RECONSTRUCTION OF HIGHER-DIMENSIONAL FUNCTION FIELDS

FEDOR BOGOMOLOV AND YURI TSCHINKEL

ABSTRACT. We determine the function fields of varieties of dimension ≥ 2 defined over the algebraic closure of $\mathbb{F}_p$, modulo purely inseparable extensions, from the quotient by the second term in the lower central series of their pro-ℓ Galois groups.

CONTENTS

INTRODUCTION

Fix two distinct primes p and ℓ. Let $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. Let X be an algebraic variety defined over k and $K = k(X)$ its function field. We will refer to X as a *model* of K; we will generally assume that X is normal and projective. Let $\mathcal{G}_K^a$ be the abelianization of the pro-ℓ-quotient $\mathcal{G}_K$ of the absolute Galois group of K. Under our assumptions on k, $\mathcal{G}_K^a$ is a torsion-free $\mathbb{Z}_\ell$-module isomorphic to $\mathbb{Z}_\ell^{\mathbb{N}}$. Let $\mathcal{G}_K^c$ be its canonical central extension - the second lower central series quotient of $\mathcal{G}_K$. It determines a set Σ_K of distinguished (primitive) finite-rank subgroups of $\mathcal{G}_K^a$: a topologically noncyclic subgroup $\sigma \in \Sigma_K$ iff

- the inverse image of σ in $\mathcal{G}_K^c$ is abelian;

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- σ is maximal: there are no subgroups $\sigma' \subset \mathcal{G}^a_K$ whose preimages in $\mathcal{G}^c_K$ are abelian and which contain σ as a proper subgroup.

Our main theorem is

Theorem 1. *Let K and L be function fields over algebraic closures of finite fields k, resp. l, of characteristic $\neq \ell$. Assume that the transcendence degree of K over k is at least two and that there exists an isomorphism*

$$\Psi = \Psi_{K,L} \,:\, \mathcal{G}^a_K \xrightarrow{\sim} \mathcal{G}^a_L \tag{1.1}$$

of abelian pro-ℓ-groups inducing a bijection of sets Σ_K and Σ_L. Then k is isomorphic to l and there exists a constant $\epsilon \in \mathbb{Z}^*_\ell$ such that $\epsilon^{-1} \cdot \Psi$ is induced from a unique, up to the composition with a power of the absolute Frobenius automorphism on $\bar{K}$, isomorphism of perfect closures

$$\bar{\Psi}^* \,:\, \bar{L} \xrightarrow{\sim} \bar{K}.$$

In this paper we implement the program outlined in [1] and [2] describing the correspondence between higher-dimensional function fields and their abelianized Galois groups. We follow closely our paper [4], where we treated in detail the case of surfaces: The isomorphism (1.1) of abelianized Galois groups induces canonically an isomorphism

$$\Psi^* \,:\, \hat{L}^* \xrightarrow{\sim} \hat{K}^*$$

between pro-ℓ-completions of multiplicative groups. One of the steps in the proof is to show that under the assumptions of Theorem 1, Ψ^* induces by restriction canonically an isomorphism

$$\Psi^* \,:\, L^*/l^* \otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} \left(K^*/k^* \otimes \mathbb{Z}_{(\ell)}\right)^\epsilon \subset \hat{K}^*, \quad \text{for some} \quad \epsilon \in \mathbb{Z}^*_\ell, \tag{1.2}$$

Here $\mathbb{Z}_{(\ell)}$ is the additive group of rational numbers with denominators coprime to ℓ and the exponent ϵ indicates the scaling of the lattice K^*/k^* by ϵ.

The proof of Theorem 1 proceeds by induction on the transcendence degree, using [4] as the inductive assumption. We first recover abelianized inertia and decomposition subgroups of divisorial valuations using the theory of commuting pairs developed in [3]. Then we apply the inductive assumption (1.2) to residue fields of divisorial valuations. This allows to prove that for every normally closed one-dimensional subfield $F = l(f) \subset L$ there exists a one-dimensional subfield $E \subset K$ such that

$$\Psi^*(F^*/l^* \otimes \mathbb{Z}_{(\ell)}) \subseteq \left(E^*/k^* \otimes \mathbb{Z}_{(\ell)}\right)^\epsilon,$$

for some constant $\epsilon \in \mathbb{Z}_\ell^*$, depending on F. The proof that ϵ is independent of F and, finally, the proof of Theorem 1 are then identical to those in dimension two in [4].

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2. BASIC ALGEBRA AND GEOMETRY OF FIELDS

Here we state some auxiliary facts used in the proof of our main theorem.

Lemma 2.1. *Every function field over an algebraically closed ground field admits a projective normal model.*

Lemma 2.2. *For every one-dimensional subfield $E \subset K$ there is a canonical sequence of maps from a sufficiently large normal projective model X of K*

$$X \xrightarrow{\pi_E} C' \xrightarrow{\mu_E} C,$$

where

- *C' and C are normal projective curves;*
- *π_E is dominant with irreducible generic fiber;*
- *μ_E is quasi-finite and dominant;*
- *$k(C')$ is the normal closure of E in K, and $k(C) = E$.*

Note that C' and C do not depend on the choice of suitable X.

A divisor D on a normal variety X is called p-irreducible if there exists an irreducible divisor $D' \subset X$ such that $D = p^n D'$, for some $n \in \mathbb{N} \cup \{0\}$.

Lemma 2.3. *Let C be a curve, $\pi : X \to C$ a surjective map with irreducible generic fiber, and $R \subset X$ an irreducible divisor surjecting onto C. Then the intersection $R \cdot \pi^{-1}(c)$ is a sum of p-irreducible divisors with disjoint support, for all but finitely many $c \in C$.*

Proof. This is a positive-characteristic version of Bertini's theorem (see, e.g., [7]). □

Lemma 2.4. *Let $\pi : T \to C$ be a separable map of degree m with branch locus $\{c_1, \dots, c_N\} \subset C$. Write*

$$\pi^{-1}(c_j) = \sum_{r=1}^{m_j} e_{j,r} t_{j,r}, \quad t_{j,r} \in T, e_{j,r} \in \mathbb{N}, \ \textit{and} \quad \sum_{r=1}^{m_j} e_{j,r} = m.$$

Let $e'_{j,r}$ be the maximal prime-to-p divisor of $e_{j,r}$. Assume that

$$\sum_{r=1}^{m_j}(e'_{j,r}-1) > m/2,$$

for all $j = 1, \dots, N$. Then

$$\mathrm{g}(T) > N - 3.$$

Proof. Hurwitz formula (for curves over a field of finite characteristics). □

Let $X \subset \mathbb{P}^N$ be a normal projective variety of dimension $n \geq 2$ over k. Consider the moduli space $\mathcal{M}(d)$ of complete intersection curves on X of multidegree $d = (d_1, \dots, d_{n-1})$. For $|d| \gg 0$ we have:

- for any codimension ≥ 2 subvariety $Z \subset X$ there is a Zariski open subset of $\mathcal{M}(d)$ such that every curve C parametrized by a point in this subset avoids Z and intersects every irreducible divisor $D \subset X$.

Such families will be called *families of flexible curves*.

A *Lefschetz pencil* is a surjective map

$$\lambda : X \to \mathbb{P}^1$$

from a normal variety with irreducible fibers and normal generic fiber.

Lemma 2.5. *Let $\lambda : X \to \mathbb{P}^1$ be a Lefschetz pencil on a normal projective variety. Then there exists an $m \in \mathbb{N}$ such that every irreducible normal fiber $D_t := \lambda^{-1}(t)$ contains a family of flexible curves of genus $\leq m$.*

Proof. There is a very ample line bundle H on X which has the same degree on all fibers D_t. We consider complete intersection curves on D_t with respect to the restriction of H. These curves are flexible on D_t and admit a uniform genus estimate from above. □

3. Galois groups

Let $\mathcal{G}^a_K$ the abelianization of the pro-ℓ-quotient $\mathcal{G}_K$ of the Galois group of a separable closure of $K = k(X)$,

$$\mathcal{G}^c_K = \mathcal{G}_K/[[\mathcal{G}_K, \mathcal{G}_K], \mathcal{G}_K] \xrightarrow{\mathrm{pr}} \mathcal{G}^a_K$$

its canonical central extension and pr the natural projection. By our assumptions, $\mathcal{G}^a_K$ is a torsion-free $\mathbb{Z}_\ell$-module.

