FLOPS ON HOLOMORPHIC SYMPLECTIC FOURFOLDS AND DETERMINANTAL CUBIC HYPERSURFACES

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1. Introduction

Let Σ be a K3 surface. Any birational map $\Sigma \dashrightarrow \Sigma$ extends to an automorphism; this follows from the uniqueness of minimal models for surfaces of non-negative Kodaira dimension. By the Torelli Theorem, the group of automorphisms of Σ is isomorphic to the group of automorphisms of $H^2(\Sigma, \mathbb{Z})$ compatible with the intersection pairing $\langle , \rangle$ and the Hodge structure on $H^2(\Sigma, \mathbb{C})$, and preserving the cone of nef (numerically eventually free) divisors. The nef cone admits an intrinsic combinatorial description (see, for example, [31]), once we specify a polarization g: A divisor h on Σ is nef if and only if $\langle h, D\rangle \geq 0$ for each divisor class D with $\langle D, D\rangle \geq -2$ and $\langle g, D\rangle > 0$. This characterization of the automorphism group has many interesting applications to arithmetic and geometric questions.

In this paper, we study certain aspects of the birational geometry of higher-dimensional analogs of K3 surfaces, i.e., irreducible holomorphic symplectic varieties F. These share many geometric properties with K3 surfaces. For example, the group $H^2(F, \mathbb{Z})$ carries a canonical integral quadratic form $(,)$, the *Beauville-Bogomolov* form (see, for example, [26]). Its definition uses the symplectic form on F but it can be characterized by the fact that the self-intersection form on $H^2(F, \mathbb{Z})$ is proportional to a power of the Beauville-Bogomolov form [26, 1.11] [18]

$$D^{\dim(F)} = c_F \, (F, F)^{\dim(F)/2} .$$

Moreover, these varieties satisfy local Torelli theorems [4] and surjectivity of the period map [26]. In contrast to the surface case, F may have numerous minimal models and may admit birational self-maps which are not regular. Furthermore, naive generalizations of the Torelli Theorem to higher dimensions are false; for counterexamples, consult [14], [35], and [32, Cor. 1.7, Thm. 4.5].

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Perhaps the best-known examples of irreducible holomorphic symplectic varieties are punctual Hilbert schemes of K3 surfaces and their deformations [4]. Here we focus on the case of length-two subschemes, which are isomorphic to the symmetric square of the K3 surface blown-up along the diagonal. These also arise as varieties of lines on cubic fourfolds [6]. By [23], given a polarization g on F, a divisor h on F is nef if $(h, \rho) \geq 0$ for each divisor class ρ satisfying

(1) $(g, \rho) > 0$; and
(2) $(\rho, \rho) \geq -2$, or $(\rho, \rho) = -10$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$. (These are called (-10)-*classes*.)

We have conjectured that these conditions are also necessary [22]. The main challenge in proving this is to show that the divisors ρ described above obstruct line bundles from being ample. For example, we expect extremal (-10)-classes ρ to be Poincaré dual to multiples of lines contained in planes $P \subset F$. The presence of such planes has implications for the birational geometry of F, as we can take the *Mukai flop* or elementary transformation along P [34, 0.7]

$$\begin{array}{ccccc} & & \mathrm{Bl}_P F \simeq \mathrm{Bl}_{P'} F' & & \\ & \swarrow & & \searrow & \\ F & & & & F'. \end{array}$$

Indeed, since P is Lagrangian, $\mathcal{N}_{P/F} \simeq \Omega^1_P$ so the exceptional divisor $E \subset \mathrm{Bl}_P F$ is isomorphic to $\mathbb{P}(\Omega^1_P)$. This admits two $\mathbb{P}^1$-bundle structures over $\mathbb{P}^2$, so we can blow down E to obtain a nonsingular variety F' birational to F. This is also an irreducible holomorphic symplectic variety, deformation equivalent to F [25, 3.4].

One especially interesting case is when there are no (0) or (-2)-classes (i.e., divisors ρ with $(\rho, \rho) = 0, -2$) but multiple (-10)-classes. Here the nef cones of birational models of F should be completely controlled by (-10)-classes. Not only are the integral extremal rays of F Poincaré dual to (-10)-classes, but this remains true for Mukai flops of F. In this situation, we expect F to admit *infinite* sequences of Mukai flops. However, Morrison [33] and Kawamata [29] have conjectured the following:

Conjecture (Finiteness of models)**.** Let F be a (simply-connected) Calabi-Yau manifold. Then there are finitely many minimal models of F up to isomorphism.

How can this be reconciled with the existence of infinite sequences of flops? The only possibility is that after a *finite* sequence of flops of F, we arrive at a variety isomorphic to F. This gives rise to birational maps $F \dashrightarrow F$ that are not automorphisms. These in turn act on $H^2(F, \mathbb{Z})$, preserving the cone of moving divisors but not the nef cone.

More specifically, consider a general cubic fourfold X containing a cubic scroll, or equivalently, a hyperplane section with six double points (see Proposition 23). The Picard lattice $\mathrm{Pic}(F)$ of the corresponding variety of lines $F = F(X)$ has rank two and the associated quadratic form represents -10 but not -2 or 0. For such fourfolds we

- compute the ample and moving cone in $\mathrm{Pic}(F)$;
- prove that F does not admit biregular automorphisms;
- exhibit a birational automorphism of infinite order explaining the chamber decomposition of the moving cone.

Our principal results are Theorems 24 and 31. The first exhibits explicit birational involutions on F and factors their indeterminacy. The second describes the action of the birational automorphism group on $H^2(F, \mathbb{Z})$.

We list some previous results in a similar vein: Miles Reid [36, 6.8] has offered examples of elliptically-fibered threefolds with an infinite number of distinct minimal models. Michael Fryers [17] classified isomorphism classes of minimal models of the general Horrocks-Mumford quintic threefold. Morrison [33] and Kawamata [29] have proven finiteness results (up to isomorphism!) for Calabi-Yau fiber spaces $F \to B$ where $0 < \dim(B) \leq \dim(F) \leq 3$. The case of Calabi-Yau manifolds of dimension ≥ 3 remains open.

The first half of the paper is devoted to classical results on cubic hypersurfaces. In Section 2 we analyze cubic threefolds Y with six ordinary double points in general position and their varieties of lines $F(Y)$. Section 3 establishes a dictionary between determinantal cubic surfaces and determinantal cubic threefolds, which generally have six ordinary double points in linear general position. Section 4 develops this to explain the geometric properties of Y, e.g., a transparent description of the components of $F(Y)$ and how they are glued together. Finally, Section 5 shows that cubic threefolds with six double points are determinantal.

The second half focuses on applications to the birational geometry of certain irreducible holomorphic symplectic varieties. Section 6 uses this information to construct birational involutions on the variety of lines F on a cubic fourfold containing Y. In Section 7 we explain the connection to our conjecture on nef cones. We close with an application to Zariski-density of rational points on F.

Throughout, the base field is algebraically closed of characteristic zero.

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2. Cubic threefolds with six double points

We assume that $Y \subset \mathbb{P}^4$ is a cubic hypersurface with ordinary double points at $p_1, \ldots, p_6$, which are in linear general position.

Lemma 1. *The cubic hypersurface Y contains no planes and the variety of lines $F(Y)$ has the expected dimension two.*

Proof. Let Y' denote a cubic threefold containing the plane

$$\Pi = \{x_0 = x_1 = 0\}.$$

Suppose G is a homogeneous cubic equation for Y'. Then we can write

$$G = x_0 Q_0 + x_1 Q_1$$

for quadratic forms Q_0 and Q_1. The singular locus of Y' contains the subscheme defined by

$$x_0 = x_1 = Q_0 = Q_1 = 0$$

which consists of four coplanar points. Thus the singularities of Y' are not in linear general position.

Suppose that $F(Y)$ has dimension > 2. As the singularities of Y are ordinary double points, there is at most a one-parameter family of lines through each singularity. Thus the generic line ℓ of $F(Y)$ is contained in a smooth hyperplane section $H \cap Y$. Consider the incidence correspondence

$$Z = \{(\ell, H) : \ell \subset Y, \ell \subset H\} \subset \mathrm{Gr}(2,5) \times \mathrm{Gr}(4,5)$$

in the partial flag variety. Since Z has dimension five the fibers of projection onto the second factor have dimension one, which is impossible as smooth cubic surfaces have a finite number of lines. □

Proposition 2. *Let Y be a cubic hypersurface with six ordinary double points $p_1, \ldots, p_6$ in linear general position. Projection from the point p_6*

$$Y \dashrightarrow \mathbb{P}^3$$

factors

$$\begin{array}{ccc} \tilde{Y} := \mathrm{Bl}_{p_6} Y & \xrightarrow{\gamma} & \mathbb{P}^3 \\ \downarrow \delta & \nearrow & \\ Y & & \end{array}$$

where γ is the blow up of a complete intersection C_6 of a smooth quadric and a cubic in $\mathbb{P}^3$, consisting of two twisted cubic curves meeting in five nodes.

Proof. The morphism γ blows down all the lines in Y incident to p_6; since p_6 is an ordinary double point, these are parametrized by a complete intersection C_6 of a smooth quadric Q (the projectivized tangent cone of Y at p_6) and a cubic in $\mathbb{P}^3$. Furthermore, an easy computation using the Jacobian criterion shows that C_6 is smooth except at the points $n_i = \gamma(p_i), i = 1, \ldots, 5$. Note that n_i corresponds to the line $\ell(p_i, p_6)$ joining p_i to p_6.

