ON THE BRAUER–MANIN OBSTRUCTION FOR INTEGRAL POINTS

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Abstract. We give examples of Brauer–Manin obstructions to integral points on open subsets of the projective plane.

1. Introduction

Let k be a number field and X a smooth projective geometrically irreducible variety over k. It is well known that the existence of points of X over all completions k_v of k does not imply the existence of a k-rational point on X, in general. This phenomenon is referred to as the failure of the Hasse principle. Examples of failure of the Hasse principle are known for genus 1 curves, cubic surfaces, etc. Even when the Hasse principle holds, rational points need not be dense in the set of adelic points of X. This phenomenon, the failure of weak approximation, also is known in many examples.

The Brauer–Manin obstruction [Man71], [Man74] often explains the failure of the Hasse principle and weak approximation. The exact sequence from class field theory

$$0 \to \mathrm{Br}(k) \longrightarrow \bigoplus_v \mathrm{Br}(k_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

(where $\sum \mathrm{inv}_v$ denotes the sum of local invariants) leads to the constraint

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} := \{\, (x_v) \in X(\mathbb{A}_k) \mid \sum \mathrm{inv}_v(\alpha|_{x_v}) = 0 \,\forall\, \alpha \in \mathrm{Br}(X) \,\}$$

on the set $X(\mathbb{A}_k)$ of adelic points on X. When $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$, then we say there is a Brauer–Manin obstruction to the Hasse principle, respectively to weak approximation, in case $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, respectively, $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

For a thorough introduction to the subject, see [Sko01]. For a survey, see [Pey05].

The study of rational points on projective hypersurfaces is equivalent to the study of integral solutions to homogeneous Diophantine equations $f(x_0, \dots, x_n) = 0$. Many interesting Diophantine problems involve non-homogeneous equations. Their solutions can be interpreted as integral points on quasi-projective varieties.

Let $\mathfrak{o}_k$ be the ring of integers of k. Let $\mathcal{X}$ be an integral model for X, i.e., a scheme, projective and flat over $\mathrm{Spec}(\mathfrak{o}_k)$ having general fiber X. Let Z be a reduced closed subscheme of X, and set $U = X \smallsetminus Z$. Then we define $\mathcal{Z}$ to be the scheme-theoretic closure of Z in $\mathcal{X}$ and set $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{Z}$. We then have $\mathcal{U}(\mathfrak{o}_k)$, the *integral points* of $\mathcal{U}$. By abuse the terminology, we say that a k-rational point of U is an integral point if its (unique) extension to an $\mathfrak{o}_k$-point of $\mathcal{X}$ has image in $\mathcal{U}$. (This notion depends on the choice of integral model $\mathcal{X}$). There are S-integral points $\mathcal{U}(\mathfrak{o}_{k,S})$ for S a finite set of non-archimedean places of k ($\mathfrak{o}_{k,S}$ denotes the ring of S-integral elements of k)

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and v-adic integral points $\mathcal{U}(\mathfrak{o}_v)$ for v a non-archimedean place of k; the latter will be identified with k_v-points of U that extend to $\mathcal{U}(\mathfrak{o}_v)$.

The Brauer–Manin obstruction has been extensively studied in the setting of projective varieties; see, e.g., [CTKS87]. Its study in the context of open varieties is more recent and originates in the work of Colliot-Thélène and Xu [CTX07], which gives a new explanation based on the Brauer–Manin obstruction for the failure of Hasse principle exhibited in [BR95] and [SX04] in the representation of integers by quadratic forms in three variables.

The insolubility of a Diophantine equation that admits solutions in k and in v-adic integers for all non-archimedean places v of k can be a manifestation of the failure of the Hasse principle, or strong approximation, for a variety. (For smooth projective varieties, strong and weak approximation are the same.) In this paper we give examples of this, that can be explained by the Brauer–Manin obstruction:

$$\mathcal{U}(\mathfrak{o}_k) \subset \big(\prod_{v\nmid\infty} \mathcal{U}(\mathfrak{o}_v) \times \prod_{v\mid\infty} U(k_v)\big)^{\mathrm{Br}(U)}, \tag{1}$$

where the set on the right is the set of tuples of adelic points, integral at all non-archimedean places, whose sum of local invariants is zero with respect to every element of $\mathrm{Br}(U)$. The formulation of the obstruction, and in particular the use of $\mathrm{Br}(U)$ rather than $\mathrm{Br}(\mathcal{U})$, follows Colliot-Thélène and Xu (see [CTX07] §1).