Definition 3.1. *We say that $\gamma, \gamma' \in \mathcal{G}_K^a$ form a commuting pair if for some (and therefore any) of their preimages $\tilde{\gamma} \in \mathrm{pr}^{-1}(\gamma), \tilde{\gamma}' \in \mathrm{pr}^{-1}(\gamma') \in \mathcal{G}_K^c$, one has $[\tilde{\gamma}, \tilde{\gamma}'] = 0$. A subgroup $\mathcal{H}$ of $\mathcal{G}_K^a$ is called liftable if any two elements in $\mathcal{H}$ form a commuting pair. A liftable subgroup is called maximal if it is not properly contained in any other liftable subgroup.*

Definition 3.2. *The fan $\Sigma_K = \{\sigma\}$ on $\mathcal{G}_K^a$ is the set of all topologically non-cyclic maximal liftable subgroups $\sigma \subset \mathcal{G}_K^a$.*

Notation 3.3. Let

$$\boldsymbol{\mu}_{\ell^n} := \{ \sqrt[\ell^n]{1} \}$$

the group of ℓ^n-th roots of unity, the collection of these groups forms a projective system under the map $x \mapsto x^\ell$. Its projective limit

$$\mathbb{Z}_\ell(1) = \varprojlim \boldsymbol{\mu}_{\ell^n}$$

is called the Tate twist of $\mathbb{Z}_\ell$. Write

$$\hat{K}^* := \varprojlim K^*/(K^*)^{\ell^n}$$

for a similar projective limit of the multiplicative group K^*.

Theorem 3.4 (Kummer theory)**.** *For every $n \in \mathbb{N}$ we have a pairing*

$$\begin{array}{rcl} [\cdot,\cdot]_n \,:\, \mathcal{G}_K^a/\ell^n \times K^*/(K^*)^{\ell^n} & \to & \boldsymbol{\mu}_{\ell^n} \\ (\mu, f) & \mapsto & [\mu, f]_n := \mu(\sqrt[\ell^n]{f})/\sqrt[\ell^n]{f} \end{array}$$

which extends to a nondegenerate pairing

$$[\cdot,\cdot] \,:\, \mathcal{G}_K^a \times \hat{K}^* \to \mathbb{Z}_\ell(1).$$

Since k is algebraically closed of characteristic $\neq \ell$ we can choose a non-canonical isomorphism of topological Galois-modules

$$\mathbb{Z}_\ell \simeq \mathbb{Z}_\ell(1).$$

From now on we will fix such a choice.

4. Valuations

In this section we recall basic definitions and facts concerning valuations, and their inertia and decomposition subgroups of Galois groups (see [5] and [8]).

A (nonarchimedean) *valuation* $\nu = (\nu, \Gamma_\nu)$ on K is a pair consisting of a totally ordered abelian group $\Gamma_\nu = (\Gamma_\nu, +)$ (the value group) and a map

$$\nu \,:\, K \to \Gamma_{\nu,\infty} := \Gamma_\nu \cup \{\infty\}$$

such that

- $\nu : K^* \to \Gamma_\nu$ is a surjective homomorphism;
- $\nu(\kappa+\kappa') \geq \min(\nu(\kappa), \nu(\kappa'))$ for all $\kappa, \kappa' \in K$;
- $\nu(0) = \infty$.

Every valuation of $K = k(X)$ restricts to the trivial valuation on $k = \overline{\mathbb{F}}_p$.

Let $\mathfrak{o}_\nu, \mathfrak{m}_\nu$ and $\boldsymbol{K}_\nu$ be the ring of ν-integers in K, the maximal ideal of $\mathfrak{o}_\nu$ and the residue field

$$\boldsymbol{K}_\nu := \mathfrak{o}_\nu/\mathfrak{m}_\nu.$$

Basic invariants of valuations are: the $\mathbb{Q}$-rank $\mathrm{rk}_{\mathbb{Q}}(\Gamma_\nu)$ of the value group Γ_ν and the transcendence degree $\mathrm{tr}\deg_k(\boldsymbol{K}_\nu)$ of the residue field. We have:

$$\mathrm{rk}_{\mathbb{Q}}(\Gamma_\nu) + \mathrm{tr}\deg_k(\boldsymbol{K}_\nu) \leq \mathrm{tr}\deg_k(K). \tag{4.1}$$

A valuation on K has an algebraic center $\mathfrak{c}_{\nu,X}$ on every projective model X of K, i.e., the irreducible subvariety whose trace on every affine chart $U \subset X$ is defined by the prime ideal $\mathfrak{m}_\nu \cap k[U]$. There exists a projective model X where the dimension of $\mathfrak{c}_{\nu,X}$ is maximal, and equal to $\mathrm{tr}\deg_k(\boldsymbol{K}_\nu)$. A valuation ν is called *divisorial* if

$$\mathrm{tr}\deg_k(\boldsymbol{K}_\nu) = \dim(X) - 1;$$

it can be realized as the discrete rank-one valuation arising from a divisor on some normal model X of K. We let $\mathcal{V}_K$ be the set of all nontrivial (nonarchimedean) valuations of K and $\mathcal{DV}_K$ the subset of its divisorial valuations.

It is useful to keep in mind the following exact sequences:

$$1 \to \mathfrak{o}_\nu^* \to K^* \to \Gamma_\nu \to 1 \tag{4.2}$$

and

$$1 \to (1+\mathfrak{m}_\nu)^* \to \mathfrak{o}_\nu^* \to \boldsymbol{K}_\nu^* \to 1. \tag{4.3}$$

For every $\nu \in \mathcal{V}_K$ we have the diagram

$$\begin{array}{ccccc} \mathcal{I}_\nu^c & \subseteq & \mathcal{D}_\nu^c & \subset & \mathcal{G}_K^c \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_\nu^a & \subseteq & \mathcal{D}_\nu^a & \subset & \mathcal{G}_K^a, \end{array}$$

where $\mathcal{I}_\nu^c, \mathcal{I}_\nu^a, \mathcal{D}_\nu^c, \mathcal{D}_\nu^a$ are the images of the inertia and the decomposition group of the valuation ν in $\mathcal{G}_K^c$, respectively, $\mathcal{G}_K^a$; the left arrow is an isomorphism and the other arrows surjections. There are canonical isomorphisms

$$\mathcal{D}_\nu^c/\mathcal{I}_\nu^c \simeq \mathcal{G}_{\boldsymbol{K}_\nu}^c \quad \text{and} \quad \mathcal{D}_\nu^a/\mathcal{I}_\nu^a \simeq \mathcal{G}_{\boldsymbol{K}_\nu}^a.$$

The group $\mathcal{D}_\nu^c$ is the centralizer of $\mathcal{I}_\nu^c = \mathcal{I}_\nu^a$ in $\mathcal{G}_\nu^c$, i.e., $\mathcal{I}_\nu^a$ is the subgroup of elements forming a commuting pair with every element of $\mathcal{D}_\nu^a$.

For divisorial valuations $\nu \in \mathcal{DV}_K$, we have

$$\mathcal{I}_\nu^c = \mathcal{I}_\nu^a \simeq \mathbb{Z}_\ell. \tag{4.4}$$

Kummer theory, combined with equations (4.2) and (4.3) yields

$$\mathcal{I}_\nu^a = \{\gamma \in \mathrm{Hom}(K^*, \mathbb{Z}_\ell) \,|\, \gamma \text{ trivial on } \mathfrak{o}_\nu^*\} = \mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell) \tag{4.5}$$

and

$$\mathcal{D}_\nu^a = \{\gamma \in \mathrm{Hom}(K^*, \mathbb{Z}_\ell) \,|\, \gamma \text{ trivial on } (1+\mathfrak{m}_\nu)^*\}. \tag{4.6}$$

In particular,

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\mathcal{I}_\nu^a) \le \mathrm{rk}_{\mathbb{Q}}(\Gamma_\nu) \le \mathrm{tr\,deg}_k(K). \tag{4.7}$$

Two valuations ν_1, ν_2 are dependent if there exists a common coarsening valuation ν (i.e., $\mathfrak{m}_\nu$ is contained in both $\mathfrak{m}_{\nu_1}, \mathfrak{m}_{\nu_2}$), in which case

$$\mathcal{D}_{\nu_1}^a, \mathcal{D}_{\nu_2}^a \subset \mathcal{D}_\nu^a.$$

For independent valuations ν_1, ν_2 we have

$$K^* = (1+\mathfrak{m}_{\nu_1})^*(1+\mathfrak{m}_{\nu_2})^*;$$

it follows that their decomposition groups have trivial intersection.