We claim that C_6 has two irreducible components E_6 and $E_6^\vee$, each smooth and rational of degree three, and $n_1, \ldots, n_5$ are nodes of C_6. Since the normalization of C_6 has genus -1 it is necessarily reducible. Consider the alternatives for the combinatorics of components: If C_6 were to contain a component of degree one then this would meet the rest of C_6 in three nodes, say n_1, n_2, n_3. Then the ordinary double points $\{p_6, p_1, p_2, p_3\} \in Y$ would all lie in a plane, contradicting our general position hypothesis. If C_6 were to contain a component of degree two then this would meet the rest of C_6 in four coplanar nodes, say n_1, n_2, n_3, n_4. Then $\{p_6, p_1, p_2, p_3, p_4\} \in Y$ would span a three-dimensional space, again contradicting our hypothesis. If C_6 contained a component of degree three and arithmetic genus one (i.e., a nodal plane cubic) then the quadric Q would be degenerate. □

Remark 3. This analysis implies that

$$n_1, n_2, n_3, n_4, n_5 \in Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

satisfy the following genericity conditions:

- the n_i are distinct;
- no two of the n_i lie on a ruling of Q;
- no four of the n_i lie on a hyperplane section of $Q \subset \mathbb{P}^3$.

Hence $S = \mathrm{Bl}_{n_1,\ldots,n_5} Q$ is isomorphic to a nonsingular cubic surface. While we will not prove this, S is isomorphic to the cubic surface constructed from Y in (7) of Section 5. In particular, S does not depend on which double point $p_i \in Y$ we choose for our projection.

Corollary 4. *The singular locus $F(Y)^{sing} \subset F(Y)$ is equal to the lines meeting the singular points $p_1, \ldots, p_6 \in Y$. The irreducible components* *of $F(Y)^{sing}$ consist of twelve smooth rational curves*

$$E_1, E_1^\vee, \ldots, E_6, E_6^\vee,$$

where $E_j \cup E_j^\vee$ parametrizes the lines through p_j. The singularities of $F(Y)^{sing}$ are the 15 lines $\ell(p_i, p_j)$ joining singularities of Y, and

$$\ell(p_i, p_j) = E_i \cap E_j = E_i \cap E_j^\vee = E_i^\vee \cap E_j = E_i^\vee \cap E_j^\vee.$$

Proof. It is a general fact [1, §1] that for any cubic hypersurface Y', the variety of lines $F(Y')$ is smooth at lines avoiding the singularities of Y'. Moreover, $F(Y')$ is singular at lines passing through an ordinary double point of Y' [11, 7.8]. The structure of the singular locus then follows from Proposition 2. □

Corollary 5. *The pair (Y, p_6) is uniquely determined up to isomorphism by the isomorphism class of the nodal curve C_6.*

Proof. The curve C_6 is a stable curve of genus four and $C_6 \hookrightarrow \mathbb{P}^3$ is its canonical embedding. We can characterize Y as the image of $\mathbb{P}^3$ under the linear series of cubics passing through C_6. □

3. Determinantal cubic surfaces and threefolds

We review determinantal representations of smooth cubic surfaces. The story begins with Grassmann [20] who showed that cubic surfaces arise as the common points of three nets of planes in $\mathbb{P}^3$, i.e., the locus where a 3×3 matrix of linear forms on $\mathbb{P}^3$ has nontrivial kernel. Schröter [37] showed that a generic surface admits such a realization and Clebsch [10] tied these representations to the structure of the lines on the cubic surface. Dickson [15] addressed the problem of expressing arbitrary smooth cubic surfaces in determinantal form. See [5, 6.4] and [8] for modern accounts and [16] for further historical discussion.

Proposition 6. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Then there exists a 3×3 matrix $M = (m_{ij})$ with entries linear forms on $\mathbb{P}^3$ such that*

$$S = \{\det(M) = 0\}.$$

Up to the left/right action of $\mathrm{GL}_3 \times \mathrm{GL}_3$, there are 72 such representations, corresponding to sextuples of disjoint lines on S.

This was extended by B. Segre [38] (cf. [5, 6.5]) to smooth cubic surfaces defined over arbitrary fields:

Proposition 7. *Let S be a smooth cubic surface defined over an arbitrary field k. Then the following conditions are equivalent:*

- *There exists a* 3×3 *matrix of linear forms over* k *such that* $S = \{\det(M) = 0\}$.
- S *contains a rational point and a sextuple of disjoint lines defined over* k.
- S *admits a birational morphism to* $\mathbb{P}^2$ *defined over* k.

We emphasize that each individual line in the sextuple need not be defined over k.

C. Segre [39, §12-14] analyzed determinantal representations of cubic threefolds:

Proposition 8. *Let* $Y \subset \mathbb{P}^4$ *be a generic cubic hypersurface realized as the determinant of a* 3×3 *matrix of linear forms. Then* Y *has six ordinary double points, in linear general position. Conversely, any cubic hypersurface with six ordinary double points in linear general position is determinantal.*

For completeness, we will provide an argument in Propositions 10 and 19.

Our main goal is to explain how all these classical theorems are related. Here is the key geometric ingredient: Let W be a vector space with a nondegenerate bilinear form $(,)$; taking orthogonal complements, we obtain a natural identification

$$\begin{array}{rcl} \mathrm{Gr}(n, W) & = & \mathrm{Gr}(\dim(W) - n, W) \\ \Lambda & \mapsto & \Lambda^{\perp}. \end{array} \tag{1}$$

Let G be a group acting linearly on W, with the natural induced action on $\mathrm{Gr}(n, W)$ and the action on $\mathrm{Gr}(\dim(W) - n, W)$ induced by (1).

We are especially interested in the case where $W = \mathrm{End}(V)$ for some vector space V of dimension d, the bilinear form $(,)$ is the trace pairing

$$(A, B) = \mathrm{tr}(AB),$$

and $\mathrm{GL}(V) \times \mathrm{GL}(V)$ acts on $\mathrm{End}(V)$ by left-right multiplication

$$(g_1, g_2) \cdot M = g_1 M g_2^t.$$

Here we are using the transpose operator

$$t : \mathrm{End}(V) \to \mathrm{End}(V)$$

obtained by dualizing and then applying the trace pairing. Consider the semidirect product

$$1 \to \mathrm{GL}(V) \times \mathrm{GL}(V) \to G \to \mathfrak{S}_2 \to 1 \tag{2}$$

where $\mathfrak{S}_2$ acts by exchanging the factors. Since

$$(g_1, g_2) \cdot M^t = g_1 M^t g_2^t = ((g_2, g_1) \cdot M)^t$$

G also acts naturally on $\mathrm{End}(V)$ and thus on the Grassmannians

$$\mathrm{Gr}(n, \mathrm{End}(V)) = \mathrm{Gr}(d^2 - n, \mathrm{End}(V)).$$

Consider the rank stratification on $\mathrm{End}(V)$

$$0 \subset \Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma_{d-1} \subset \mathrm{End}(V),$$

which is invariant under the group actions. We have the incidence correspondence

$$\begin{aligned} &\{(W_1, W_2, A) : \; W_1 \subset \ker(A), \; \mathrm{im}(A) \subset W_2\} \\ &\qquad \subset \mathrm{Gr}(d-k, V) \times \mathrm{Gr}(k, V) \times \mathrm{End}(V), \end{aligned}$$

a vector bundle of rank k^2 over $\mathrm{Gr}(d-k, V) \times \mathrm{Gr}(k, V)$. The projection onto $\mathrm{End}(V)$ induces a birational morphism to Σ_k, invertible away from Σ_{k-1}. In particular, Σ_k is ruled by k^2-dimensional vector spaces; through each $A \in \Sigma_k \setminus \Sigma_{k-1}$ there passes a unique ruling

$$\mathcal{R}_A := \{M \in \mathrm{End}(V) : \; \ker(A) \subset \ker(M), \; \mathrm{im}(M) \subset \mathrm{im}(A)\}.$$

Recall the description of the tangent space of Σ_k (see, for instance, [3, pp. 68-69]): When A has rank $< k$ then $T_A\Sigma_k = \mathrm{End}(V)$; furthermore, Σ_{k-1} is the singular locus of Σ_k. If A has rank k then

$$T_A\Sigma_k = \{M \in \mathrm{End}(V) : M(\ker(A)) \subset \mathrm{im}(A)\};$$

note that $T_A \supset \mathcal{R}_A$. We can express

$$(T_A\Sigma_k)^{\perp} = \{N \in \mathrm{End}(V) : NA = AN = 0\},$$

which is a linear subspace of dimension $(d-k)^2$ in Σ_{d-k}. Thus for each matrix B satisfying

$$\mathrm{im}(B) = \ker(A) \quad \text{and} \quad \ker(B) = \mathrm{im}(A),$$

we have $(T_A\Sigma_k)^{\perp} = \mathcal{R}_B$.

Proposition 9. *Assume V is a three-dimensional vector space and $\Lambda \subset \mathrm{End}(V)$ is a four-dimensional subspace. Let $\Lambda^{\perp} \subset \mathrm{End}(V)$ denote the orthogonal complement of Λ with respect to the trace pairing. Then the following conditions are equivalent:*

- *Λ is tangent to Σ_2 at a smooth point or intersects Σ_1 nontrivially;*
- *$\Lambda^{\perp}$ is tangent to Σ_1 at a nonzero point or is tangent to Σ_2 at a smooth point.*

In other words, Λ is transverse to the rank strata if and only if $\Lambda^{\perp}$ is transverse to the rank strata.