This note is inspired by lectures of J.-L. Colliot-Thélène on his joint work with F. Xu. As in their work, we take U to be the complement of a geometrically irreducible smooth divisor D on a surface X. In their work, X is a quadric surface and D a hyperplane section. Here for simplicity we take $k = \mathbb{Q}$ and $X = \mathbb{P}^2$, so that if homogeneous $f(x,y,z) \in \mathbb{Z}[x,y,z]$ defines D (and $\mathcal{D}$), then elements of $\mathcal{U}(\mathbb{Z})$ correspond to triples of integers $(x:y:z)$ (up to a factor ± 1) such that $f(x,y,z) = \pm 1$, where $\mathcal{U} = \mathbb{P}^2_{\mathbb{Z}} \smallsetminus \mathcal{D}$. The geometric Brauer group of U is understood by an exact sequence [AM92] that reduces in this case to an isomorphism, the *ramification map*:

$$\mathrm{Br}(U \otimes \overline{\mathbb{Q}}) \xrightarrow{\sim} H^1(D \otimes \overline{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}). \tag{2}$$

Our $D \subset \mathbb{P}^2$ will admit unramified coverings over $\mathbb{Q}$, such that known constructions of algebras representing ramified Brauer group elements can be carried out over $\mathbb{Q}$. For the local analysis we must restrict the Brauer group elements to points of $U(\mathbb{R})$ and to the p-adic integral points in $U(\mathbb{Q}_p)$. These correspond to triples $(x:y:z)$ of reals satisfying $f(x,y,z) \in \mathbb{R}^*$, respectively p-adic integers satisfying $f(x,y,z) \in \mathbb{Z}_p^*$.

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2. Cubic

This section is devoted to the following example, concerning rational and integral points on the complement of a plane cubic curve over $\mathbb{Q}$.

Example 1. *For the Diophantine equation*

$$y^2 z - (4x - z)(16x^2 + 20xz + 7z^2) = 1. \tag{3}$$

we have:

(i) *There are solutions in p-adic integers for all primes p.*
(ii) *There are solutions in $\mathbb{Q}$.*
(iii) *There are no solutions in $\mathbb{Z}$.*

There is the rational solution $(1/4, 1, 1)$, which is a p-adic integer solution for $p \neq 2$. A 2-adic integer solution is $(0, 0, \sqrt[3]{1/7})$, so statements (i) and (ii) are established.

The proof of (iii) is more subtle and uses the Brauer group. Letting the polynomial

$$f(x, y, z) = y^2 z - (4x - z)(16x^2 + 20xz + 7z^2)$$

define the divisor $D \subset \mathbb{P}^2$ (and $\mathcal{D} \subset \mathbb{P}^2_{\mathbb{Z}}$) and setting $U = \mathbb{P}^2 \smallsetminus D$ (and $\mathcal{U} = \mathbb{P}^2_{\mathbb{Z}} \smallsetminus \mathcal{D}$) we will exhibit a 2-torsion ramified Brauer group element $A \in \mathrm{Br}(U)$ whose pull-back to $\mathrm{Br}(U \otimes \overline{\mathbb{Q}})$ has prescribed (nontrivial) image under the ramification map. With A, we are able to deduce from (1) a congruence condition on integral points on U that is incompatible with (3).

Remark 1. Since f is homogeneous of odd degree, the insolubility of (3) in integers implies $\mathcal{U}(\mathbb{Z}) = \emptyset$. So, statements (i)–(iii) of Example 1 signify a Brauer–Manin obstruction to the Hasse principle over $\mathbb{Z}$.