In [3, Proposition 6.4.1, Lemma 6.4.3 and Corollary 6.4.4] we proved:

Proposition 4.1. *Every topologically noncyclic liftable subgroup σ of $\mathcal{G}_K^a$ contains a subgroup $\sigma' \subseteq \sigma$ such that there exists a valuation $\nu \in \mathcal{V}_K$ with*

$$\sigma' \subseteq \mathcal{I}_\nu^a, \quad \sigma \subseteq \mathcal{D}_\nu^a,$$

and σ/σ' topologically cyclic.

Corollary 4.2. *For every $\sigma \in \Sigma_K$ one has*

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\sigma) \le \mathrm{tr\,deg}_k(K).$$

Proof. By (4.7),

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\mathcal{I}_\nu^a) \le \mathrm{tr\,deg}_k(K).$$

We are done if $\sigma = \sigma'$. Otherwise, $\mathcal{D}_\nu^a/\mathcal{I}_\nu^a$ is nontrivial and $\mathrm{tr\,deg}_k(\boldsymbol{K}_\nu) \ge 1$. In this case, (4.7) and (4.1) yield that

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\sigma') \le \mathrm{tr\,deg}_k(K) - 1,$$

and the claim follows. □

Corollary 4.3. *Assume that for $\sigma_1, \sigma_2 \in \Sigma_K$ one has*

$$\sigma_1 \cap \sigma_2 \neq 0.$$

Then there exists a valuation $\nu \in \mathcal{V}_K$ such that

$$\sigma_1, \sigma_2 \subset \mathcal{D}^a_\nu.$$

Proof. The valuations cannot be independent. Thus there exists a common coarsening. □

This allows to recover the abelianized decomposition and inertia groups of valuations in terms of Σ_K. Here is one possible description for divisorial valuations, a straightforward generalization of the two-dimensional case treated in [4, Proposition 8.3]:

Lemma 4.4. *Let $K = k(X)$ be the function field of an algebraic variety of dimension $n \geq 2$. Let $\sigma_1, \sigma_2 \in \Sigma_K$ be liftable subgroups of rank n such that $\mathcal{I} := \sigma_1 \cap \sigma_2$ is topologically cyclic. Then there exists a unique divisorial valuation ν such that $\mathcal{I} = \mathcal{I}^a_\nu$. The corresponding decomposition group $\mathcal{D}^a_\nu \subset \mathcal{G}^a_K$ is the subgroup of elements forming a commuting pair with a topological generator of $\mathcal{I}^a_\nu$.*

Proof. Let $\nu_1, \nu_2 \in \mathcal{V}_K$ be the valuations associated to σ_1, σ_2 in Proposition 4.1. By Corollary 4.3, there exists a valuation $\nu \in \mathcal{V}_K$ such that

$$\sigma_j \subset \mathcal{D}^a_{\nu_j} \subset \mathcal{D}^a_\nu, \quad \text{for} \quad j = 1, 2.$$

Let $\mathcal{I}^a_\nu$ be the corresponding inertia subgroup, the subgroup of elements commuting with all of $\mathcal{D}^a_\nu$. In particular, $\mathcal{I}^a_\nu$ commutes with all elements of σ_1 and σ_2. Since σ_1, σ_2 are maximal liftable subgroups of $\mathcal{G}^a_K$, we obtain that

$$\mathcal{I}^a_\nu \subseteq \sigma_1 \cap \sigma_2 = \mathcal{I} \simeq \mathbb{Z}_\ell.$$

Note that $\mathcal{I}^a_\nu$ cannot be trivial; otherwise, the residue field $\boldsymbol{K}_\nu$ would contain a liftable subgroup of rank n, and have transcendence degree n, by Corollary 4.2, which is impossible. It follows that $\mathrm{rk}_{\mathbb{Z}_\ell}(\mathcal{I}^a_\nu) = 1$ and $\mathrm{tr}\deg_k(\boldsymbol{K}_\nu) \leq n - 1$.

Now we apply Corollary 4.2 to

$$\bar{\sigma}_j := \sigma_j / \mathcal{I}^a_\nu \subset \mathcal{G}^a_{\boldsymbol{K}_\nu}, \quad \text{for} \quad j = 1, 2,$$

liftable subgroups of rank $n - 1$. It follows that $\mathrm{tr}\deg_k(\boldsymbol{K}_\nu) \geq n - 1$, thus equal to $n - 1$, i.e., ν is a divisorial valuation.

Conversely, an inertia subgroup $\mathcal{I}^a_\nu$ can be embedded into maximal liftable subgroups σ_1, σ_2 as above, e.g., by considering “flag” valuation with value group $\mathbb{Z}^n$, with disjoint centers supported on the corresponding divisor $D = D_\nu \subset X$. □

The following is useful for the visualization of composite valuations:

Lemma 4.5. *Let $\nu \in \mathcal{DV}_K$ be a divisorial valuation. There is a bijection between liftable subgroups $\sigma \in \Sigma_K$ with the property that*

$$\mathcal{I}^a_\nu \subset \sigma \subseteq \mathcal{D}^a_\nu$$

and liftable subgroups $\sigma_\nu \in \Sigma_{\boldsymbol{K}_\nu}$.

Proof. We apply [4, Corollary 8.2] (whose proof is valid for arbitrary function fields): Let ν be a valuation of K and $\iota_\nu \in \mathcal{I}^a_\nu$. Let $\gamma \in \mathcal{G}^a_K$ be such that ι_ν and γ form a commuting pair. Then $\gamma \in \mathcal{D}^a_\nu$. □

In summary, under the assumptions of Theorem 1, we have obtained:

- an isomorphism of completions $\Psi^* : \hat{L}^* \xrightarrow{\sim} \hat{K}^*$ canonically induced, by Kummer theory, from the isomorphism $\Psi : \mathcal{G}^a_K \xrightarrow{\sim} \mathcal{G}^a_L$;
- a bijection on the set of inertia (and decomposition) subgroups of divisorial valuations

$$\mathcal{G}^a_K \supset \mathcal{I}^a_\nu \xrightarrow{\Psi} \mathcal{I}^a_\nu \subset \mathcal{G}^a_L.$$

Note that $K^*/k^* \subset \hat{K}^*$ determines a canonical (up to a sign) topological generator $\delta_{\nu,K} \in \mathcal{I}^a_\nu$, for all $\nu \in \mathcal{DV}_K$, by the condition that the restriction takes all integer values

$$\delta_{\nu,K} : K^*/k^* \to \mathbb{Z} \subset \mathbb{Z}_\ell$$

i.e., that there exist elements $f \in K^*/k^*$ such that $\delta_{\nu,K}(f) = 1$. A topological generator of the procyclic group $\mathcal{I}^a_\nu \simeq \mathbb{Z}_\ell$ is defined up to the action of $\mathbb{Z}^*_\ell$. We conclude that there exist constants

$$\varepsilon_\nu \in \mathbb{Z}^*_\ell, \quad \nu \in \mathcal{DV}_K = \mathcal{DV}_L$$

such that

$$\Psi(\delta_{\nu,K}) = \varepsilon_\nu \cdot \delta_{\nu,L}, \quad \forall\, \nu \in \mathcal{DV}_K. \tag{4.8}$$

The main difficulty is to show that there exists a conformally *unique* $\mathbb{Z}_{(\ell)}$-lattice, i.e., a constant $\epsilon \in \mathbb{Z}^*_\ell$, unique modulo $\mathbb{Z}^*_{(\ell)}$, such that

$$\varepsilon_\nu = \epsilon, \quad \forall \nu \in \mathcal{DV}_K.$$

A proof of this fact will be carried out in Section 6.

