oplus_\tau H^i_{k,s}(\tau, (\mathbb{Z}/2)^{r_\tau}),$$

where τ runs over the set of 2-elementary abelian subgroups of $\mathcal{W}$, modulo conjugation, and $r_\tau \in \mathbb{N}$.

Proof. — The quasi-reflections in the standard faithful representation of $\mathcal{W}$ have order 2. Their centralizers are products of powers of $\mathbb{Z}/2$ with smaller Weyl groups. It suffices to apply induction. □

Remark 4.6. — It is possible to obtain a more precise vanishing result following the approach for $\mathcal{W} = \mathfrak{S}_n$ in [8].

5. Comparison with Serre's negligible classes

Stable cohomology was defined by the first author in [2] and [3]. J.P. Serre defined a related but somewhat different notion [13, p. 170].

In his terminology, *negligible* elements $\alpha \in H^*(G, M)$ are those which are killed under every surjective homomorphism $\mathcal{G}_K \to G$, where $K = k(\tilde{X})$ is the function field of a quotient $\tilde{X} = G\backslash X^\circ$. Negligible elements form an ideal in the total ring $H^*(G, M)$.

We are considering a smaller set of homomorphisms $\mathcal{G}_K \to G$, namely from Galois groups of fields of type $K = k(V)^G$, and the ideal of negligible classes defined by Serre is smaller. The resulting groups are different (for example, for $\mathbb{Z}/2$-coefficients). The quotient ring obtained by Serre's construction surjects onto the ring $H^*_{k,s}(G, M)$, for any algebraically closed k.

6. Unramified cohomology

Let $K = k(X)$ be a function field over an algebraically closed field k, and M an étale sheaf on X. For every divisorial valuation $\nu \in \mathrm{Val}_K$ of K we have a split exact sequence

$$1 \to \mathcal{I}_\nu \to \mathcal{G}_{K_\nu} \to \mathcal{G}_{\boldsymbol{K}_\nu} \to 1$$

where K_ν is the completion of K with respect to ν, $\boldsymbol{K}_\nu$ the residue field and $\mathcal{I}_\nu$ is the inertia group. This gives an exact sequence in Galois cohomology

$$H^i(\mathcal{G}_{K_\nu}, M^{\mathcal{I}_\nu}) \to H^i(\mathcal{G}_{K_\nu}, M) \xrightarrow{\delta_\nu} H^{i-1}(\mathcal{G}_{\boldsymbol{K}_\nu}, {}_{\mathcal{I}_\nu}M)$$

where $M^{\mathcal{I}_\nu}$, resp. ${}_{\mathcal{I}_\nu}M$, are the sheaves of invariants, resp. coinvariants. Unramified cohomology is defined by

$$H^i_{k,un}(\mathcal{G}_K, M) := \cap_{\nu \in \mathrm{Val}_K} \mathrm{Ker}(\delta_\nu) \subset H^i(\mathcal{G}_K, M).$$

LEMMA 6.1. — *Let $\pi : X \to Y$ be a surjective morphism of algebraic varieties over k, and M an étale sheaf on Y. Then there is a natural homomorphism:*

$$\pi^* : H^i_{k,un}(Y, M) \to H^i_{k,un}(X, \pi^*(M)).$$

Moreover, if π is finite, then there is a natural homomorphism

$$\pi_* : H^i_{k,un}(X, \pi^*(M)) \to H^i_{k,un}(Y, M)$$

and the composition $\pi_* \circ \pi^*$ is multiplication by the degree of π.

Proof. — We have an embedding $\pi^* : k(X) \hookrightarrow k(Y)$ of function fields and a the corresponding map $\pi_* : \mathcal{G}_{k(Y)} \to \mathcal{G}_{k(X)}$ of Galois groups. A divisorial valuation ν of $k(Y)$ is either trivial on $\pi^*(k(X))$ or defines a divisorial valuation ν^* on $k(X)$. If ν is trivial on $\pi^*(k(X))$ then $\pi_*(\mathcal{I}_\nu)$ for the inertia subgroup $\mathcal{I}_\nu \subset \mathcal{G}_{k(Y)_\nu}$ and hence δ_ν is zero on $\pi^* H^*(\mathcal{G}_{k(X)}, M)$. If ν on $\pi^*(k(X))$ coincides with ν^* then under the induced map $\pi_* : \mathcal{G}_{k(Y)_\nu} \to \mathcal{G}_{k(X)_{\nu'}}$ we have $\pi_*(\mathcal{I}_\nu) \subset \mathcal{I}_{\nu^*}$. Thus $\delta_{\nu^*}(\alpha) = 0$ implies that $\delta_\nu \pi^*(\alpha) = 0$ for all $\alpha \in H^*_{k,un}(\mathcal{G}_{k(X)}, M)$. □