Remark 2. The algebraic fundamental group of $U \otimes \overline{\mathbb{Q}}$ is cyclic of order 3 by [Zar29] Theorem 8, [SGA1] (XII.5.2), and [SGA1] (X.1.8) with appeal to (XIII.4.6) instead of (X.1.7). Hence a universal cover of $U \otimes \overline{\mathbb{Q}}$ is $\widetilde{U} \otimes \overline{\mathbb{Q}}$, where $\widetilde{U}$ is the affine open subscheme defined by $t \neq 0$ of the cubic surface $\widetilde{X}$ with defining equation $f(x, y, z) = t^3$, and the covering map is given by

$$(x : y : z : t) \mapsto (x : y : z).$$

For the degree 3 extension of function fields $\overline{\mathbb{Q}}(U) \to \overline{\mathbb{Q}}(\widetilde{U})$, there are restriction and corestriction maps on Brauer groups, and restriction followed by corestriction acts as multiplication by 3 on $\mathrm{Br}(\overline{\mathbb{Q}}(U))$. Nontriviality of the pull-back of A to $\mathrm{Br}(U \otimes \overline{\mathbb{Q}})$ (see Lemma 1, below) implies: the 2-torsion element A remains nontrivial, hence *ramified* (since $\mathrm{Br}(\widetilde{X} \otimes \overline{\mathbb{Q}}) = 0$), upon pull-back to the geometric universal cover.

Remark 3. We can state an integer-point analogue of a basic obstruction to rational points coming from torsor theory and check that it does not obstruct integral points on U. Letting the group μ_3 of cube roots of unity act on the variety $\widetilde{U}$ of Remark 2 by multiplication on the t coordinate gives $\widetilde{U}$ the structure of U-torsor under μ_3. Torsor theory, described in [Sko01] §2.2, supplies arithmetic twists $\pi_\gamma \colon \widetilde{U}_\gamma \to U$ indexed by $\gamma \in H^1(\mathbb{Q}, \mu_3)$, with the property that $U(\mathbb{Q})$ is the disjoint union of the images in U of the $\widetilde{U}_\gamma(\mathbb{Q})$. Obstructions based on the consequence that $U(\mathbb{Q}) = \emptyset$ if $\widetilde{U}_\gamma(\mathbb{Q}) = \emptyset$ for all γ are called *descent obstructions*, cf. [Sko01] §5.3. An easy integer-point descent obstruction states: *if, for every $\gamma \in H^1(\mathbb{Q}, \mu_3)$, there exists a prime p_γ such that $\pi_\gamma(\widetilde{U}_\gamma(\mathbb{Q}_{p_\gamma}))$ contains no p_γ-adic integral points, then $\mathcal{U}(\mathbb{Z}) = \emptyset$.* We know *a priori* that there are only finitely many classes γ such that $\pi_\gamma(\widetilde{U}_\gamma(\mathbb{Q}_{p_\gamma}))$ contains a p_γ-adic integral point (see the next paragraph). For the untwisted $\widetilde{U}_0 = \widetilde{U}$ we have by statement (i) that $\pi_0(\widetilde{U}_0(\mathbb{Q}_p))$ contains p-adic integral points for every prime p.

Proposition 5.3.2 of [Sko01] tells us that the descent obstruction to rational points on a *proper* variety over a number field k reduces to analysis on just finitely many arithmetic twists of a given torsor. In the proof, properness is used only for the assertion that every k-rational point extends to an $\mathfrak{o}_k$-point of a fixed integral model. So we can apply the argument in the setting of integral points on an open subvariety. Viewing μ_3 as a smooth group scheme over $\mathrm{Spec}(\mathbb{Z}[1/3])$, then for suitable N, $\widetilde{U}$ extends to some $\mathcal{U} \otimes \mathbb{Z}[1/3N]$-torsor $\widetilde{\mathcal{U}} \otimes \mathbb{Z}[1/3N]$ under μ_3. For a prime $p \nmid 3N$ and point $s \in U(\mathbb{Q}_p)$, the classes γ such that $s \in \pi_\gamma(\widetilde{U}_\gamma(\mathbb{Q}_p))$ are precisely those whose image under $H^1(\mathbb{Q},\mu_3)\to H^1(\mathbb{Q}_p,\mu_3)$ is the class of the restriction $\widetilde{U}|_s$ of the U-torsor $\widetilde{U}$ to s. If the point s is a p-adic integral point then $\widetilde{U}|_s$ is isomorphic to the restriction of a $\mathrm{Spec}(\mathbb{Z}_p)$-torsor under μ_3. The proof of [Sko01] Proposition 5.3.2, establishes that for a number field k, linear algebraic group G over k, finite set S of non-archimedean places of k, and smooth group scheme $\mathcal{G}_S\to\mathrm{Spec}(\mathfrak{o}_{k,S})$ extending G, there are only finitely many $\gamma\in H^1(k,G)$ whose image in $H^1(k_v,G)$ is the restriction of a class in $H^1(\mathfrak{o}_v,\mathcal{G}_S)$ for every non-archimedean place $v\notin S$.