Let ν be a divisorial valuation. Passing to ℓ-adic completions in sequence (4.2) we obtain an exact sequence

$$1 \to \hat{\mathfrak{o}}_\nu^* \to \hat{K}^* \xrightarrow{\hat{\nu}} \mathbb{Z}_\ell \to 0.$$

The sequence (4.3) gives rise to a surjective homomorphism

$$\hat{\mathfrak{o}}_\nu^* \to \hat{\boldsymbol{K}}_\nu^*.$$

Combining these, we obtain a surjective homomorphism

$$\mathrm{res}_\nu \ : \ \mathrm{Ker}(\hat{\nu}) \to \hat{\boldsymbol{K}}_\nu^*. \tag{4.9}$$

This homomorphism has a Galois-theoretic description, via duality arising from Kummer theory: We have

$$\mathcal{I}_\nu^a \subset \mathcal{D}_\nu^a \subset \mathcal{G}_K^a,$$

and

$$\hat{\boldsymbol{K}}_\nu^* = \mathrm{Hom}(\mathcal{G}^a_{\boldsymbol{K}_\nu}, \mathbb{Z}_\ell) = \mathrm{Hom}(\mathcal{D}_\nu^a/\mathcal{I}_\nu^a, \mathbb{Z}_\ell);$$

each $\hat{f} \in \mathrm{Ker}(\nu) \subset \hat{K}^* = \mathrm{Hom}(\mathcal{G}_K^a, \mathbb{Z}_\ell)$ gives rise to a well-defined element in $\mathrm{Hom}(\mathcal{D}_\nu^a/\mathcal{I}_\nu^a, \mathbb{Z}_\ell)$.

5. ℓ-ADIC ANALYSIS: GENERALITIES

Here we recall the main issues arising in the analysis of ℓ-adic completions of functions, divisors, and Picard groups of normal projective models X of function fields $K = k(X)$ (see [4, Section 11] for more details).

We have an exact sequence

$$0 \to K^*/k^* \xrightarrow{\mathrm{div}_X} \mathrm{Div}(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \to 0, \tag{5.1}$$

where $\mathrm{Div}(X)$ is the group of Cartier divisors of X and $\mathrm{Pic}(X)$ is the Picard group. Write $\mathrm{Div}^0(X)$ for the group of divisors algebraically equivalent to zero, in particular, of degree zero upon restriction to every curve $C \to X$. We will identify an element $f \in K^*/k^*$ with its image under div_X. Let

$$\widehat{\mathrm{Div}}(X)$$

be the pro-ℓ-completion of $\mathrm{Div}(X)$ and put

$$\mathrm{Div}(X)_\ell := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \subset \widehat{\mathrm{Div}}(X).$$

Every element $\hat{f} \in \hat{K}^*$ has a representation

$$\hat{f} = (f_n)_{n \in \mathbb{N}} \text{ or } \hat{f} = f_0 f_1^\ell f_2^{\ell^2} \cdots ,$$

with $f_n \in K^*$. We have homomorphisms

$$\begin{array}{rcll} \mathrm{div}_X & : & \hat{K}^* & \to & \widehat{\mathrm{Div}}(X), \\ & & \hat{f} & \mapsto & \mathrm{div}_X(\hat{f}) := \sum_{n\in\mathbb{N}\cup\{0\}} \ell^n \cdot \mathrm{div}_X(f_n) = \sum_m \hat{a}_m D_m, \end{array}$$

where $D_m \subset X$ are irreducible divisors,

$$\hat{a}_m = \sum_{n\in\mathbb{N}\cup\{0\}} a_{nm}\ell^n \in \mathbb{Z}_\ell, \quad a_{nm} \in \mathbb{Z}.$$

Equation (5.1) gives rise to an exact sequence

$$0 \to K^*/k^* \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{div}_X} \mathrm{Div}^0(X)_\ell \xrightarrow{\varphi_\ell} \mathrm{Pic}^0(X)\{\ell\} \to 0, \tag{5.2}$$

where

$$\mathrm{Div}^0(X)_\ell := \mathrm{Div}(X)^0 \otimes \mathbb{Z}_\ell, \quad \text{and} \quad \mathrm{Pic}^0(X)\{\ell\} = \mathrm{Pic}^0(X) \otimes \mathbb{Z}_\ell$$

is the ℓ-primary component of the torsion group $\mathrm{Pic}^0(X)$. The assignment

$$\mathcal{T}_\ell(X) := \varprojlim \mathrm{Tor}_1(\mathbb{Z}/\ell^n, \mathrm{Pic}^0(X)\{\ell\}).$$

is functorial:

$$Y \to X \quad \Rightarrow \quad \mathcal{T}_\ell(X) \to \mathcal{T}_\ell(Y). \tag{5.3}$$

We have $\mathcal{T}_\ell(X) \simeq \mathbb{Z}_\ell^{2\mathrm{g}}$, where g is the dimension of $\mathrm{Pic}^0(X)$. Let $\widehat{\mathrm{Div}}^0(X)$ be the ℓ-completion of $\mathrm{Div}^0(X)$. Note that $\widehat{\mathrm{Div}}^0(X)$ embedds into $\widehat{\mathrm{Div}}(X)$ since the Néron-Severi group $\mathrm{NS}(X)$ is finitely-generated. Passing to pro-ℓ-completions in (5.2) we obtain an exact sequence:

$$0 \to \mathcal{T}_\ell(X) \to \hat{K}^* \xrightarrow{\mathrm{div}_X} \widehat{\mathrm{Div}}^0(X) \longrightarrow 0, \tag{5.4}$$

since $\mathrm{Pic}^0(X)$ is an ℓ-divisible group. Note that all groups in this sequence are torsion-free. We have a diagram

(5.5)

$$\begin{array}{ccccccccccc} & & 0 & \to & K^*/k^* \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{div}_X} & \mathrm{Div}^0(X)_\ell & \xrightarrow{\varphi_\ell} & \mathrm{Pic}^0(X)\{\ell\} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{T}_\ell(X) & \to & \hat{K}^* & \xrightarrow{\mathrm{div}_X} & \widehat{\mathrm{Div}}^0(X) & \xrightarrow{\hat{\varphi}} & 0. & & \end{array}$$

Every $\nu \in \mathcal{DV}_K$ gives rise to a homomorphism

$$\hat{\nu} \,:\, \hat{K}^* \to \mathbb{Z}_\ell.$$

On a normal model X, where $\nu = \nu_D$ for some divisor $D \subset X$, $\hat{\nu}(\hat{f})$ is the ℓ-adic coefficient at D of $\mathrm{div}(\hat{f})$.

The following lemma generalizes [4, Lemmas 11.2 and 11.4] to normal varieties.

Lemma 5.1. *Let K be a function field over k. Then there exists a normal projective model X of K such that for all birational maps $\tilde{X} \to X$ from a normal variety $\tilde{X}$ one has a canonical isomorphism*

$$\mathcal{T}_\ell(X) \to \mathcal{T}_\ell(\tilde{X}).$$

In particular, $\mathcal{T}_\ell(X)$ is an invariant of K. Moreover, we have

$$\mathcal{T}_\ell(X) = \mathcal{T}_\ell(K) = \cap_{\nu \in \mathcal{DV}_K} \mathrm{Ker}(\hat{\nu}) \subset \hat{K}^*. \tag{5.6}$$

Proof. For any projective X, its Albanese $\mathrm{Alb}(X)$ is an abelian variety endowed with a universal morphism $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$, i.e., for any map $\alpha\colon X \to A$ to an abelian variety there exists a map $h\colon \mathrm{Alb}(X) \to A$ such that $h \circ \mathrm{alb} = \alpha$ (see [6, Chapter II, pp. 41–50] for the background). This construction is functorial with respect to morphisms between projective varieties. Thus, given a function field K there exits a natural tower $\mathrm{Alb}(X_i)$ of such varieties for a projective system of projective normal models. This tower is bounded since all $\mathrm{Alb}(X_i)$ are dominated by the Jacobian of a sufficiently general curve $C \subset X_i$. Thus there exists a maximal abelian variety $\mathrm{Alb}(K)$ dominating all $\mathrm{Alb}(X)$. It suffices to observe that $\mathcal{T}_\ell(X) = \mathcal{T}_\ell(\mathrm{Alb}(K))$.

The second claim follows from the exactness of the sequence (5.4) and the fact that every divisorial valuation can be realized as a divisor on a normal model X of K. □

Lemma 5.2. *Let $K = k(X)$ be the function field of a normal projective variety $X \subset \mathbb{P}^N$ of dimension ≥ 3. For every divisorial valuation $\nu \in \mathcal{DV}_K$ there is a canonical homomorphism:*

$$\xi_{\nu,\ell} : \mathcal{T}_\ell(K) \to \mathcal{T}_\ell(\boldsymbol{K}_\nu).$$

Assume that ν corresponds to an irreducible normal hyperplane section of X. Then $\xi_{\nu,\ell}$ is an isomorphism.