Let G be a finite group and V a faithful G-representation as above. Let $K = k(V)^G$ be the function field of the quotient. We can consider its stable cohomology groups $H^i_{k,s}(G, M)$ as subgroups of $H^i(\mathcal{G}_K, M)$. Define *unramified cohomology groups*

$$H_{k,un}(G, M) := H^i_{k,s}(G, M) \cap H^i_{k,un}(\mathcal{G}_K, M).$$

PROPOSITION 6.2. — *Assume that* $\mathrm{char}(k) \nmid |M|$. *Then the unramified cohomology groups $H^i_{k,un}(G, M)$*

- *do not depend on the representation V;*
- *are functorial in G.*

Proof. — Let V, V' be two faithful representations of G and consider the diagram:

$$\begin{array}{ccc} G\backslash(V \times V') & \to & G\backslash V' \\ \downarrow & & \\ G\backslash V & & \end{array}$$

The case of constant coefficients follows from [7] and the observation that both arrows in the above diagram are natural vector bundles on the quotients. A small modification of the argument proves the claim for a general G-module M. □

7. General vanishing results

In this section, we work over $k = \bar{\mathbb{F}}_p$. Here we collect general arguments proving triviality of stable and unramified cohomology groups.

THEOREM 7.1. — *Let G be a finite group and M a finite p-torsion G-module. Then, $H^i_{k,s}(G, M) = 0$, for all $i > 1$.*

Proof. — See [13, Chapter 2, Proposition 3]. □

The main reason for introducing unramified cohomology group is:

THEOREM 7.2. — *Let V be a faithful representation of G. If $K = k(V)^G$ is a purely transcendental extension of k then, for all $i > 0$, we have*

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0.$$

THEOREM 7.3 (Lang). — *Let G be an algebraic group over k. Let F be an automorphism of $\mathsf{G}(k)$ which is a composition of an element in $\mathrm{Aut}(\mathsf{G})(k)$ and a Frobenius of $k = \bar{\mathbb{F}}_p$. Let $G = G^F \subset \mathsf{G}(k)$ be the finite subgroup fixed by F. Then $G\backslash\mathsf{G} \simeq \mathsf{G}$, hence is a rational variety.*

Proof. — Consider the map

$$\begin{array}{rcl} \tau : \mathsf{G} & \to & \mathsf{G} \\ x & \to & F(x)^{-1}x. \end{array}$$

The action of τ on the Lie algebra of G is surjective with kernel a finite subgroup $G = G^F$. Note that τ coincides with the composition

$$\tau : \mathsf{G} \to G\backslash\mathsf{G} \to \mathsf{G}.$$

Indeed, if $\tau(x) = \tau(y)$ then $F(x)^{-1}x = F(y)^{-1}y$, or $F(xy^{-1})^{-1}xy^{-1} = 1$, or $xy^{-1} \in G$. If follows that $x = gy, g \in G$. The converse is clear. Thus $G\backslash\mathsf{G}$ is rational. □

LEMMA 7.4. — *Let ℓ be a prime and $\pi : X \to Y$ a separable morphism k-varieties of finite degree prime to ℓ. Assume that $H^i_{k,un}(X, \mathbb{Z}/\ell) = 0$. Then $H^i_{k,un}(Y, \mathbb{Z}/\ell) = 0$.*

Proof. — Immediate from Lemma 6.1: the degree $\deg(\pi)$ is prime to ℓ and multiplication by $\deg(\pi)$ is invertible on $H^i_{k,un}(Y, \mathbb{Z}/\ell)$. □

This lemma will be applied to $Y = G\backslash V$. The goal will be to construct X with vanishing unramified cohomology.