Proof. Suppose that Λ is tangent to Σ_2 at a rank-two matrix A_0, i.e.,

$$A_0 \in \Lambda \subset T_{A_0}\Sigma_2. \tag{3}$$

Let B_0 be a matrix with $\ker(B_0) = \mathrm{im}(A_0)$ and $\mathrm{im}(B_0) = \ker(A_0)$, which is unique up to scalars; it follows that

$$(T_{A_0}\Sigma_2)^{\perp} = \mathcal{R}_{B_0}, \quad (T_{B_0}\Sigma_1)^{\perp} = \mathcal{R}_{A_0}.$$

Dualizing (3) we obtain

$$\mathcal{R}_{B_0} = (T_{A_0}\Sigma_2)^{\perp} \subset \Lambda^{\perp} \subset A_0^{\perp}.$$

We also have

$$\mathcal{R}_{B_0} \subset T_{B_0}\Sigma_1 = \mathcal{R}_{A_0}^{\perp} \subset A_0^{\perp}.$$

Since

$$\dim \mathcal{R}_{B_0} = 1, \quad \dim T_{B_0}\Sigma_1 = \dim \Lambda^{\perp} = 5, \quad \dim A_0^{\perp} = 8,$$

we deduce

$$\mathrm{span}(B_0) \subsetneq \Lambda^{\perp} \cap T_{B_0}\Sigma_1.$$

Thus $\Lambda^{\perp}$ fails to intersect Σ_1 transversely at B_0.

Now suppose Λ is incident to Σ_1 at a rank-one matrix B_0, i.e.,

$$\mathcal{R}_{B_0} \subset \Lambda.$$

Again, let A_0 be a matrix with $\ker(A_0) = \mathrm{im}(B_0)$ and $\mathrm{im}(A_0) = \ker(B_0)$. Dualizing, we have $\Lambda^{\perp} \subset B_0^{\perp} = T_{A_0}\Sigma_2$ and the intersection

$$\Lambda^{\perp} \cap \mathcal{R}_{A_0} \subset T_{A_0}\Sigma_2$$

is nonzero. Pick a nonzero matrix

$$C \in \Lambda^{\perp} \cap \mathcal{R}_{A_0}.$$

If C has rank two then $\mathrm{im}(C) = \mathrm{im}(A_0)$ and $\ker(C) = \ker(A_0)$, thus $\mathcal{R}_C = \mathcal{R}_{A_0}$ and $T_C\Sigma_2 = T_{A_0}\Sigma_2$. Here $\Lambda^{\perp}$ fails to be transverse to Σ_2 at C. If C has rank one then

$$\begin{aligned} T_C\Sigma_1 &= \{M \in \mathrm{End}(V) : M(\ker(C)) \subset \mathrm{im}(C)\} \\ &\subset \{M \in \mathrm{End}(V) : M(\ker(A_0)) \subset \mathrm{im}(A_0)\} \\ &= T_{A_0}\Sigma_2 = B_0^{\perp}. \end{aligned}$$

Since $\Lambda^{\perp} \subset B_0^{\perp}$ as well, the intersection

$$T_C\Sigma_1 \cap \Lambda^{\perp} \subset B_0^{\perp}$$

has dimension at least two, i.e., $\Lambda^{\perp}$ meets Σ_1 nontransversely at C.

Conversely, suppose that $\Lambda^{\perp}$ is tangent to Σ_1 at B_0, i.e.,

$$\mathrm{span}(B_0) \subsetneq \Lambda^{\perp} \cap T_{B_0}\Sigma_1.$$

Let A_0 denote a matrix with $\mathrm{im}(A_0) = \ker(B_0)$ and $\ker(A_0) = \mathrm{im}(B_0)$. Dualizing, we find that

$$\Lambda + \mathcal{R}_{A_0} \subsetneq B_0^{\perp} \quad \text{and} \quad \Lambda \cap \mathcal{R}_{A_0} \neq 0.$$

Let C be a nonzero matrix in this intersection. If C has rank one then Λ meets Σ_1 nontrivially and we're done. If C has rank two then it has the same image and kernel as A_0, whence $T_C\Sigma_2 = B_0^{\perp}$ and $\Lambda \subset T_C\Sigma_2$, i.e., Σ_2 and Λ fail to be transverse at C.

Now suppose $\Lambda^{\perp}$ is tangent to Σ_2 at a matrix A_0 of rank two, i.e.,

$$A_0 \subset \Lambda^{\perp} \subset T_{A_0}\Sigma_2.$$

Again, choose B_0 to be a matrix with $\mathrm{im}(B_0) = \ker(A_0)$ and $\ker(B_0) = \mathrm{im}(A_0)$ so that $B_0 \in \mathcal{R}_{B_0} = T_{A_0}\Sigma_2^{\perp}$. Dualizing yields $B_0 \in \Lambda$, i.e., Λ intersects Σ_1. □

We use this to interpret our determinantal expressions for cubic hypersurfaces. Recall that $\dim(V) = 3$. Tensor multiplication

$$V \times V^{\vee} \to \mathrm{End}(V)$$

gives the Segre embedding

$$\mathbb{P}(V) \times \mathbb{P}(V^{\vee}) \hookrightarrow \mathbb{P}(\mathrm{End}(V)),$$

whose image has degree six and coincides with $\mathbb{P}(\Sigma_1)$. Given a four-dimensional subspace

$$\Lambda \subset \mathrm{End}(V)$$

the intersection

$$S := \mathbb{P}(\Lambda \cap \Sigma_2) \subset \mathbb{P}(\Lambda) \simeq \mathbb{P}^3 \tag{4}$$

is a determinantal cubic surface. It is smooth precisely when Λ meets Σ_2 transversely at smooth points. Then we obtain an embedding

$$\begin{array}{rcl} S & \hookrightarrow & \mathbb{P}(V) \times \mathbb{P}(V^{\vee}) \\ s & \mapsto & (\ker(s), \mathrm{im}(s)) \end{array}$$

such that the projections induce the blow-up realizations of S (cf. Proposition 6)

$$\beta : S \to \mathbb{P}(V), \quad \beta^{\vee} : S \to \mathbb{P}(V^{\vee}).$$

Let $\Lambda^{\perp}$ be the orthogonal complement to Λ with respect to the trace pairing. Then

$$Y := \mathbb{P}(\Lambda^{\perp} \cap \Sigma_2) \subset \mathbb{P}(\Lambda^{\perp}) \simeq \mathbb{P}^4 \tag{5}$$

is a determinantal cubic threefold. It is necessarily singular along the points of $\mathbb{P}(\Lambda^{\perp} \cap \Sigma_1)$. If $\mathbb{P}(\Lambda^{\perp})$ intersects $\mathbb{P}(\Sigma_1)$ and the smooth points of $\mathbb{P}(\Sigma_2)$ transversely then the Bezout theorem implies that the singular locus of Y is

$$\{p_1, \dots, p_6\} := \mathbb{P}(\Sigma_1 \cap \Lambda^\perp).$$

Note that these give a sextuple of points in $\mathbb{P}(V) \times \mathbb{P}(V^\vee) \simeq \mathbb{P}^2 \times \mathbb{P}^2$; a straightforward cohomology computation shows these are in linear general position in $\mathbb{P}(\mathrm{End}(V))$.

Proposition 10. *Let S and Y be determinantal cubic hypersurfaces defined by Equations 4 and 5 above. Then Y is a cubic threefold with six ordinary double points in linear general position if and only if S is a smooth cubic surface. We thus obtain a identification*

$$\left\{ \begin{array}{c} \textit{determinantal cubic} \\ \textit{threefolds with six} \\ \textit{ordinary double points} \\ \textit{in linear general position} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{determinantal cubic surfaces} \\ \textit{without singularities} \end{array} \right\}$$

that is equivariant with respect to the action of G.

Indeed, Proposition 9 says we can identify the open subsets in

$$\mathrm{Gr}(4, \mathrm{End}(V)) = \mathrm{Gr}(5, \mathrm{End}(V))$$

where our transversality conditions hold.

4. Geometric applications of the determinantal description

In this section, we assume that S and Y satisfy the conclusions of Proposition 10. The determinantal description allows a transparent derivation of many of the key properties of Y.

Proposition 11. *Let $F(Y)$ denote the variety of lines on Y. We have a natural surjective morphism*

$$\nu : \mathbb{P}(V) \sqcup S \sqcup \mathbb{P}(V^\vee) \to F(Y)$$

that maps each component birationally onto its image.

Proof. For each point $[v] \in \mathbb{P}(V)$, let

$$\ell_{[v]} = \{y = [\phi] : \phi(v) = 0\} = \{y = [\phi] : v \in \ker(\phi)\} \subset Y$$

where $\phi \in \mathrm{End}(V)$ represents $y \in Y \subset \mathbb{P}(\Lambda^\perp)$. This is a linear subspace of codimension at most three in $\mathbb{P}(\Lambda^\perp)$. Indeed, elements of

$$\Lambda^\perp \cap \{M : v \subset \ker(M)\}$$

automatically have vanishing determinants. Lemma 1 guarantees Y does not contain any planes, so we conclude that $\ell_{[v]}$ is a line.

Similarly, for $[v^\vee] \in \mathbb{P}(V^\vee)$ we also get lines

$$\ell_{[v^\vee]} = \{y = [\phi] : v^\vee \circ \phi = 0\} = \{y = [\phi] : v^\vee \in \ker(\phi^t)\} \subset Y.$$

Given $s = [\sigma] \in S$ with $\sigma \in \Lambda$, we have the locus

$$\ell_s = \{y = [\phi] : \sigma\phi\sigma = 0\} \subset Y.$$

Since σ has rank two, this condition translates into the vanishing of the 2×2 matrix of the induced map

$$\mathrm{im}(\sigma) \xrightarrow{\phi} V/\ker(\sigma).$$

However, the orthogonality assumption $\mathrm{tr}(\sigma\phi) = 0$ implies that there are only three independent linear conditions. In particular, ℓ_s is a line in Y.