Proof. The map is induced from a canonical map of Albanese varieties (see [4, Lemma 11.2]). It suffices to apply Lefschetz' theorem. □

Lemma 5.3. *Let $\lambda : X \to \mathbb{P}^1$ be a Lefschetz pencil on a normal variety of dimension ≥ 3 and $D_t = \lambda^{-1}(t)$. Then:*

(1) *For all but finitely many $t \in \mathbb{P}^1$,*

$$\xi_{D_t,\ell} : \mathcal{T}_\ell(X) \xrightarrow{\sim} \mathcal{T}_\ell(D_t),$$

is an isomorphism.

(2) *For any* $t \in \mathbb{P}^1$ *and any surjection* $D_t \to C_t$ *onto a smooth projective curve we have* $\mathsf{g}(C_t) \le \mathrm{rk}_{\mathbb{Z}_\ell}(\mathcal{T}_\ell(X))$.

Proof. Follows from standard facts for general hyperplane sections of normal varieties (see Lemma 5.2). □

Lemma 5.4. *Let* X *be a normal variety,* C *a curve, and* $\pi : X \to C$ *a surjective map with irreducible fibers. Assume that* $\hat{f} \in \mathrm{Ker}(\hat{\nu})$ *and that* $\mathrm{res}_\nu(\hat{f}) = 1 \in \hat{\boldsymbol{K}}^*_\nu$*, for infinitely many* $\nu \in \mathcal{DV}_K$ *corresponding to fibers of* π*. Then* $\hat{f}$ *is induced from* $\widehat{k(C)}^*$.

Proof. Assume that $\hat{f} \mod \ell^n$, for some $n \in \mathbb{N}$, contains a summand corresponding to a horizontal divisor R. By Lemma 2.3, R intersects all but finitely many fibers p^m-transversally. In particular, $\mathrm{div}_X(\hat{f})$ intersects infinitely many fibers nontrivially, contradiction to the assumption. Thus $\mathrm{div}_X(\hat{f})$ is a sum of vertical divisors.

Hence $\hat{f} = \tau + \hat{g}$, where $\hat{g} \in \widehat{k(C)}^*$, and $\tau \in \mathcal{T}_\ell(K)$. The triviality of τ on fibers $D_c = \pi^{-1}(c)$ implies that τ is induced from the image of X in $\mathrm{Alb}(X)/\mathrm{Alb}(D_c)$. In particular, the triviality on infinitely many fibers implies that it is induced from the Jacobian $J(C)$ and hence $\hat{f} \in \widehat{k(C)}^*$. □

Notation 5.5. Let X be a normal projective model of K. For $\hat{f} \in \hat{K}^*$ with

$$\mathrm{div}_X(\hat{f}) = \sum_m \hat{a}_m D_m$$

we put

$$\begin{array}{rlcl} \mathrm{supp}_K(\hat{f}) & := \{ \quad \nu \in \mathcal{DV}_K & | & \hat{f} \text{ nontrivial on } \mathcal{I}^a_\nu \ \}; \\ \mathrm{supp}_X(\hat{f}) & := \{ \quad D_m \subset X & | & \hat{a}_m \neq 0 \ \}; \\ \mathrm{fibr}(\hat{f}) & := \{ \quad \nu \in \mathcal{DV}_K & | & \hat{f} \in \mathrm{Ker}(\hat{\nu}) \text{ and } \mathrm{res}_\nu(\hat{f}) = 1 \in \hat{\boldsymbol{K}}^*_\nu \ \}, \end{array}$$

where res_ν is the projection from Equation (4.9). Note that the *finiteness* of $\mathrm{supp}_X(\hat{f})$ does not depend on the choice of the normal model X. Put

$$\mathrm{supp}'_K(\hat{f}) := \mathrm{fibr}(\hat{f}) \cup \mathrm{supp}_K(\hat{f}).$$

If X is a normal model of K write

$$\mathrm{supp}'_X(\hat{f}) \subset \mathrm{supp}'_K(\hat{f})$$

for the subset of divisorial valuations realized by divisors on X. We have

$$\mathrm{supp}'_K(\hat{f}) = \cup_X \ \mathrm{supp}'_X(\hat{f}).$$

Definition 5.6. *A K-divisor is a function*

$$\mathcal{DV}_K \to \mathbb{Z}_\ell.$$

Each $\hat{f} \in \hat{K}^*$ defines a K-divisor by

$$\mathrm{div}_K(\hat{f}) \colon \quad \nu \mapsto [\delta_{\nu,K}, \hat{f}].$$

The different notions of support for elements in $\hat{K}^*$ introduced in Notation 5.5 extend naturally to K-divisors. The divisor of $\hat{f}$ on a normal model X of K coincides with the restriction of $\mathrm{div}_K(\hat{f})$ to the set of divisorial valuations of K which are realized by divisors on X. In particular, it has finite support on X modulo ℓ^n, for any $n \in \mathbb{N}$. (This fails for general K-divisors.)

Let $E \subset K$ be a one-dimensional subfield and $\pi_E : X \to C$ the corresponding surjective map with irreducible generic fiber. For all nontrivial $\hat{f}_1, \hat{f}_2 \in \hat{E}^*$, we have

$$\mathrm{supp}'_K(\hat{f}_1) = \mathrm{supp}'_K(\hat{f}_2).$$

This gives a well-defined invariant of $\hat{E}^*$. We have a decomposition

$$\mathrm{supp}'_K(\hat{E}^*) = \sqcup_{c \in C} \mathrm{supp}'_{K,c}(\hat{E}^*), \tag{5.7}$$

where $\mathrm{supp}'_{K,c}(\hat{E}^*)$ are minimal nonempty subsets of the form

$$\mathrm{supp}_K(\hat{f}_1) \cap \mathrm{supp}_K(\hat{f}_2)$$

contained in $\mathrm{supp}'_K(\hat{E}^*)$; these correspond to sets of irreducible divisors supported in $\pi_E^{-1}(c)$, for $c \in C(k)$. Note that $\mathrm{supp}'_K(\hat{E}^*)$ depends only on the normal closure of E in K. On the other hand, the decomposition (5.7) is preserved only under purely inseparable extensions of E. We formalize this discussion in the following definition.

Definition 5.7. *A formal projection is a triple*

$$\pi_{\hat{E}} = (C, \{R_c\}_{c \in C}, Q),$$

where C is an infinite set, $\{R_c\}_{c \in C}$ is a set of K-divisors, and $Q \subset \hat{K}^*$ a subgroup of $\mathbb{Z}_\ell$-rank at least two satisfying the following properties:

(1) *for all $\hat{f}_1, \hat{f}_2 \in Q$ one has $\mathrm{supp}'_K(\hat{f}_1) = \mathrm{supp}'_K(\hat{f}_2)$;*
(2) $\mathrm{supp}_K(R_{c_1}) \cap \mathrm{supp}_K(R_{c_2}) = \emptyset$*, for all pairs of distinct $c_1, c_2 \in C$;*
(3) *for all nontrivial $\hat{f} \in Q$ one has*

$$\mathrm{div}_K(\hat{f}) = \sum_{c \in C} a_c R_c, \quad a_c \in \mathbb{Z}_\ell,$$

and

$$\cup_{c\in C}\mathrm{supp}_K(R_c) = \mathrm{supp}'_K(\hat{f});$$

(4) *for all* $c_1, c_2 \in C$ *there exists an* $m \in \mathbb{N}$ *such that*

$$m(R_{c_1} - R_{c_2}) = \mathrm{div}_K(\hat{f}),$$

for some $\hat{f} \in Q$.

Example 5.8. A one-dimensional subfield $E = k(C) \subset K$ defines a formal projection $\pi_{\hat{E}} = (C, \{R_c\}_{c\in C}, Q)$, with C the set of k-points of the image of π_E, R_c the intrinsic K-divisors over $c \in C$, and $Q = \hat{E}^*$.

Note that for normally closed subfields $E \subset K$, the corresponding subgroup Q is maximal, for subgroups of $\hat{K}^*$ appearing in formal projections.

Lemma 5.9. *For any model* X *of* K*, the support of the formal divisor* R_c *on* X *is finite* $\mod \ell^n$.