Corollary 7.5. — *Let $G = G^F \subset \mathsf{G}(k)$ be as above. Let X be a G-linear variety over k. Assume that there exist a variety Y over k, with trivial G-action, and a G-equivariant finite morphism*

$$\pi \, : \, \mathsf{G} \times Y \to X.$$

Let S be the set of all primes dividing the degree of π. Then

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0, \quad \text{for all } i > 0 \quad \text{and} \quad \ell \notin S \cup \{p\}.$$

Proof. — Any affine connected algebraic group over $k = \bar{\mathbb{F}}_p$ is rational. By Lang's theorem, the quotient $G\backslash\mathsf{G}$ is isomorphic to G, and thus rational. For primes $\ell \notin S \cup \{p\}$ not dividing the degree of π, the induced map on cohomology is injective. This concludes the proof. □

Theorem 7.6. — *Let G be a Lie group over k. Let $G = G^F \subset \mathsf{G}(k)$ be a finite subgroup. Put*

$$s(G) := \begin{cases} \{p, 2\} & \text{for } G \text{ of type } C \text{ or } D_n, n \geq 5; \\ \{p, 2, 3\} & \text{for } G \text{ of type } D_4, F_4, E_6, E_7; \\ \{p\} & \text{otherwise.} \end{cases}$$

Then

$$H^i_{k,un}(G, \mathbb{Z}/\ell) = 0$$

for all $\ell \notin s(G)$.

We have a natural homomorphism

$$H^i(G, \mathbb{Z}/\ell) \to H^i(G\backslash\mathsf{G}, \mathbb{Z}/\ell).$$

By Lang's theorem 7.3, $G\backslash\mathsf{G} \simeq \mathsf{G}$, as algebraic varieties. Thus we get a homomorphism

$$\rho \, : \, H^i(G, \mathbb{Z}/\ell) \to H^i_{et}(\mathsf{G}, \mathbb{Z}/\ell).$$

Assume that G is semi-simple. Then $\mathrm{Pic}(\mathsf{G}) \simeq \pi_1(\mathsf{G})$ is a finite group and $\mathsf{G} = \tilde{\mathsf{G}}/\mathrm{Pic}(G)$, where $\tilde{\mathsf{G}}$ is the universal cover of G. We obtain a natural homomorphism

$$\eta \, : \, H^i_{et}(\mathrm{Pic}(\mathsf{G}), \mathbb{Z}/\ell) \to H^i_{et}(\mathsf{G}, \mathbb{Z}/\ell).$$

Theorem 7.7. — *Let G be a semi-simple Lie group over k. Let $G = G^F \subset \mathsf{G}(k)$ be a finite subgroup. Consider the diagram*

$$\begin{array}{ccc} & & H^i_{et}(\mathrm{Pic}(\mathsf{G}), \mathbb{Z}/\ell) \\ & & \Big\downarrow{\scriptstyle \sigma^i_* \circ \eta} \\ H^i(G, \mathbb{Z}/\ell) & \xrightarrow{\sigma^i_* \circ \rho} & H^i_{k,s}(\mathsf{G}, \mathbb{Z}/\ell). \end{array}$$

Then the image of ρ is contained in the image of η.

Proof. — Standard computation using restriction of the fibration $\mathsf{G} \to \mathsf{G}/\mathsf{T}$ to $\mathsf{G}^\circ = \mathsf{T} \times \mathbb{A}^N$ and the transgression homomorphism. □

Corollary 7.8. — *Assume that G is simply-connected and that the natural translation action of G on itself is affine. Then*

$$H^i_{k,s}(\mathsf{G}, \mathbb{Z}/\ell) = 0,$$

for all $i > 0$ and $\ell \nmid q$.