Combining these three constructions, we obtain the morphism ν. We next show that ν is surjective. Lemma 1 implies that $F(Y)$ is two-dimensional. A standard intersection theory computation [19, 14.7.13] shows that $\deg F(Y) = 45$ (with respect to the Plücker embedding of the Grassmannian). However, we can compute the pull back

$$\nu^*\mathcal{O}_{F(Y)}(1) = (\mathcal{O}_{\mathbb{P}(V)}(3), \mathcal{O}_S(3), \mathcal{O}_{\mathbb{P}(V^\vee)}(3))$$

which means that

$$\deg(\mathbb{P}(V)) = \deg(\mathbb{P}(V^\vee)) = 9, \quad \deg(S) = 27.$$

Thus all the components of $F(Y)$ are in the image of ν; furthermore, ν maps each component birationally onto its image. □

Corollary 12. *Retain the notation of Proposition 11 and let $y \in Y$ be a nonsingular point. The components of $F(Y)$ dominated by $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$ each admit a unique line passing through y. The component dominated by S admits four lines passing through y.*

Proof. The first statement is easily verified using linear algebra. The second can be deduced from the fact that a generic $y \in Y$ lies on six lines in Y. □

Proposition 13. *The morphism*

$$\nu : \mathbb{P}(V) \sqcup S \sqcup \mathbb{P}(V^\vee) \to F(Y)$$

induces the following identifications: Consider the distinguished double-six on S

$$\{E_1, \dots, E_6; E_1^\vee, \dots, E_6^\vee\},$$

with each 6-tuple blowing down to a collections of points

$$\{q_1, \dots, q_6\} \subset \mathbb{P}(V), \quad \{q_1^\vee, \dots, q_6^\vee\} \subset \mathbb{P}(V^\vee).$$

Let $\{\ell_1, \dots, \ell_6\}$ and $\{\ell_1^\vee, \dots, \ell_6^\vee\}$ be the lines in $\mathbb{P}(V^\vee)$ and $\mathbb{P}(V)$ dual to these points. We have natural isomorphisms for each i:

$$\psi_i : \ell_i \xrightarrow{\sim} E_i, \quad \psi_i(\ell_i \cap \ell_j) = E_i \cap E_j^\vee \subset S,$$
$$\psi_i^\vee : \ell_i^\vee \xrightarrow{\sim} E_i^\vee, \quad \psi_i(\ell_i^\vee \cap \ell_j^\vee) = E_i^\vee \cap E_j \subset S.$$

Proof. We break up the argument into two lemmas:

Lemma 14. *The morphism ν maps $E_i, E_i^\vee \subset S, \ell_i \subset \mathbb{P}(V^\vee)$ and $\ell_i^\vee \subset \mathbb{P}(V)$ to the locus C_i of lines passing through p_i. Furthermore, E_i and ℓ_i parametrize $y = [\phi] \in Y$ such that $\operatorname{im}(\phi) \supset \operatorname{im}(p_i)$; $E_i^\vee$ and $\ell_i^\vee$ parametrize $y = [\phi]$ such that $\ker(\phi) \subset \ker(p_i)$. Here we regard the singularity $p_i \in Y$ as an element $\Lambda^\perp \cap \Sigma_1$.*

Proof. The determinantal description of S identifies

$$E_i = \{s \in \Lambda : \ker(s) = \operatorname{im}(p_i)\}.$$

Similarly, we have

$$E_i^\vee = \{s \in \Lambda : \operatorname{im}(s) = \ker(p_i)\}.$$

On the other hand,

$$\ell_i = \{\operatorname{im}(s) : s \in \Lambda \text{ with } \ker(s) = \operatorname{im}(p_i)\} \subset \mathbb{P}(V^\vee)$$

and

$$\ell_i^\vee = \{\ker(s)) : s \in \Lambda \text{ with } \operatorname{im}(s) = \ker(p_i)\} \subset \mathbb{P}(V).$$

Thus for $s_i = [\sigma] \in E_i$

$$\nu(s_i) = [\{y = [\phi] : \phi(\operatorname{im}(\sigma)) \subset \ker(\sigma) = \operatorname{im}(p_i)\}]$$

which is a line through p_i. On the other hand, for $v_i^\vee \in \ell_i$ (where $v_i^\vee \in V^\vee$ satisfies $v_i^\vee(\operatorname{im}(p_i)) = 0$) we have

$$\nu(v_i^\vee) = [\{y = [\phi] : v_i^\vee(\operatorname{im}(\phi)) = 0\}]$$

which is also a line through p_i. As we vary $s_i \in E_i$ and $v_i^\vee \in \ell_i$, we get the locus of $y = [\phi]$ such that $\operatorname{im}(\phi) \supset \operatorname{im}(p_i)$.

The analogous statements for $E_i^\vee$ and $\ell_i^\vee$ are proven similarly. □

There is an obvious identification

$$\ell_i = \mathbb{P}(q_i^\perp) = \mathbb{P}((V/q_i)^\vee) = \mathbb{P}(V/q_i) = \mathbb{P}(\operatorname{Hom}(q_i, V/q_i)) \\ = \mathbb{P}(T_{q_i}\mathbb{P}(V)) = E_i; \tag{6}$$

note that if W is a two-dimensional vector space then the isomorphism $W = W^\vee \otimes \bigwedge^2 W$ induces a natural isomorphism $\mathbb{P}(W) = \mathbb{P}(W^\vee)$. This is *not* the gluing inducing ν. However, note that this takes the points $\ell_i \cap \ell_j$ to the intersections $E_i \cap \mathfrak{l}_{ij}$, where $\mathfrak{l}_{ij}$ is the proper transform of the line joining q_i and q_j. Using (6), it suffices to express ψ_i and $\psi_i^\vee$ as automorphisms of E_i and $E_i^\vee$.

The gluings $\ell_i \simeq E_i$ and $\ell_i^\vee \simeq E_i^\vee$ will be obtained from the following:

Lemma 15. *There exists a projectivity $\psi_i : E_i \to E_i$ mapping $E_i \cap \mathfrak{l}_{ij}$ to $E_i \cap E_j^\vee$ for each $j \neq i$. The analogous statement holds for $E_i^\vee$.*

Proof. For notational simplicity we take $i = 1$. Consider the conic bundle $S \to \mathbb{P}^1$ given by the pencil of cubics on $\mathbb{P}(V)$ double at q_1 and containing $q_2, \dots, q_6$. The degenerate fibers are

$$\mathfrak{l}_{1j} \cup E_j^\vee, \quad j = 2, \dots, 6.$$

The curve E_1 is a bi-section of this conic bundle, so there is a covering involution $\psi_1 : E_1 \to E_1$ taking $\mathfrak{l}_{1j} \cap E_1$ to $E_j^\vee \cap E_1$. □

It remains to check that this is in fact the identification induced by ν. However, we know from Corollary 4 that ν glues the points $E_i \cap E_j^\vee$, $E_i^\vee \cap E_j$, to $[\ell(p_i, p_j)]$, the line in Y joining p_i and p_j. Now ℓ_i and ℓ_j meet in $\mathbb{P}(V^\vee)$, and $\ell_i^\vee$ and $\ell_j^\vee$ meet in $\mathbb{P}(V)$; thus these points must also be mapped by ν to $[\ell(p_i, p_j)]$. In general, the isomorphisms ψ_i and $\psi_i^\vee$ are the unique ones identifying all these points. This finishes the proof of Proposition 13. □

Proposition 16. *The morphism ν is obtained by gluing $\mathbb{P}(V), S,$ and $\mathbb{P}(V^\vee)$ using the identifications described in Proposition 13.*

Proof. Let F' denote the surface obtained by gluing $\mathbb{P}(V), S,$ and $\mathbb{P}(V^\vee)$ using the identifications ψ_i and $\psi_i^\vee$. Again, F' contains twelve distinguished rational curves

$$\ell_i = E_i, \ell_i^\vee = E_i, \ i = 1, \dots, 6$$

and *fifteen* distinguished points

$$\ell_i \cap \ell_j = E_i \cap E_j^\vee = \ell_i^\vee \cap \ell_j^\vee = E_i^\vee \cap E_j,$$

which map surjectively onto $F(Y)^{sing}$ (by Corollary 4).

We have already seen that ν factors through F'; it only remains to prove that the induced morphism $F' \to F(Y)$ is an isomorphism. Our analysis of the gluings over $F(Y)^{sing}$ shows that ξ is a bijection over $F(Y)^{sing}$.

We first check that ν is the normalization of $F(Y)$. Proposition 11 shows that ν maps each irreducible component birationally onto its image. It follows that the restrictions

$$\mathbb{P}(V) \to \nu(\mathbb{P}(V)), \quad \mathbb{P}(V^\vee) \to \nu(\mathbb{P}(V^\vee))$$

are normalization maps. Consider the factorization of $\nu|S$ through the normalization of its image

$$S \to \nu(S)' \to \nu(S);$$

this reverses the identifications induced by the ψ_i and $\psi_i^\vee$. The images of the six lines $E_1, \dots, E_6$, (and $E_1^\vee, \dots, E_6^\vee$) in $\nu(S)'$ are pairwise disjoint. Hence $S \to \nu(S)'$ contracts no curves and thus is an isomorphism; S is the normalization of $\nu(S)$.

This analysis implies $F' \to F(Y)$ is bijective.

The Fano scheme $F(Y)$ is defined by the degeneracy locus of a vector bundle over the Grassmannian $\mathrm{Gr}(2,5)$, with the expected dimension (by Lemma 1). Thus $F(Y)$ is a local complete intersection scheme and is Cohen-Macaulay; hence it has no embedded points and is seminormal. The universal property of seminormalization then implies that $F' \to F(Y)$ is an isomorphism. □

The determinantal description offers a transparent construction for the cubic scrolls on Y. Each point $[v] \in \mathbb{P}(V)$ determines a line in $\mathbb{P}(V^\vee)$, which may be interpreted as a ruled surface $T_v \subset Y$ using the analysis of the components of $F(Y)$ in Proposition 11:

Proposition 17. *For each $[v] \in \mathbb{P}(V)$, the locus*

$$T_v = \{y \in Y : v \in \mathrm{im}(y)\}$$

is a cubic scroll. The ruling arises from the morphism

$$\begin{array}{rcl} T_v & \to & \mathbb{P}(V/\mathrm{span}(v)) \\ y & \mapsto & \mathrm{im}(y) \end{array}$$

with fibers $\ell_{[v^\vee]}$, where $v^\vee \neq 0 \in V^\vee$ with $v^\vee(v) = 0$.