Proof. The support of $\mathrm{div}_X(\hat{f}) \mod \ell^n$ is finite for all $n \in \mathbb{N}$. Now observe that the K-divisors R_c have disjoint support in $\mathrm{supp}'_K(Q)$, thus have no components in common. □

6. One-dimensional subfields

We recall the setup of Theorem 1:

$$\Psi : \mathcal{G}^a_K \to \mathcal{G}^a_L.$$

Our goal here is to show:

$$\begin{array}{ccc} \hat{L}^* & \xrightarrow{\Psi^*} & \hat{K}^* \\ \uparrow & & \uparrow \\ L^*/l^* & \longrightarrow & (K^*/k^*)^{\epsilon} \end{array}$$

for some constant ϵ. We know that $g \in K^*/k^* \otimes \mathbb{Z}_\ell$ have finite support $\mathrm{supp}_X(g)$, on every normal model X of K. In the second half of this section we will prove:

Proposition 6.1 (Finiteness of support)**.** *For all* $f \in L^*/l^*$ *and all normal models* X *of* K *the support* $\mathrm{supp}_X(\Psi^*(f))$ *is finite.*

Assuming this, we will prove:

Proposition 6.2 (Image of Ψ^*)**.** *For all* $f \in L^*/l^*$ *there exist a function* $g \in K^*/k^*$ *and constants* $N \in \mathbb{N}$*,* $\alpha \in \mathbb{Z}_\ell$ *such that*

$$\Psi^*(f)^N = g^\alpha. \tag{6.1}$$

Moreover, there exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ *such that*

$$\Psi^*(l(f)^*/l^* \otimes \mathbb{Z}_{(\ell)}) \subseteq \left(k(g)^*/k^* \otimes \mathbb{Z}_{(\ell)}\right)^{\epsilon}.$$

Considerations in Section 4 imply that under the assumptions of Theorem 1 we have a canonical commutative diagram, for every $\nu \in \mathcal{DV}_K$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_\ell(L) & \longrightarrow & \mathrm{Ker}(\hat{\nu}) & \longrightarrow & \hat{\boldsymbol{L}}_\nu^* & \supset & \boldsymbol{L}_\nu^*/l^* \otimes \mathbb{Z}_{(\ell)} \\ & & \downarrow{\scriptstyle \Psi^*} & & \|{\scriptstyle \Psi_\nu^*} & & \downarrow{\scriptstyle \Psi_\nu^*} & & \downarrow{\scriptstyle \Psi_\nu^*} \\ 0 & \longrightarrow & \mathcal{T}_\ell(K) & \longrightarrow & \mathrm{Ker}(\hat{\nu}) & \longrightarrow & \hat{\boldsymbol{K}}_\nu^* & \supset & \left(\boldsymbol{K}_\nu^*/k^* \otimes \mathbb{Z}_{(\ell)}\right)^{\epsilon}, \end{array}$$

for some constant $\epsilon \in \mathbb{Z}_\ell^*$, depending on ν. By [4, Proposition 12.10], the left vertical map is an isomorphism canonically induced by Ψ. In both proofs (Finiteness of support and Image of Ψ^*) we will apply the inductive assumption (1.2) to residue fields of appropriate divisorial valuations.

Proof of Proposition 6.2. Let X be a normal projective model of K and put $\hat{f} := \Psi^*(f)$. By Proposition 6.1, we may assume that $\mathrm{supp}_X(\hat{f})$ is finite, i.e.,

$$\mathrm{div}(\hat{f}) = \sum_{j \in J} d_j D_j,$$

where J is a finite set, $d_j \in \mathbb{Z}_\ell$ and D_i are irreducible divisors on X. A priori, we do not know that $\mathrm{div}(\hat{f}) \in \mathrm{Div}^0(X)_\ell$. Since the support of $\hat{f}$ on X is finite, $\mathrm{div}_X(\hat{f}) \in \mathrm{Div}^0(X)_\ell \subset \widehat{\mathrm{Div}^0}(X)$, as $\widehat{\mathrm{Div}^0}(X) \cap \mathrm{Div}(X)_\ell = \mathrm{Div}^0(X)_\ell$. Furthermore, since $\mathrm{Pic}^0(X)$ is torsion, there exists an $N \in \mathbb{N}$ such that

$$\mathrm{div}_X(\hat{f}^N) \in K^*/k^* \otimes \mathbb{Z}_\ell \subseteq \mathrm{Div}^0(X)_\ell.$$

By (5.5), we have

$$\hat{f}^N = t_{\hat{f}} \cdot \prod_{i \in I} g_i^{a_i},$$

with I a finite set, $a_i \in \mathbb{Z}_\ell$ linearly independent over $\mathbb{Z}_{(\ell)}$, $g_i \in K^*/k^*$ multiplicatively independent, and $t_{\hat{f}} \in \mathcal{T}_\ell(K)$.

The projective model X contains a hyperplane section $D \subset X$ such that

$$\mathcal{T}_\ell(K) = \mathcal{T}_\ell(X) = \mathcal{T}_\ell(D),$$

under the natural restriction isomorphism $\xi_{D,\ell}$ from Lemma 5.3, and the restrictions of g_i to D are multiplicatively independent in $k(D)^*/k^* = \boldsymbol{K}_\nu^*/k^*$, where $\nu = \nu_D$.

By the construction and the inductive assumption, we have $\mathrm{res}_\nu(\hat{f}^N) = g_\nu^{b_\nu}$, where $b_\nu \in \mathbb{Z}_\ell$, $g_\nu \in \boldsymbol{K}_\nu^*$:

$$\mathrm{res}_\nu(\hat{f}^N) = \mathrm{res}_\nu(t_{\hat{f}}) \cdot \prod_{i\in I} \mathrm{res}_\nu(g_i)^{a_i} = g_\nu^{b_\nu}.$$

In particular, $\mathrm{res}_\nu(t_{\hat{f}}) = 1$ and hence $t_{\hat{f}} = 1$. Since $\mathrm{res}_\nu(g_i) \in \boldsymbol{K}_\nu^*$ are independent, it follows that $\#I = 1$ and

$$\hat{f}^N = g^a, \quad g \in K^*/k^*, \quad a \in \mathbb{Z}_\ell.$$

This proves the first claim.

The function $g \in K^*/k^*$ defines a map $\pi : X \to C$ from some normal model of K onto a curve, with generically irreducible fibers. For each $h \in l(f)^*/l^*$, consider $\mathrm{div}_X(\Psi^*(h)) \in \widehat{\mathrm{Div}}^0(X)$. Then divisors in $\mathrm{div}_X(\Psi^*(h))$ are π-vertical. Indeed, the restriction of g to a π-horizontal component D would be defined and nontrivial. On the other hand, the restriction of f to D is either not defined or trivial, contradiction. By Lemma 5.4, $\Psi^*(h) \in \widehat{k(C)}^* \supseteq \widehat{k(g)}^*$.

Let $\nu = \nu_D$ be a divisorial valuation such that f is defined and nontrivial on D. Then

$$f \in \boldsymbol{L}_\nu^*/l^* \text{ and } g \in \boldsymbol{K}_\nu^*/k^*,$$

and

$$\hat{\boldsymbol{L}}_\nu^* \supset \widehat{l(f)}^* \xrightarrow{\Psi_\nu^*} \widehat{k(g)}^* \subset \hat{\boldsymbol{K}}_\nu^*.$$

By the inductive assumption, this implies that there exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that

$$\Psi_\nu^*(l(f)^*/l^* \otimes \mathbb{Z}_{(\ell)}) \subseteq \left(k(g)^*/k^* \otimes \mathbb{Z}_{(\ell)}\right)^\epsilon,$$

(see, e.g., [4, Proposition 13.1]). □

We now prove Proposition 6.1. Fix a normal projective model Y of L. The subfield $F = l(f)$ determines a surjective map $\pi_F : Y \to C$ with irreducible generic fibers. For each $c \in C$ we have an intrinsically defined formal sum

$$R_c = \sum_{\nu \in \mathcal{DV}_{L,c}} a_{c,\nu} R_{c,\nu}, \quad a_{c,\nu} \in \mathbb{N} \cup \{0\}, \tag{6.2}$$

where $\mathcal{DV}_{L,c} \subset \mathcal{DV}_L = \mathcal{DV}_K$ is the subset of divisorial valuations supported in the fiber over c, $R_{c,\nu}$ is a divisor on some model $\tilde{Y} \to Y$ realizing ν, and $a_{c,\nu}$ are local degrees. Note that R_c do not depend on the model Y, and that R_{c_1} and R_{c_2} have no common components, for $c_1 \neq c_2$. Furthermore, the sets $\mathcal{DV}_{L,c}$ have an intrinsic Galois-theoretic characterization in terms of $\hat{F}^*$: these are minimal nonempty subsets of the form

$$\mathrm{supp}_K(\hat{f}_1) \cap \mathrm{supp}_K(\hat{f}_2), \quad f_1, f_2 \in \hat{F}^*,$$

contained in $\mathrm{supp}'_K(\hat{F}^*)$.