8. Reduction to Sylow subgroups

Let G be a finite group. For $H \subset G$ let $\mathfrak{N}_G(H)$ denote the normalizer of H. Let $\mathrm{Syl}_\ell(G)$ be an ℓ-Sylow subgroup of G. Recall the following classical result (see, e.g., [1, Section III.5]):

$$H^i(G, \mathbb{Z}/\ell) = H^i(\mathrm{Syl}_\ell(G), \mathbb{Z}/\ell)^{\mathfrak{N}_G(\mathrm{Syl}_\ell(G))}.$$

Theorem 8.1. — *Let G be a finite group. Let ℓ be a prime distinct from the characteristic of k. Then there is an isomorphism*

$$H^i_{k,s}(G, \mathbb{Z}/\ell) \xrightarrow{\sim} H^i_{k,s}(\mathrm{Syl}_\ell(G), \mathbb{Z}/\ell)^{\mathfrak{N}_G(\mathrm{Syl}_\ell(G))}.$$

Similarly,

$$H^i_{k,un}(G, \mathbb{Z}/\ell) \xrightarrow{\sim} H^i_{k,un}(\mathrm{Syl}_\ell(G), \mathbb{Z}/\ell)^{\mathfrak{N}_G(\mathrm{Syl}_\ell(G))}.$$

Proof. — Let V be a faithful representation of G over k. Then the map

$$\pi \,:\, \mathrm{Syl}_\ell(G)\backslash V \to G\backslash V$$

is a finite, separable and surjective map of degree prime to ℓ. Hence $\pi_* \circ \pi^*$ is invertible in cohomology. This implies the first claim.

The fact that local ramification indices of π are coprime to ℓ implies the second claim. □

LEMMA 8.2. — *Let G, H be finite groups. Let $\rho : H \to W$ be a faithful k-representation of H. Assume that $H\backslash W$ is stably rational. Let U be a faithful representation of $\tilde{G} := H \wr_S G$, where S is a finite G-set. Then $\tilde{G}\backslash U$ is stably birationally equivalent to $G\backslash V$, where V is a faithful representation of G.*

Proof. — Put

$$\rho_S = \oplus_{s\in S}\rho_s \, : \, H_S := \prod_{s\in S} H_s \to \mathrm{Aut}(W_S), \quad W_S = \oplus_{s\in S} W_s,$$

where $H_s = H$, $W_s = W$, for all $s \in S$, and $\rho_s = \rho$ on the factor H_s and trivial on $H_{s'}$, for $s' \neq s$. We construct $U := V' \oplus W_S$, where V' is a faithful representation of G and extend the action of G to V_S via the G-action on S. This gives a representation of $\tilde{G} = H \wr_S G$ in $\mathrm{Aut}(U)$. The quotient space is a fibration over $G\backslash V'$ with fibers $(H\backslash W)^{|S|}$. We can assume that $H\backslash W$ is rational. The action of G on $(H\backslash W)^{|S|}$ permutes the coordinates. It follows that $\tilde{G}\backslash U$ is birationally equivalent to a vector bundle over $G\backslash V'$. □

COROLLARY 8.3. — *Let $G = \mathrm{Syl}_\ell(\mathfrak{S}_n)$ and let V be a faithful representation of G. Then $G\backslash V$ is stably rational.*

Proof. — The ℓ-Sylow subgroups of $\mathfrak{S}_n$ are products of wreath products of groups $\mathbb{Z}/\ell \wr \cdots \wr \mathbb{Z}/\ell$ (see [1, VI.1]). The quotient $H\backslash W$ is rational, for a faithful representation W of $H = \mathbb{Z}/\ell$. We apply induction to conclude that the quotient $G\backslash V$ is stably rational. □

COROLLARY 8.4. — *Let $G = \mathrm{Syl}_\ell(\mathsf{GL}_n(\mathbb{F}_q))$, with $\ell \nmid q$, and let V be a faithful representation of G. Then $G\backslash V$ is stably rational.*

Proof. — The structure of ℓ-Sylow subgroups of $\mathsf{GL}_n(\mathbb{F}_q)$ is known (see [1, VII.4]: it is also a product of iterated wreath products of cyclic ℓ-groups. Thus we can apply Lemma 8.2. □

COROLLARY 8.5. — *Let k be an algebraically closed field of characteristic zero. Then*

$$H^i_{k,un}(\mathsf{GL}_n(\mathbb{F}_q), \mathbb{Z}/\ell) = 0 \quad \textit{for all} \;\; i > 0, \;\; \textit{and} \;\; \ell \nmid q.$$

REMARK 8.6. — Similar computations can be performed for some other finite groups of Lie type, e.g., for $\mathsf{O}^{\pm}_{2m}(\mathbb{F}_q)$ and $\mathsf{Sp}_n(\mathbb{F}_q)$.

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