Similarly, for $[v^\vee] \in \mathbb{P}(V^\vee)$ the locus

$$T_{v^\vee} = \{y \in Y : v^\vee(\ker(y)) = 0\}$$

is a cubic scroll. Each union

$$T_v \cup T_{v^\vee} = Y \cap Q$$

where Q is a quadric hypersurface.

If $s \in S$ and ℓ_s denotes the corresponding line in Y then $\ell_s \subset T_{\beta(s)}$ (resp. $T_{\beta^\vee(s)}$) is a section of the ruling.

Proof. Choose a basis v, v', v'' of V such that $v^\vee(v') = v^\vee(v'') = 0$. The matrices in the closure of the locus of rank-two matrices with image containing v can be written

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where the bottom two rows are linearly dependent. This defines a closed subset in $\mathbb{P}(\mathrm{End}(V))$. Geometrically, this is a cone over the

Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$$

with a vertex a projective plane. Intersecting this with $\Lambda^\perp$ yields a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2$, which is a cubic scroll.

On the other hand, the closure of the locus of rank-two matrices with kernel annihilated by $v^\vee$ are those whose right two columns are linearly dependent. The union of these two loci is given by the intersection

$$\{\det(B) = 0\} \cap \{b_{22}b_{33} - b_{23}b_{32} = 0\} \subset \mathbb{P}(\mathrm{End}(V)),$$

i.e., the intersection of $\mathbb{P}(\Sigma_2)$ with a quadric hypersurface.

Recall that ℓ_s was defined in the proof of Proposition 11

$$\ell_s = \{y = [\phi] : \sigma\phi\sigma = 0\}.$$

Fix a ruling in $T_{\beta(s)}$: Regarding $\beta(s) = \ker(\sigma)$ as a line in V, we choose a two-dimensional subspace $\ker(\sigma) \subset U \subset V$, and consider the matrices ϕ with image U. This imposes one additional linear constraint on the matrix entries of ϕ, so each ruling meets ℓ_s in one point.

□

Proposition 18. *For each $s \in S$,*

$$T_{\beta(s)} \cap T_{\beta^\vee(s)} = \ell_s \cup R_s$$

where $R_s \subset Y$ is a twisted quartic curve passing through the singularities $p_1, \ldots, p_6$.

Proof. Generically, the cubic scrolls are nonsingular and isomorphic to $\mathbb{P}^2$ blown up at one point, in which case $\ell_s \subset T_{\beta(s)}$ is the exceptional curve. Let R_s denote the union of components of the intersection other than ℓ_s. We have shown that $T_{\beta s} \cup T_{\beta^\vee(s)}$ is a complete intersection of a quadric and cubic in $\mathbb{P}^4$, and thus is a singular K3 surface. Adjunction shows that

$$K_{T_{\beta(s)}} + R_s + \ell_s \equiv 0;$$

hence R_s has degree four and genus zero. □

5. Cubic threefolds with six double points are determinantal

Here we complete C. Segre's determinantal construction of cubic threefolds with six double points:

Proposition 19. *Each cubic threefold with six ordinary double points in linear general position is determinantal.*

We prove Proposition 19 using the geometry of the twisted quartic curves in a determinantal cubic threefold, following [12, 3.2-3.4]. One key tool is the *Segre threefold* $\mathfrak{S} \subset \mathbb{P}^4$; we recall its basic properties:

- Given $p_1, \dots, p_6 \in \mathbb{P}^4$ in linear general position, the linear series of cubics double at these points induces a morphism
$$\varpi : \mathrm{Bl}_{p_1,\dots,p_6}\mathbb{P}^4 \to \mathbb{P}^4$$
with image $\mathfrak{S}$ and fibers twisted quartic curves containing the points $p_1, \dots, p_6$. If we choose $p_1 = [1,0,0,0,0]$,$p_2 = [0,1,0,0,0]$, $p_3 = [0,0,1,0,0]$, $p_4 = [0,0,0,1,0]$,$p_5 = [0,0,0,0,1]$, and $p_6 = [1,1,1,1,1]$ then the cubics double at these points are
$$y_0 = (x_3 - x_4)x_0(x_1 - x_2), \quad y_1 = (x_4 - x_0)x_1(x_2 - x_3),$$
$$y_2 = (x_0 - x_1)x_2(x_3 - x_4), \quad y_3 = (x_1 - x_2)x_3(x_4 - x_1),$$
$$y_4 = (x_2 - x_3)x_4(x_0 - x_1)$$
which satisfy
$$y_0y_1y_2 + y_1y_2y_3 + y_2y_3y_4 + y_3y_4y_0 + y_4y_0y_1 = 0.$$
- $\mathfrak{S}$ contains 10 ordinary double points and 15 planes.
- The nonsingular twisted quartic curves map to an open subset of $\mathfrak{S}$ that is isomorphic to $\mathcal{M}_{0,6}$, the moduli space of genus-zero curves with six marked points. The morphism ϖ is the universal family over $\mathcal{M}_{0,6}$.
- The inclusion $\mathcal{M}_{0,6} \hookrightarrow \mathfrak{S}$ extends to an isomorphism [24]
$$(\mathbb{P}^1)^6 /\!/ \mathrm{SL}_2 \xrightarrow{\sim} \mathfrak{S}$$
from the GIT quotient of six points in $\mathbb{P}^1$ with the symmetric linearization.

Thus we have a morphism

$$(7) \quad \left\{ \begin{array}{c} \text{cubic threefolds with} \\ \text{ordinary double points} \\ \text{at } p_1, \dots, p_6 \end{array} \right\} \to \left\{ \begin{array}{c} \text{smooth cubic surfaces} \\ \text{arising as hyperplane} \\ \text{sections of } \mathfrak{S} \subset \mathbb{P}^4 \end{array} \right\}$$

Remark 20. In general, M. Kapranov [28] [27, 4.3] has shown that $\mathcal{M}_{0,n-1}$ can be identified with the rational normal curves in $\mathbb{P}^{n-3}$ passing through points $p_1, \dots, p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position. The rational normal curves are the universal curve, with $p_1, \dots, p_{n-1}$ tracing out the marked points. Identifying the universal curve over $\overline{\mathcal{M}}_{0,n-1}$ with $\overline{\mathcal{M}}_{0,n}$, there is a morphism

$$\begin{array}{rcl} \overline{\mathcal{M}}_{0,n} & \to & \mathbb{P}^{n-3} \\ (C, p_1, \dots, p_n) & \mapsto & p_n, \end{array}$$

factoring through $\mathrm{Bl}_{p_1,\dots,p_{n-1}}\mathbb{P}^{n-3}$.

Recall our previous notation: Let

$$\{E_1, \dots, E_6; E_1^\vee, \dots, E_6^\vee\}$$

denote the double-six on S, $\beta : S \to \mathbb{P}(V)$ and $\beta^\vee : S \to \mathbb{P}(V^\vee)$ the associated contractions, and

$$\{q_1, \dots, q_6; q_1^\vee, \dots, q_6^\vee\}$$

the images of the exceptional divisors. There is an involution of the Picard lattice taking E_i to $E_i^\vee$ for $i = 1, \dots, 6$. If the Picard lattice is presented

$$\mathbb{Z}L + \mathbb{Z}E_1 + \dots + \mathbb{Z}E_6, \quad E_i^2 = -1, E_iE_j = \delta_{ij}, L^2 = 1, LE_i = 0,$$

the involution takes the form

$$E_i \mapsto 2L - E_j - E_k - E_a - E_b - E_c = E_i^\vee, \tag{8}$$

where $\{i, j, k, a, b, c\}$ is a permutation of the indices $\{1, \dots, 6\}$.

We shall need a version of Cremona's hexahedral construction [13, 16]:

Proposition 21. *Let $S^\circ \subset S$ denote the complement to the lines in S. For $s \in S^\circ$ consider the images of $q_1, \dots, q_6$ and $q_1^\vee, \dots, q_6^\vee$ under the projections*

$$\mathbb{P}(V) \dashrightarrow \mathbb{P}(V/\beta(s)) \quad \mathbb{P}(V^\vee) \dashrightarrow \mathbb{P}(V^\vee/\beta^\vee(s)), \tag{9}$$

which determine elements $j(s), j^\vee(s) \in \mathcal{M}_{0,6}$. Then we have the following:

- *$j(s) = j^\vee(s)$ for each $s \in S^\circ$;*
- *there exists an extension $j : S \to \mathfrak{S}$;*
- *the image of j is a hyperplane section of $\mathfrak{S} \subset \mathbb{P}^4$;*
- *conversely, each smooth hyperplane section $S \subset \mathfrak{S}$ is a cubic surface with a distinguished ordered double-six.*

Thus we obtain an identification

$$\left\{ \begin{array}{c} \textit{smooth cubic surfaces} \\ \textit{with a double-six of} \\ \textit{ordered lines} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{smooth hyperplane} \\ \textit{sections of } \mathfrak{S} \subset \mathbb{P}^4 \end{array} \right\}.$$

Proof. Fix $s \in S^\circ$ and consider the *degree-two* Del Pezzo surface $S' := \mathrm{Bl}_s(S) = \mathrm{Bl}_{q_1,\dots,q_7}(\mathbb{P}(V))$. The projections (9) induce conic bundles

$$\varphi : S' \to \mathbb{P}^1, \quad \varphi^\vee : S' \to \mathbb{P}^1$$

with degenerate fibers corresponding to the images of $q_1, \dots, q_6$ and $q_1^\vee, \dots, q_6^\vee$ respectively. However, each degree-two Del Pezzo admits a canonical involution, i.e. the covering involution of the anticanonical morphism $S' \to \mathbb{P}^2$. Moreover, φ and $\varphi^\vee$ are conjugate under this involution and thus have the same degenerate fibers. We conclude that $j(s) = j^\vee(s)$ in $\mathcal{M}_{0,6}$.