Returning to the verification of (iii), D is an elliptic curve over $\mathbb{Q}$, and its group structure is readily computed to be $D(\mathbb{Q})\cong\mathbb{Z}/2\mathbb{Z}$. Hence there is a unique (up to isomorphism) nontrivial unramified cover $\widetilde{D}\otimes\overline{\mathbb{Q}}\to D\otimes\overline{\mathbb{Q}}$ that can be obtained by base change from some $\widetilde{D}\to D$ defined over $\mathbb{Q}$. The cover has degree 2.

Lemma 1. *The class in* $\mathrm{Br}(\mathbb{Q}(U))$ *of the quaternion algebra*

$$(y^2z^{-2}-(4x-z)(16x^2+20xz+7z^2)z^{-3},(4x-z)z^{-1}) \tag{4}$$

is the restriction of a Brauer group element $A\in\mathrm{Br}(U)$*, such that the pull-back of* A *to* $\mathrm{Br}(U\otimes\overline{\mathbb{Q}})$ *is sent by the ramification map to the class of the extension* $\overline{\mathbb{Q}}(D)\to\overline{\mathbb{Q}}(\widetilde{D})$.

Proof. The assertion about $\mathrm{Br}(U)$ uses the fact, immediate from [Gro68] (III.6.2), that the image of $\mathrm{Br}(U)$ in $\mathrm{Br}(\mathbb{Q}(U))$ is the kernel of a prescribed map $\mathrm{Br}(\mathbb{Q}(U))\to\bigoplus_{u\in U^{(1)}}H^1(\mathbf{k}_u,\mathbb{Q}/\mathbb{Z})$, where the sum is over generic points u of codimension 1 subvarieties of U, with $\mathbf{k}_u$ the residue field at such a point u. The argument of [AM92] §3 supplies a concrete description of this map, also called ramification map (since the base field $\mathbb{Q}$ is not algebraically closed, the right-hand 0 in the exact sequence [AM92] (3.2) needs to be replaced with $H^3(\mathbb{Q}(U),\mathbb{G}_m)$ in the present setting). Given rational functions f, $g\in\mathbb{Q}(U)^*$, if f and g are both units in $\mathcal{O}_{U,u}$ then $(f,g)\in\mathrm{Br}(U)$ lies in the kernel of the ramification map at u, while if f is a uniformising element for the discrete valuation ring $\mathcal{O}_{U,u}$ and g is a unit in $\mathcal{O}_{U,u}$, then (f,g) lies in the kernel of the ramification map when g is a square in $\mathbf{k}_u$ and otherwise is sent to the class of $\mathbf{k}_u\to\mathbf{k}_u(g^{1/2})$. Immediately from (4), then, A lies in the kernel of the ramification map at the line $4x-z=0$. We rewrite the class of (4) in $\mathrm{Br}(\mathbb{Q}(U))$ as

$$\begin{aligned}&(-zy^{-1}+(4x-z)(16x^2+20xz+7z^2)y^{-3},zy^{-1})\\&\qquad+(z^2y^{-2}-(4x-z)(16x^2+20xz+7z^2)zy^{-4},(4x-z)y^{-1})\end{aligned}$$

to see that A lies in the kernel of the ramification map at $z=0$. At other $u\in U^{(1)}$, the functions in (4) are both units in $\mathcal{O}_{U,u}$.

Upon base change to $\overline{\mathbb{Q}}$, the algebra (4) becomes the explicitly given symbol algebra in [Jac01], proof of the Theorem. There the image under the ramification map is computed and found to be as claimed. □

The local analysis of the algebra A at p-adic integral points of $\mathcal{U}$ is simplified by the following observation.

Lemma 2. *The element* $A\in\mathrm{Br}(U)$ *from Lemma 1 is the restriction of an element of* $\mathrm{Br}(\mathcal{U}\otimes\mathbb{Z}[1/2])$.