For each model $\tilde{Y} \to Y$ we have a map

$$R_c \mapsto R_{\tilde{Y},c} := \sum_{\nu \,:\, D_\nu \in \mathrm{Div}(\tilde{Y})} a_{c,\nu} R_{c,\nu},$$

the fiber over c. The divisor of a function $f \in F^*/l^*$ on this model can be written as a finite sum

$$\mathrm{div}_{\tilde{Y}}(f) = \sum n_c R_{\tilde{Y},c}, \quad n_c \in \mathbb{Z}.$$

Given $\{\delta_{\nu,L}\}$ as in Section 4, each $\hat{f} \in \hat{L}^*$ defines a $\mathbb{Z}_\ell$-valued function on $\mathcal{DV}_L$ by the Kummer-pairing from Theorem 3.4

$$\begin{array}{ccc} \mathcal{DV}_L & \to & \mathbb{Z}_\ell \\ \nu & \mapsto & [\delta_{\nu,L}, \hat{f}]. \end{array} \tag{6.3}$$

Similarly, each R_c defines a function on $\mathcal{DV}_L$ by setting

$$\nu \mapsto \delta_{\nu,L} \cdot R_c = \delta_{\nu,L}(t),$$

where t is a local parameter along c if ν is supported over c, and $\nu \mapsto 0$, otherwise.

For $\hat{f} \in \hat{F}^* \subset \hat{L}^*$ write

$$\mathrm{div}_C(\hat{f}) = \sum_{c \in C} b_{\hat{f},c} c, \quad b_{f,c} \in \mathbb{Z}_\ell,$$

with "decreasing" coefficients $b_{\hat{f},c}$. Then (6.3) is given by

$$\nu \mapsto b_{\hat{f},c} a_{\nu,c}.$$

We face the following difficulty: we don't know the image $\Psi^*(F^*/l^*)$ in $\hat{K}^*$, and in particular, we don't know that $\Psi^*(R_c)$, resp. $\Psi^*(R_{\tilde{Y},c})$, as functions on $\mathcal{DV}_K$, correspond to fibers of any fibration on a model X of K. However, as explained in Section 4, we know the "action" of Ψ^* on the coefficients in Equation (6.2):

$$a_{c,\nu} \mapsto \varepsilon_\nu^{-1} a_{c,\nu}.$$

Lemma 6.3. *Either there is a nonconstant $f \in F^*/l^*$ such that $\mathrm{supp}_X(\Psi^*(f))$ is finite or there is at most one $c \in C$, where C corresponds to F, such that $\Psi^*(R_c)$ has finite support on every model X of K.*

Proof. Let $c_1, c_2 \in C$ be distinct points such that

$$\mathrm{supp}_X(\Psi^*(R_c)) \cup \mathrm{supp}_X(\Psi^*(R_{c'}))$$

is finite. Then there is a function f with divisor supported in this set, thus finite $\mathrm{supp}_X(\Psi^*(f))$. □

Proof of Proposition 6.1. By contradiction. Assume that $\mathrm{supp}_X(\Psi^*(f))$ is infinite. An argument as in the proof of Proposition 6.2 shows that the same holds for every nonconstant $h \in l(f)^*/l^*$.

Fix a Lefschetz pencil $\lambda : X \to \mathbb{P}^1$ such that for almost all fibers D_t of λ we have a well-defined

$$\mathrm{res}_\nu : l(f)^*/l^* \to \boldsymbol{L}^*_{\nu_t} \xrightarrow{\Psi^*} \widehat{\boldsymbol{K}}^*_{\nu_t},$$

where ν_t is the divisorial valuation corresponding to D_t. By the inductive assumption, there exist one-dimensional closed subfields $E_t = k(C_t) \subset k(D_t) = \boldsymbol{K}_{\nu_t}$ such that

$$\Psi^*(\mathrm{res}_{\nu_t}(l(f)^*/l^*) \otimes \mathbb{Z}_{(\ell)}) \subseteq \left(E_t^* \otimes \mathbb{Z}_{(\ell)}\right)^{\epsilon_t}, \quad \epsilon_t \in \mathbb{Z}_\ell^*.$$

We have an induced surjective map

$$\pi_t : D_t \to C_t$$

as in Lemma 2.2. Passing to a finite purely-inseparable cover of C_t we may assume that π_t is separable (this effects the constant ϵ by multiplication by a power of p which is in $\mathbb{Z}_\ell^*$). We identify the sets $C(k)$ and $C_t(k)$, set-theoretically.

Fix a family of flexible curves $\{T_t\}$ uniformly on all but finitely many D_t as in Lemma 2.5 and let m be the bound on the genus of these curves obtained in this Lemma. Put $N := m + 4$ and choose $c_1, \ldots, c_N \in C_t(k) = C(k)$ such that $\mathrm{supp}_X(R_{c_j})$ is infinite for all j, this is possible by Lemma 6.3.

For each c_j express the fiber over c_j as

$$R_{c_j} := \sum_{e=0}^{\infty} \ell^e R_{c_j,e}, \quad R_{c_j,e} := \sum_{i \in I_{e,j}} \epsilon_{i,e,j} R_{i,e,j},$$

where $I_{e,j}$ are finite, and $R_{i,e,j}$ irreducible divisors over c_j, and $\epsilon_{i,e,j} \in \mathbb{Z}_\ell^*$ (see Lemma 5.9). Let $S_{c_j,e} = \cup R_{i,e,j}$ be the support of $R_{c_j,e}$. Note that T_t intersect all $S_{c_j,e}$ and write $d_{j,e} := \deg(S_{c_j,e} \cdot T_t)$ for the degree of the intersection. Choose M such that for all $j = 1, \dots, N$ one has

$$d_{j,0} < \sum_{e=1}^{M} d_{j,e}, \tag{6.4}$$

this is possible since the number of components over all c_j is infinite. Using Lemma 2.3 choose t so that the intersections

$$R_{i,e,j,t} := D_t \cdot R_{i,e,j}$$

are p-irreducible and pairwise distinct, this holds for all but finitely many t. Choose a flexible curve $T_t \subset D_t$ such that

- T_t does not pass through the points of indeterminacy of $\pi_t : D_t \to C_t$;
- T_t is not contained in any of the $R_{i,e,j,t}$;
- T_t does not pass through pairwise intersections of these divisors.

Consider the restriction

$$\pi_t : T_t \to C_t.$$

By the choice of T_t, the number of nonramified points over each c_j is at most $d_{j,0}$. On the other hand, the ramification index over c_j is at least $\ell \cdot \sum_{e=1}^{m} d_{j,e}$. By the choice (6.4), combined with Hurwitz formula in Lemma 2.4, we obtain that $\mathrm{g}(T_t) > m$, contradicting the universal bound. □

Proposition 6.4. *There exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that*

$$\Psi^*(L^*/l^* \otimes \mathbb{Z}_{(\ell)}) = (K^*/k^* \otimes \mathbb{Z}_{(\ell)})^{\epsilon}. \tag{6.5}$$

Proof. By Proposition 6.2, for each one-dimensional subfield $F = l(f) \subset L$ there exists a one-dimensional subfield $E = k(g)$ and a constant $\epsilon_F \in \mathbb{Z}_\ell^*$ such that

$$\Psi^*(F^*/l^* \otimes \mathbb{Z}_{(\ell)}) \subseteq (E^*/k^* \otimes \mathbb{Z}_{(\ell)})^{\epsilon_F}.$$

We claim that ϵ_F does not depend on F, modulo $\mathbb{Z}_{(\ell)}^*$. For $f_1, f_2 \in L^*/l^* \subset \hat{L}^*$ and $f_3 := f_1 f_2$ let $\Psi^*(f_j) = g_j^{\alpha_j}$, for $g_j \in K^*/k^* \subset \hat{K}^*$ and $\alpha_j \in \mathbb{Z}_\ell^*$.