We extend j to S: Assume first that $\beta(s) \neq q_1, \dots, q_6$. We still have a conic bundle $\varphi : S' \to \mathbb{P}^1$ but the images of q_i and q_j in $\mathbb{P}^1$ coincide if $\beta(s) \in \mathfrak{l}_{ij}$, the line joining q_i and q_j. However, since no three of the q_i are collinear at most two points may coincide, so the image of $(q_1, \dots, q_6)$ is a GIT-semistable point of $(\mathbb{P}^1)^6$; this yields a well-defined point on $\mathfrak{S}$. If $\beta(s) = q_1$ then we can identify $E_1 = \mathbb{P}(T_{q_1}\mathbb{P}(V)) = \mathbb{P}(V/q_1)$ and the images of the $q_j, j = 2, \dots, 6$ in $\mathbb{P}(V/q_1)$ with the intersections of the proper transforms of the $\mathfrak{l}_{1j}$ with E_1. The rule

$$j(s) = (s, \mathfrak{l}_{12} \cap E_1, \dots, \mathfrak{l}_{16} \cap E_1)$$

extends the definition of j over $E_1 \subset S$. (This argument is very similar to the proof of Lemma 15.)

A straightforward degree computation shows that j maps S to a hyperplane section of $\mathfrak{S}$.

For the final statement, the fifteen planes of $\mathfrak{S}$ cut out fifteen ordered lines of S. The remaining lines form a double-six. $\square$

Let Y' be a cubic threefold with ordinary double points at $p_1, \dots, p_6$; ϖ induces a rational map $Y' \dashrightarrow S$ contracting the twisted quartic curves in Y' to points of a smooth hyperplane section $i : S \hookrightarrow \mathfrak{S}$ with a distinguished double-six. After ordering the two sextuples of disjoint lines, Proposition 10 yields a determinantal cubic hypersurface Y with ordinary double points at $p_1, \dots, p_6$ corresponding to the marked cubic surface S. By Proposition 18, the image of Y under ϖ is a hyperplane section $i_2 : S \hookrightarrow \mathfrak{S}$ with the planes of $\mathfrak{S}$ tracing out the corresponding 15 lines of S. Proposition 21 implies $i_1(S) = i_2(S)$ and thus $Y \simeq Y'$; this yields an inverse to the morphism (7).

Remark 22. The natural map

$$\left\{ \begin{array}{c} \text{determinantal cubic threefolds} \\ \text{with ordinary double points} \\ \text{at } p_1, \dots, p_6 \end{array} \right\} \to \left\{ \begin{array}{c} \text{cubic threefolds with} \\ \text{ordinary double points} \\ \text{at } p_1, \dots, p_6 \end{array} \right\}$$

is not an isomorphism. Under our identifications, these correspond to

$$\left\{ \begin{array}{c} \text{determinantal cubic surfaces} \\ \text{with a sextuple of ordered} \\ \text{lines} \end{array} \right\} \to \left\{ \begin{array}{c} \text{cubic surfaces with a} \\ \text{double-six of ordered} \\ \text{lines} \end{array} \right\}$$

which has degree two. Indeed, this reflects the involution (8) interchanging the sextuples of our double-six.

6. Constructing flops

Let X be a smooth cubic fourfold with hyperplane class h.

Proposition 23. *Assume that X admits a hyperplane section $Y \subset X$ with six ordinary double points in linear general position. Then X contains two families of cubic scrolls T and $T^\vee$, whose cycle classes satisfy*

$$[T] + [T^\vee] = 2h^2.$$

Conversely, if X is a smooth cubic fourfold containing a smooth cubic scroll T then the hyperplane section

$$Y = X \cap \operatorname{span}(T)$$

has at least six double points, counted with multiplicities.

Proof. Assume that X admits a hyperplane section Y as above. We may assume that Y is determinantal by Proposition 8. Proposition 17 guarantees that X contains two families of cubic scrolls, each parametrized by $\mathbb{P}^2$. Given T and $T^\vee$ from different families, we have

$$T \cup T^\vee = Y \cap Q$$

for some quadric hypersurface in $\mathbb{P}^3$. The equation on cycle classes follows.

Now suppose that X contains a smooth cubic scroll T spanning the hyperplane section Y. In suitable coordinates,

$$T = \{[x_0, \dots, x_4] : \operatorname{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \end{pmatrix} = 1\}$$

and thus there exist linear forms y_0, y_1, y_2 in $x_0, \dots, x_4$ such that

$$Y = \{[x_0, \dots, x_4] : \det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 \end{pmatrix} = 0\},$$

i.e., Y is determinantal. The double points correspond to the matrices of rank one (cf. Proposition 9). □

Theorem 24. *Let X be a smooth cubic fourfold not containing a plane, and F its variety of lines. Assume that X admits a hyperplane section Y with six ordinary double points in linear general position. Write*

$$F(Y) = P \cup S' \cup P^\vee$$

with normalization $\mathbb{P}^2 \sqcup S \sqcup \mathbb{P}^2$. Then there exist birational involutions

$$\iota, \iota^\vee : F \dashrightarrow F$$

which are regular away from $P \cup S'$ and $P^\vee \cup S'$ respectively.

Precisely, ι is factored as follows:

(1) *Flop P to get a new holomorphic symplectic fourfold F_1; the proper transform S_1 of S is a plane in F_1.*
(2) *Flop the S_1 in F_1 to get F_2, which is isomorphic to F.*

Proof. We construct ι: Let $[m] \in F$ be a line not contained in $F(Y)$. Then $m \cap Y = \{y\}$, a nonsingular point of Y. By Corollary 12, there exists a unique line $\ell^\vee \in P^\vee$ containing y. Let Π denote the plane spanned by $\ell^\vee$ and m; by assumption, $\Pi \not\subset X$. Thus we have

$$\Pi \cap X = m \cup \ell^\vee \cup \bar{m} \tag{10}$$

for some line $\bar{m} \in F$. Setting $\iota(m) = \bar{m}$, we get a morphism

$$\iota : F \setminus F(Y) \to F.$$

Since (10) is symmetric in m and $\bar{m}$, ι is an involution.

As constructed, ι is not well-defined along $F(Y)$. It remains to show that it extends to $m \in P^\vee \setminus (P \cup S)$. Proposition 13 implies that m does not contain any singularities of Y. The normal bundle to m in X is one of the following [11, Proposition 6.19]

$$\mathcal{N}_{m/X} \simeq \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1), \mathcal{O}(-1) \oplus \mathcal{O}(1)^{\oplus 2}.$$

Since m does not contain any of the singularities of Y, Corollary 12 implies that we have the first case. But then there exists a distinguished plane Π with

$$\Pi \cap X = 2m \cup \bar{m},$$

i.e., Π corresponds to the directions associated with the $\mathcal{O}(1)$-summand. Consider the correspondence

$$Z = \{(m, \ell^\vee, \Pi) : \Pi \cap X \supset m \cap \ell^\vee\} \subset F \times P^\vee \times \mathrm{Gr}(2,5);$$

the normal bundle computation guarantees that the projection

$$Z \to F$$

is an isomorphism along $P^\vee \setminus (P \cup S)$. By definition, ι is regular on Z and thus at the generic point of $P^\vee$.

We will use the following notation for our factorization

$$\begin{array}{ccccccccc} & & F_{01} & & & & F_{12} & & \\ & \overset{\beta_{10}}{\swarrow} & & \overset{\beta_{01}}{\searrow} & & \overset{\beta_{21}}{\swarrow} & & \overset{\beta_{12}}{\searrow} & \\ F = F_0 & & & & F_1 & & & & F_2 \\ & \overset{\gamma_{10}}{\searrow} & & \overset{\gamma_{01}}{\swarrow} & & \overset{\gamma_{21}}{\searrow} & & \overset{\gamma_{12}}{\swarrow} & \\ & & \bar{F}_{01} & & & & \bar{F}_{12} & & \end{array}$$

where β_{10} blows up P, β_{01} blows down the exceptional divisor of β_{10}, β_{21} blows up S_1 (the proper transform of S), and β_{12} blows down the exceptional divisor of β_{12}. Here $\bar{F}_{01}$ and $\bar{F}_{12}$ denote the singular varieties obtained by contracting P and S_1 to a point. In other words, F_2 is obtained from F_0 by two Mukai flops. Moreover, we will show that $\iota : F \dashrightarrow F$ is resolved on passage to F_2, so the induced

$$\iota_2 : F_2 \to F \tag{11}$$

is necessarily an isomorphism.

Let $P_1 \subset F_1$ denote the plane that results from flopping P.

Lemma 25. *S_1 is isomorphic to $\mathbb{P}^2$ and meets P_1 transversely at six points.*

Proof. Proposition 13 describes how S' and P intersect: S' has two smooth branches meeting transversely in F, each of which meets $P \simeq \mathbb{P}^2$ in a line. If S° is the smooth locus of S' then

$$P \cap S^\circ \subset S^\circ$$

is Cartier, hence $\beta_{10}^{-1}(S^\circ) \simeq S^\circ$. However, S' fails to be Cohen-Macaulay at the points of $S' \setminus S^\circ$, so any Cartier divisor through these points would necessarily have an embedded point. In particular,

$$P \cap S' \subset S'$$

is not Cartier at singular points of S' and β_{10} necessarily modifies S' at these points.

We claim that the proper transform S_{01} of S' in F_{01} is just S. The easiest way to see this is through a local computation. At each singularity of S' choose local coordinates $\{x_1, x_2, x_3, x_4\}$ such that

$$S' = \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}, \quad P = \{x_2 = x_3 = 0\}.$$

The blow-up of P has homogeneous equation

$$Ax_3 = Bx_2$$

and thus the proper transforms of the components of S' are disjoint and mapped isomorphically onto their images.