Proof. Lemma 1 implies that for suitable N there exists a scheme $\mathcal{U}_1$ of the form $\mathcal{U}\otimes\mathbb{Z}[1/2N]$ and $\mathcal{A}_1\in\mathrm{Br}(\mathcal{U}_1)$ extending A. If we define $\mathcal{U}_2$ to be the complement in $\mathcal{U}\otimes\mathbb{Z}[1/2]$ of the union of $\mathcal{D}$ and the scheme defined by $(4x-z)z=0$, then the rational functions in (4) are units in $\mathcal{O}_{\mathcal{U}_2}$, hence (4) defines $\mathcal{A}_2\in\mathrm{Br}(\mathcal{U}_2)$ also extending A. The restrictions of $\mathcal{A}_1$ and $\mathcal{A}_2$ to $\mathrm{Br}(\mathcal{U}_1 \cap \mathcal{U}_2)$ are equal since $\mathcal{U} \otimes \mathbb{Z}[1/2]$ is regular and $\mathcal{A}_1$ and $\mathcal{A}_2$ extend the same element of $\mathrm{Br}(\mathbb{Q}(U))$. So, by the Mayer-Vietoris sequence, A extends to an element of $\mathrm{Br}(\mathcal{U}_1 \cup \mathcal{U}_2)$. By a purity result of Gabber for three-dimensional regular schemes, [Gab81] Theorem 2′, the restriction map $\mathrm{Br}(\mathcal{U} \otimes \mathbb{Z}[1/2]) \to \mathrm{Br}(\mathcal{U}_1 \cup \mathcal{U}_2)$ is an isomorphism. □

At any 2-adic integral point $(x : y : z)$ of $\mathcal{U}$ satisfying

$$y \equiv 0 \pmod 2 \qquad \text{and} \qquad z \equiv 1 \pmod 2, \tag{5}$$

direct evaluation reveals that A is nonzero at $(x : y : z) \in U(\mathbb{Q}_2)$. By Lemma 2, A vanishes at all p-adic integral points of $\mathcal{U}$, for any odd prime p. Since $U(\mathbb{R})$ is connected, the behaviour of A at real points of U is revealed by evaluation at a single point, and we find that A vanishes at real points. By the condition (1), $\mathcal{U}$ has no integral points satisfying (5). Any integer solution to (3) would have to satisfy (5), as we see by reduction modulo 2. Since any integer solution to (3) would determine an integral point on $\mathcal{U}$, statement (iii) is established.

3. Quartic

In this section, we study the complement of a quartic curve in the projective plane over $\mathbb{Q}$.

Example 2. *For the Diophantine equation*

$$-2x^4 - y^4 + 18z^4 = 1 \tag{6}$$

we have:

(i) *There are solutions in p-adic integers for all primes p.*
(ii) *There are solutions in $\mathbb{Q}$.*
(iii) *There are no solutions in $\mathbb{Z}$.*

The rational solution $(1/2, 0, 1/2)$ is a p-adic integer solution for $p \neq 2$, and a 2-adic integer solution is $(0, \sqrt[4]{17}, 1)$. So, (i) and (ii) are established.

Let $D \subset \mathbb{P}^2$ and $\mathcal{D} \subset \mathbb{P}^2_{\mathbb{Z}}$ be defined by the equation

$$f(x, y, z) = -2x^4 - y^4 + 18z^4,$$

and set $U = \mathbb{P}^2 \smallsetminus D$ and $\mathcal{U} = \mathbb{P}^2_{\mathbb{Z}} \smallsetminus \mathcal{D}$. We will exhibit 2-torsion $A \in \mathrm{Br}(U)$ and use the constraint (1) dictated by A to establish (iii).

The equation for D can be expressed in the form

$$(4x^2 - y^2)^2 + 2(x^2 + 2y^2 + 9z^2)(x^2 + 2y^2 - 9z^2) = 0.$$

This shows that the extension of function fields $\mathbb{Q}(D) \to \mathbb{Q}(D)(\sqrt{x^2 + 2y^2 + 9z^2}/x)$ corresponds to an unramified degree 2 cover $\widetilde{D} \to D$.

Lemma 3. *The class in* $\mathrm{Br}(\mathbb{Q}(U))$ *of the quaternion algebra*[1]

$$(fh, -gh), \tag{7}$$

with

$$g = -28x^2 - 36xy + 7y^2 + 72z^2 \quad \text{and} \quad h = -25x^2 + 16xy - 22y^2 + 81z^2,$$

is the restriction of an element $A \in \mathrm{Br}(U)$. The pull-back of A to $\mathrm{Br}(U \otimes \overline{\mathbb{Q}})$ is sent by the ramification map to the class of the extension $\overline{\mathbb{Q}}(D) \to \overline{\mathbb{Q}}(\widetilde{D})$.