We want to show that α_j/α_i are rational and hence contained in $\mathbb{Z}_{(\ell)}^*$. We have an equality of K-divisors:

$$\mathrm{div}_K(\Psi^*(f_1)) + \mathrm{div}_K(\Psi^*(f_2)) = \mathrm{div}_K(\Psi^*(f_3)),$$

since $f_1 f_2 = f_3 \in L^*/l^*$. We have

$$\alpha_j \mathrm{div}_K(g_j) = \mathrm{div}_K(\Psi^*(f_j)).$$

For every model X of K we have

$$\alpha_1 \mathrm{div}_X(g_1) + \alpha_2 \mathrm{div}_X(g_2) = \alpha_3 \mathrm{div}_X(g_3),$$

where div_X is obtained from div_K by removing contributions from all divisorial valuations which are not represented by divisors on a normal projective model X. This can be rewritten as an equality between coefficients for irreducible divisors D_r on such models X:

$$\alpha_1 n_{1,r} + \alpha_2 n_{2,r} = \alpha_3 n_{3,r}, \tag{6.6}$$

where $n_{j,r}$ is the multiplicity of a component D_r in the divisor of g_j on X. The rank of the matrix $(n_{j,r})$ cannot be equal to 3, due to the relation (6.6). If the rank is 2, then the ratios α_j/α_i are all $\mathbb{Q}$-rational, and hence are contained in $\mathbb{Z}_{(\ell)}^*$. If the rank is one, all g_j are powers of the same element, and the same property holds for f_j; hence $\alpha_i = \alpha_j$.

Applying the same arguments to the inverse isomorphism $(\Psi^*)^{-1}$ we obtain the claim.

□

7. Proof

In this section we prove our main theorem.

Step 1. We have a nondegenerate pairing

$$\mathcal{G}_K^a \times \hat{K}^* \to \mathbb{Z}_\ell(1).$$

This induces canonically an isomorphism

$$\Psi^* : \hat{L}^* \to \hat{K}^*.$$

Step 2. By assumption, $\Psi : \mathcal{G}_K^a \to \mathcal{G}_L^a$ is bijective on the set of liftable subgroups, in particular, it maps liftable subgroups $\sigma \in \Sigma_K$ to a liftable subgroups of the same rank. In Section 4 we identify intrinsically the inertia and decomposition groups of divisorial valuations:

$$\mathcal{I}_\nu^a \subset \mathcal{D}_\nu^a \subset \mathcal{G}_K^a :$$

every liftable subgroup $\sigma \in \Sigma_K$ contains an inertia element of a divisorial valuation (which is also contained in at least one other $\sigma' \in \Sigma_K$). The corresponding decomposition group is the "centralizer" of the (topologically) cyclic inertia group (the set of all elements which "commute" with inertia). This identifies $\mathcal{DV}_K = \mathcal{DV}_L$.

Step 3. By [4, Section 17, Step 7 and 8], when $K = k(X)$ and $L = l(Y)$ are function fields of surfaces over algebraic closures of finite fields of characteristic $\neq \ell$, the existence of an isomorphism

$$\Psi : \mathcal{G}^a_K \to \mathcal{G}^a_L$$

identifying Σ_K and Σ_L, implies the existence of a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that Ψ^* restricts to an isomorphism

$$L^*/l^* \otimes \mathbb{Z}_\ell \supset \cup_{n\in\mathbb{N}}(L^*/l^*)^{1/p^n} \simeq \cup_{n\in\mathbb{N}}(K^*/k^*)^{\epsilon/p^n} \subset K^*/k^* \otimes \mathbb{Z}_\ell.$$

By the induction hypothesis, we may assume that this isomorphism holds for all function fields of transcendence degree $\leq n-1$: Once we have identified decomposition and inertia subgroups of divisorial valuations, we have, for each $\nu \in \mathcal{DV}_K$, an intrinsically defined sublattice

$$\Psi^*(\boldsymbol{L}_\nu^*/l^* \otimes \mathbb{Z}[\frac{1}{p}]) = (\boldsymbol{K}_\nu^*/k^*)^\epsilon \otimes \mathbb{Z}[\frac{1}{p}] \subset \hat{\boldsymbol{K}}_\nu^* \tag{7.1}$$

of elements of the form g^ϵ, with $g \in \boldsymbol{K}_\nu^*/k^*$ and $\epsilon \in \mathbb{Z}_\ell^*$ in the completion of the multiplicative group of the residue field.

Step 4. Proposition 6.1 states that for any $f \in L^*/l^*$ the support of $\Psi^*(f)$ is finite on every projective model of K. The proof of this fact in Section 6 uses the induction hypothesis formulated in Step 3, for ν corresponding to divisors in a general Lefschetz pencil, depending on $f \in L^*/l^*$. Then Proposition 6.2 implies that $\Psi^*(f)$ has the form g^ϵ, for some $g \in K^*/k^*$, and Proposition 6.4 says that we have an isomorphism:

$$\epsilon^{-1} \cdot \Psi^* : L^*/l^* \otimes \mathbb{Z}_{(\ell)} \to K^*/k^* \otimes \mathbb{Z}_{(\ell)}$$

which maps multiplicative groups of one-dimensional subfields L into multiplicative groups of one-dimensional subfields of K, modulo $\mathbb{Z}_{(\ell)}$.

Step 5. If $f_1, f_2 \in L^*$ are algebraically dependent then

$$\mathrm{supp}'_L(f_1) = \mathrm{supp}'_L(f_2)$$

(see Section 5 for the definition). Conversely, if f_1, f_2 are algebraically independent then the map

$$(f_1, f_2) \colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$$

is dominant and hence there is an irreducible divisor $D \subset Y$, e.g., in the preimage of $(\mathbb{P}^1, 0)$, such that the restriction of f_1 to D is nonconstant in the residue field $\boldsymbol{K}_D$ of D (modulo a sufficiently high power of ℓ), and in its completion $\hat{\boldsymbol{K}}^*_D$. It follows that

$$\mathrm{supp}'_L(f_1) \neq \mathrm{supp}'_L(f_2).$$

This property is preserved under Ψ^*.

Step 6. Assume that $f_1, f_2 \in L^*$ are algebraically independent elements and let

$$g_j = \epsilon^{-1} \cdot \Psi^*(f_j) \in K^*/k^* \otimes \mathbb{Z}_{(\ell)}.$$

By Step 5, g_1, g_2 (or rather their integral powers contained in K^*) are also algebraically independent.

Step 7. Let $F = F_{f_1,f_2}$ be the smallest subfield of L containing $l(f_1, f_2)$ and such that for any $f \in F$ the normal closure of $l(f)$ in L is contained in F. The group F^*/l^* can be characterized as the minimal subgroup $S = S(f_1, f_2)$ of L^*/l^* containing f_1, f_2 and closed under the following operation: if $f \in S$ then the multiplicative group of the normal closure of $l(f)$ in L, modulo l^*, is contained in S. Note that the subgroup S is also closed under "addition", in the following sense: if $h_1, h_2 \in L^*$ are such that $h_1 + h_2 \neq 0$, and their images in L^*/l^* are in S, then the image of $h_1 + h_2$ is also in S. Indeed, the one-dimensional field $l(h_1/h_2)$ contains $(h_1/h_2) + 1 = (h_1 + h_2)/h_2$ and hence $h_1 + h_2$, since S is a multiplicative subgroup. This implies that the preimage of S in L^* is F^*. The homomorphism $\epsilon^{-1} \cdot \Psi^*$ preserves this property.

Step 8. We can now apply Proposition 2.13 and Lemma 2.14 in [4] to multiplicative subgroups of such two-dimensional subfields $F = F_{f_1,f_2}$, i.e., for each such two-dimensional subfield $F \subset L$ there exists a two-dimensional subfield $E = E_{g_1,g_2} \subset K$ such that the intersection

$$\epsilon^{-1} \cdot \Psi^*(F^*/l^*) \cap K^*/k^*$$

is the multiplicative group of a subfield $E' \subseteq E$, with E/E' purely inseparable. It follows that

$$R := \epsilon^{-1} \cdot \Psi^*(L^*/l^*) \cap K^*/k^*$$

has the same property, i.e., there is a purely inseparable extension K/K' such that the multiplicative group of K' is R.

Thus $\epsilon^{-1} \cdot \Psi^*$ defines an isomorphism between perfect closures of K and L. Since we can modify ϵ^{-1} by arbitrary p-primary multiples, the initial map Ψ^*

defines a unique isomorphism between perfect closures of K and L modulo integral powers of the absolute Frobenius endomorphism.

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, N.Y.U., 251 MERCER STR., NEW YORK, NY 10012, U.S.A. AND STEKLOV MATHEMATICAL INSTITUTE, GUBKINA STR. 8, 119991, MOSCOW, RUSSIA
E-mail address: bogomolo@cims.nyu.edu

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, N.Y.U., 251 MERCER STR., NEW YORK, NY 10012, U.S.A.
E-mail address: tschinkel@cims.nyu.edu