We next show that β_{01} contracts the double-six in S corresponding to the intersection of P with S'. The key ingredient is the local description of proper-transforms of Lagrangian submanifolds under Mukai flops given in [30, §4.2]: Locally, a holomorphic-symplectic fourfold containing a plane looks like the total space of the cotangent-bundle of $\mathbb{P}^2$. In the cotangent bundle, a complex Lagrangian submanifold is modelled locally as the conormal sheaf $\mathcal{N}_V^*$ of a complex submanifold $V \subset \mathbb{P}^2$. The Mukai flop is realized as the cotangent bundle of the dual plane $\check{\mathbb{P}}^2$; the proper transform of the Lagrangian submanifold looks locally like the conormal sheaf $\mathcal{N}_{\check{V}}^*$ of the projective dual $\check{V} \subset \check{\mathbb{P}}^2$. Since each branch of S' looks locally like $\mathcal{N}_\ell^*$ for a line $\ell \subset \mathbb{P}^2$, its proper transform looks locally like $\mathcal{N}_{[\ell]}^*$, where $[\ell] \in \check{\mathbb{P}}^2$ classifies ℓ. In particular, the proper transform S_1 meets P_1 in six points and $\beta_{01} : S_{01} \to S_1$ blows down the double-six (corresponding to $S' \cap P$) to these points.

The analysis in Proposition 13 implies $S_1 \simeq \mathbb{P}^2$, and thus is a Lagrangian plane in F_1. □

We define F_2 as the Mukai flop of this plane and let S_2 denote the resulting plane, P_2 the proper transform of P_1, and

$$\phi : F_2 \dashrightarrow F$$

the composition of Mukai flops. Now P_2 is isomorphic to a cubic surface, meeting S_2 along a double-six as described before.

It remains to show that ι is resolved on F_2. Let g denote the polarization on $F \subset \mathrm{Gr}(2,5)$ induced by the hyperplane class on the Grassmannian. The pull back $\iota^* g$ is globally generated away $S \cup P$, the indeterminacy of ι. We shall use the following result of Boucksom [7]: Let Z be an irreducible holomorphic symplectic variety and f a divisor class on Z; then f is ample if and only if f meets each rational curve on Z positively. In particular, $\iota^* g$ necessarily meets some rational curves supported in $S \cup P$ negatively. Consider the map on one-cycles modulo homological equivalence induced by normalization and inclusion

$$j_* : N_1(\tilde{S}, \mathbb{Z}) \oplus N_1(P, \mathbb{Z}) \to N_1(F, \mathbb{Z}).$$

This has a nontrivial kernel: The description in Proposition 13 implies that

$$j_*(E_i, 0) = j_*(0, [\mathrm{line}]), \quad i = 1, \ldots, 6,$$

hence j_* has rank two, and the image of the effective curve classes is spanned by $\lambda_1 = j_*(0, [\mathrm{line}])$ and $\lambda_2 = j_*(\beta_{01}^*[\mathrm{line}], 0)$. In particular, all the effective curve classes where $\iota^* g$ is negative can be expressed as linear combinations of λ_1 and λ_2 with nonnegative coefficients.

We now analyze the pull-back $f = \phi^* \iota^* g$. Since ϕ is an isomorphism away from $P_2 \cup S_2$, any curve of F_2 not contained in $P_2 \cup S_2$ meets f positively. Just as before, the image of the effective curve classes in $P_2 \cup S_2$ are a cone generated by two classes λ_1' and λ_2'. However, using the identifications

$$H_2(F_2, \mathbb{Q}) \xrightarrow{\sim} H^2(F_2, \mathbb{Q}) \xrightarrow{\iota^*} H^2(F, \mathbb{Q}) \xrightarrow{\sim} H_2(F, \mathbb{Q})$$

we see that $\lambda_1' = -\lambda_1$ and $\lambda_2' = -\lambda_2$. It follows that f is positive along *all* curves classes in F_2 and thus is ample.

To reiterate, the composition (cf. Equation 11)

$$\iota_2 = \iota \circ \phi : F_2 \dashrightarrow F$$

is a birational map of smooth projective varieties, and takes the ample divisor g to an ample divisor f. It follows that ϕ is an isomorphism. □

Remark 26. The strategy of our argument is due to Burns-Hu-Luo [9], who prove that any birational morphism of irreducible holomorphic symplectic varieties with *normal* exceptional loci is a sequence of Mukai flops. The normality assumption can be eliminated (cf. [40, 1.2]).

Proposition 27. *Retain the notation of Theorem 24. Let* $[v] \in \mathbb{P}(V)$ *and* $T^\vee := T_v \subset Y$ *denote one of the cubic scrolls described in Proposition 17. The divisor*

$$\tau^\vee = \{[m] \in F : m \cap T^\vee \neq \emptyset\} \subset F$$

is invariant under ι.

Proof. By Proposition 17, the ruling of $T^\vee$ is the rational curve

$$\lambda_1^\vee = \{[v^\vee] : v^\vee(v) = 0\} \subset P^\vee \subset F(Y) \subset F.$$

If m is incident to $T^\vee$ then m meets some ruling $\ell_{[v^\vee]} \subset T^\vee$ where $[v^\vee] \in \lambda_1^\vee$. Thus $\ell_{[v^\vee]}$ coincides with the line $\ell^\vee$ used in the construction of ι. Since the lines $\{m, \ell_{[v^\vee]}, \iota(m)\}$ are coplanar, we have $\iota(m) \cap \ell_{[v^\vee]} \neq \emptyset$ and $\iota(m) \in \tau^\vee$. □

7. Cones of moving and ample divisors

Let X be a smooth cubic fourfold and F its variety of lines. The incidence correspondence

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\psi} \\ X & & F \end{array}$$

induces the *Abel Jacobi map* of integral Hodge structures [6]

$$\alpha = \psi_* \phi^* : H^4(X) \to H^2(F).$$

This is compatible with quadratic forms: Writing $g = \alpha(h^2)$ we have

$$(g, g) = 2\left\langle h^2, h^2 \right\rangle = 6$$

and

$$(\alpha(z_1), \alpha(z_2)) = -\langle z_1, z_2 \rangle \text{ for all } z_1, z_2 \in (h^2)^\perp.$$

For general cubic fourfolds we have

$$H^4(X, \mathbb{Z}) \cap H^{2,2}(X, \mathbb{C}) = \mathbb{Z}h^2,$$

but special cubic fourfolds admit additional algebraic cycles [21]:

Proposition 28. *Let $\mathcal{C}$ denote the moduli space of smooth cubic fourfolds and $\mathcal{C}_d \subset \mathcal{C}$ the cubic fourfolds X admitting a rank-two saturated lattice of Hodge cycles*

$$h^2 \in K_d \subset H^4(X,\mathbb{Z}) \cap H^{2,2}(X,\mathbb{C})$$

of discriminant d. Then $\mathcal{C}_d$ is nonempty if and only if

$$d \equiv 0,2 \pmod 6, \quad d > 6.$$

In this case, $\mathcal{C}_d$ is an irreducible divisor in $\mathcal{C}$.

Assume that X contains a smooth cubic scroll T and let Y denote the hyperplane section of X containing T. The intersection form $\langle , \rangle$ on the middle cohomology of X restricts to

$$K_{12} := \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 3 \\ T & 3 & 7 \end{array}$$

a lattice of discriminant 12. Moreover, $\mathcal{C}_{12}$ is the closure of the locus of cubic fourfolds containing a cubic scroll or a hyperplane section with six double points in general position.

Let $\tau = \alpha(T)$ so that the Beauville-Bogomolov form restricts to

$$J_{12} := \begin{array}{c|cc} & g & \tau \\ \hline g & 6 & 6 \\ \tau & 6 & 2 \end{array}$$

a lattice of discriminant -24. We summarize elementary properties of this lattice:

(1) The elements of $J_{12} \otimes_{\mathbb{Z}} \mathbb{R}$ with nonnegative Beauville form are a union of convex cones $\mathcal{P} \cup -\mathcal{P}$ where

$$\mathcal{P} = \mathrm{Cone}\left(g - (3-\sqrt{6})\tau, (3+\sqrt{6})\tau - g\right).$$

(2) J_{12} does not represent -2 or 0.

(3) The automorphism group of J_{12} is isomorphic to the direct product of $\langle \pm 1 \rangle$ and the infinite dihedral group

$$\Gamma := \left\langle R_1, R_2 : R_1^2 = R_2^2 = 1 \right\rangle$$

where the reflections R_1, R_2 can be written

$$\begin{array}{rclrcl} R_1(g) & = & g & R_1(\tau) & = & 2g - \tau \\ R_2(g) & = & -g + 6\tau & R_2(\tau) & = & \tau. \end{array}$$

Γ consists of the automorphisms taking $\mathcal{P}$ to itself.

(4) J_{12} represents -10. We list (-10)-classes with positive intersection with g:

$$\begin{array}{rcl|rcl}
\multicolumn{3}{c}{\text{class}} & \multicolumn{3}{c}{\text{intersection with } g} \\
\hline
 & \vdots & & & \vdots & \\
\rho_3^\vee & = & 16\tau - 3g & (\rho_3^\vee, g) & = & 78 \\
\rho_2^\vee & = & 4\tau - g & (\rho_2^\vee, g) & = & 18 \\
\rho_1^\vee & = & 2\tau - g & (\rho_1^\vee, g) & = & 6 \\
\rho_1 & = & 3g - 2\tau & (\rho_1, g) & = & 6 \\
\rho_2 & = & 7g - 4\tau & (\rho_2, g) & = & 18 \\
\rho_3 & = & 29g - 16\tau & (\rho_3, g) & = & 78 \\
 & \vdots & & & \vdots &
\end{array}$$

For $j \geq 3$ we define recursively

$$\rho_j = (R_1 R_2)\rho_{j-2} \quad \text{and} \quad \rho_j^\vee = (R_2 R_1)\rho_{j-2}^\vee.$$

(5) The element $R_1 R_2$ has infinite order and acts on the (-10)-classes with orbits:

$$\begin{array}{cc}
\{\ldots, \rho_3^\vee, \rho_1^\vee, -\rho_2, \ldots\} & \{\ldots, -\rho_3^\vee, -\rho_1^\vee, \rho_2, \ldots\} \\
\{\ldots, -\rho_2^\vee, \rho_1, \rho_3 \ldots\} & \{\ldots, \rho_2^\vee, -\rho_1, -\rho_3 \ldots\}.
\end{array}$$

The element R_1 has order two and acts via

$$R_1(\rho_i) = \rho_i^\vee$$

for each i.