[1] We take the liberty here of writing homogeneous functions of even degree rather than rational functions.

Proof. For the first claim it suffices by [Gro68](III.6.2) to show that A is in the kernel of the ramification map at the divisors on U defined by g and by h. Since $(fh, -gh) = (f, g) + (f, -h) + (g, h)$ in $\mathrm{Br}(\mathbb{Q}(U))$, this reduces to the assertions that fg is a square modulo h and fh is a square modulo g, and these assertions can be verified directly.

The algebra (7), upon base change to $\overline{\mathbb{Q}}$, results by applying the recipe of [KRTY06], proof of Proposition 1.3(iii), to the ramification locus $D \otimes \overline{\mathbb{Q}}$ (after transforming by a linear change of coordinates that eliminates the y^4 term from f), wherein the image under the ramification map is calculated and found to be as claimed here. □

Lemma 4. *The element $A \in \mathrm{Br}(U)$ from Lemma 3 is the restriction of an element of $\mathrm{Br}(\mathcal{U} \otimes \mathbb{Z}[1/2])$.*

Proof. The element A extends to $\mathcal{U} \otimes \mathbb{Z}[1/N]$ for some integer N, as well as to the open subscheme of $\mathcal{U} \otimes \mathbb{Z}[1/2]$ where f, g, and h are nonvanishing, and hence to their union by the Mayer-Vietoris sequence. Gabber's purity result [Gab81] Theorem 2′ completes the proof. □

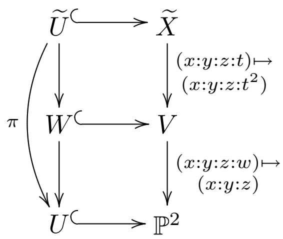
By direct evaluation, we see that for $(x : y : z) \in \mathcal{U}(\mathbb{Z}_2)$ satisfying

$$x \equiv 0 \pmod 2, \qquad y \equiv 1 \pmod 2, \qquad \text{and} \qquad z \equiv 1 \pmod 2, \tag{8}$$

A is nonzero at $(x : y : z) \in U(\mathbb{Q}_2)$. By Lemma 4, A vanishes at p-adic integral points of $\mathcal{U}$ for p odd. Evaluation at one point in each of the two connected components of $U(\mathbb{R})$ reveals that A vanishes at all real points of U. So, the constraint (1) dictates that $\mathcal{U}$ has no integral points satisfying (8). Since any integer solution to (6) would have to satisfy (8), we have established (iii).

Remark 4. Since $(0 : 1 : 0) \in \mathcal{U}(\mathbb{Z})$, Example 2 furnishes a Brauer–Manin obstruction to strong approximation of integral points, as formulated in [HT08] §2.

Remark 5. There is a tower of projective varieties and open subvarieties



where $\widetilde{X}$ is defined by $f(x, y, z) = t^4$ in $\mathbb{P}^4$, V is defined by $f(x, y, z) = w^2$ in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, and where $\widetilde{U} \subset \widetilde{X}$ and $W \subset V$ are defined by $t \neq 0$ and $w \neq 0$ respectively. By [Zar29] Theorem 8, [SGA1] (XII.5.2), and [SGA1] (X.1.8) with appeal to (XIII.4.6) instead of (X.1.7), $\widetilde{U} \otimes \overline{\mathbb{Q}}$ is a geometric universal cover of U. The pull-back of A to $\mathrm{Br}(W)$ is seen by direct evaluation to lie in the kernel of the ramification map on V, hence the pull-back of A to the geometric universal cover is *unramified*, and in fact, trivial (since $\mathrm{Br}(V \otimes \overline{\mathbb{Q}}) = 0$).

Remark 6. Statement (iii) is not, to the authors' knowledge, a consequence of any descent obstruction coming from U-torsors under finite algebraic groups. Remark 5 exhibits a U-torsor $\widetilde{U}$ under the group μ_4 of fourth roots of unity, acting by multiplication on the t coordinate. For any prime p, the set $\pi(\widetilde{U}(\mathbb{Q}_p))$ contains p-adic integral points, and some of these indeed satisfy (8) when $p = 2$.

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