Proposition 29. *Let X be a smooth cubic fourfold with variety of lines F. Assume that X contains a smooth cubic scroll T with*

$$H^4(X, \mathbb{Z}) \cap H^{2,2}(X, \mathbb{C}) = \mathbb{Z}h^2 + \mathbb{Z}T$$

or equivalently

$$H^2(F, \mathbb{Z}) \cap H^{1,1}(F, \mathbb{C}) = \mathbb{Z}g + \mathbb{Z}\tau.$$

The nef cone of F equals the cone dual to $\mathrm{Cone}(\rho_1, \rho_1^\vee)$*, i.e.,* $\mathrm{Cone}(\alpha_1, \alpha_1^\vee)$ *where $\alpha_1 = 7g - 3\tau$ and $\alpha_1^\vee = g + 3\tau$.*

Proof. The main theorem of [23] asserts that a divisor class f on F is nef if $(f, \rho) \geq 0$ for each divisor class ρ on F satisfying $(g, \rho) > 0$ and either $(\rho, \rho) \geq -2$ or $(\rho, \rho) - 10$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$. Thus the nef cone contains $\mathrm{Cone}(\alpha_1, \alpha_1^\vee)$.

It remains to show that α_1 and $\alpha_1^\vee$ are at the boundary of the nef cone. We show they induce nontrivial contractions of F. It suffices to prove this for a generic cubic fourfold containing a cubic scroll, so we may assume Y has exactly six ordinary double points in linear general position. It follows that there exist Lagrangian planes

$$P, P^\vee \subset Y \subset F$$

such that lines $\lambda_1 \subset P$ and $\lambda_1^\vee \subset P^\vee$ both have degree three. The classes of these lines are dual to ρ_1 and $\rho_1^\vee$ respectively. The divisors α_1 and $\alpha_1^\vee$ induce small contractions

$$\gamma_{10} : F = F_0 \to \bar{F}_{01}, \quad \gamma_{01}^\vee : F_0 \to \bar{F}_{10}^\vee$$

of P and $P^\vee$ respectively. □

We may flop the plane P (resp. $P^\vee$) in F to obtain new holomorphic symplectic fourfold F_1 (resp. $F_1^\vee$). The birational maps between F, F_1, and $F_1^\vee$ induce identifications of their Picard groups. The first step in analyzing their nef cones is to enumerate the orbit of α_1 under Γ:

class			intersection with g		
	$\vdots$			$\vdots$	
$\alpha_2^\vee = R_2R_1(\alpha_1)$	$=$	$9\tau - g$	$(\alpha_2^\vee, g)$	$=$	48
$\alpha_1^\vee = R_1(\alpha_1)$	$=$	$g + 3\tau$	$(\alpha_1^\vee, g)$	$=$	24
α_1	$=$	$7g - 3\tau$	(α_1, g)	$=$	24
$\alpha_2 = R_1R_2(\alpha_1^\vee)$	$=$	$17g - 9\tau$	(α_2, g)	$=$	48
	$\vdots$			$\vdots$	

For $j \geq 3$ we define recursively

$$\alpha_j = R_1R_2(\alpha_{j-2}) \text{ and } \alpha_j^\vee = R_2R_1(\alpha_{j-2}^\vee).$$

Observe that $(\alpha_i^\vee, \rho_i^\vee) = (\alpha_i, \rho_i) = 0$ for each $i \geq 1$.

Proposition 30. *The nef cone of F_1 (resp. $F_1^\vee$) equals* $\mathrm{Cone}(\alpha_1, \alpha_2)$ *resp.* $\mathrm{Cone}(\alpha_2^\vee, \alpha_1^\vee)$.

Figure 1 illustrates the relative positions of the nef cones of F and its flop F_1.

Proof. Since F_1 is the flop of F along P, it is clear that α_1 (which induces the contraction of P) is one generator of the nef cone. The factorization given in Theorem 24 shows that ι induces a *regular* involution on F_1. Proposition 27 implies that ι fixes the divisor

$$\tau^\vee = \alpha(T^\vee) = \alpha(2h^2 - T) = 2g - \tau,$$

where the middle equality uses Proposition 23. Consequently, ι acts on J_{12} via the reflection R_3 through the line orthogonal to $2g - \tau$

$$R_3(g) = 11g - 6\tau, \quad R_3(\tau) = 20g - 11\tau.$$

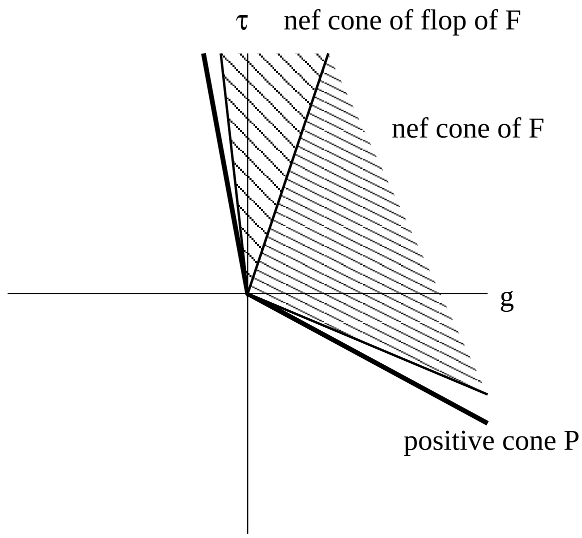

FIGURE 1. The nef cones of F and $F_1^\vee$

The second generator of the nef cone of F_1 is thus

$$R_3(\alpha_1) = 17g - 9\tau = \alpha_2.$$

□

Theorem 24 gives isomorphisms between every second model, so Proposition 29 and 30 suffice to describe the nef cone of *every* model of F (see Figure 2). We summarize our whole discussion:

Theorem 31. *Suppose that X is a smooth cubic fourfold containing a smooth cubic scroll T with*

$$H^4(X,\mathbb{Z}) \cap H^{2,2}(X,\mathbb{C}) = \mathbb{Z}h^2 + \mathbb{Z}T$$

and let $F = F_0$ denote the variety of lines on X. Then we have an infinite sequence of Mukai flops

$$\cdots F_2^\vee \dashrightarrow F_1^\vee \dashrightarrow F_0 \dashrightarrow F_1 \dashrightarrow F_2 \cdots$$

with isomorphisms between every other flop in this sequence

$$\cdots F_2^\vee \xrightarrow{\sim} F_0 \xrightarrow{\sim} F_2 \cdots \quad \textit{and} \quad \cdots F_1^\vee \xrightarrow{\sim} F_1 \cdots$$

The positive cone of F can be expressed as the union of the nef cones of the models $\{\cdots, F_1^\vee, F_0, F_1, \cdots\}$:

$$\cdots, \mathrm{Cone}(\alpha_2^\vee, \alpha_1^\vee),\ \mathrm{Cone}(\alpha_1^\vee, \alpha_1),\ \mathrm{Cone}(\alpha_1, \alpha_2), \cdots$$

The isomorphisms induce an action of $\mathbb{Z} \simeq \langle R_1 R_2 \rangle \subset \Gamma$ on the Picard group of F.

Figure 2 is a schematic illustration of the partition of the positive cone into ample cones for isomorphism classes of minimal models. This verifies the conjectures of [22] for the birational models of F. It

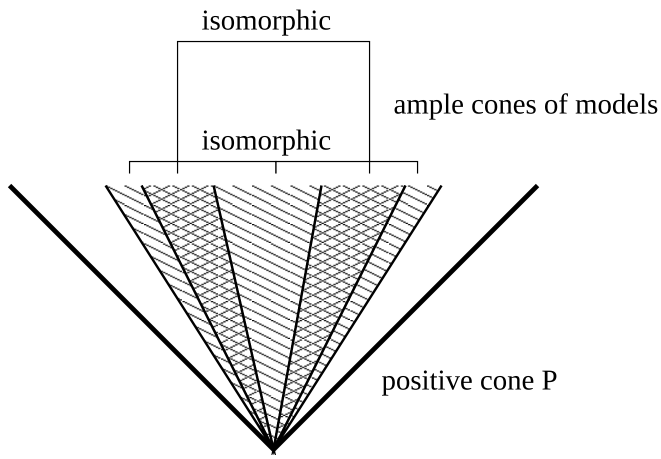

FIGURE 2. Partition of the positive cone into ample cones for various minimal models

also illustrates the Finiteness of Models Conjecture of Kawamata and Morrison–here we have two birational models up to isomorphism.

Remark 32 (Application to rational points)**.** Let F be the variety of lines of a cubic fourfold X containing a cubic scroll T, both defined over a field k. Assume that the hyperplane section containing T has precisely six ordinary double points in linear general position and X does not contain a plane. Then k-rational points on F are Zariski dense. Indeed, the infinite collection of Lagrangian planes defined over k is Zariski dense.

If the Picard group of F has rank two then

- F does not admit regular automorphisms, and
- F is not birational to an abelian fibration.

Potential density of rational points on varieties of lines on *generic* cubic fourfolds over number fields has recently been established in [2].

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