UNIVERSAL TORSORS OVER DEL PEZZO SURFACES AND RATIONAL POINTS

by

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ABSTRACT. — We discuss Manin's conjecture concerning the distribution of rational points of bounded height on Del Pezzo surfaces, and its refinement by Peyre, and explain applications of universal torsors to counting problems. To illustrate the method, we provide a proof of Manin's conjecture for the unique split singular quartic Del Pezzo surface with a singularity of type $\mathbf{D}_4$.

Contents

1. Introduction

Let $f \in \mathbb{Z}[x_0, \ldots, x_n]$ be a form of degree d. By the circle method,

$$N(f, B) := \#\{\mathbf{x} \in \mathbb{Z}^{n+1}/\pm \mid \max_j(|x_j|) \leqslant B\} \sim c \cdot B^{n+1-d}$$

with $c \in \mathbb{R}_{>0}$, provided $d \ll n$, and $f(\mathbf{x})$ is solvable over all completions of $\mathbb{Q}$ (see [**Bir62**]). Let $X = X_f \subset \mathbb{P}^n$ be a smooth hypersurface over $\mathbb{Q}$, given by $f(\mathbf{x}) = 0$. It follows that

$$N(X, -K_X, B) = \#\{\mathbf{x} \in X(\mathbb{Q}) \mid H_{-K_X}(\mathbf{x}) \leqslant B\} \sim C \cdot B, \tag{1.1}$$

as $B \to \infty$. Here $X(\mathbb{Q})$ is the set of rational points on X, represented by primitive vectors $\mathbb{Z}^{n+1}_{\mathrm{prim}} \setminus 0$, modulo ± 1, and

$$H_{-K_X}(\mathbf{x}) := \max_j(|x_j|)^{n+1-d}, \quad \text{for } \mathbf{x} = (x_0, \dots, x_n) \in (\mathbb{Z}^{n+1}_{\mathrm{prim}} \setminus 0)/\pm. \tag{1.2}$$

is the *anticanonical height* of a primitive representative.

In 1989 Manin initiated a program towards understanding connections between certain geometric invariants of algebraic varieties over number fields and their arithmetic properties, in particular, the distribution of rational points of bounded height, see [**FMT89**] and [**BM90**]. The main goal is an extension of the asymptotic formula (1.1) to other algebraic varieties of *small* degree, called Fano varieties, which are not necessarily realizable as hypersurfaces in projective space. It became apparent, that in general, to obtain a geometric interpretation of asymptotic results, it may be necessary to restrict to appropriate Zariski open subsets of X and to allow finite field extensions.

Of particular interest are Del Pezzo surfaces, i.e., geometrically rational surfaces S whose anticanonical class $-K_S$ is ample. Prime examples are cubic surfaces $S_3 \subset \mathbb{P}^3$ or degree 4 surfaces, i.e., intersections of two quadrics $S_4 := Q_1 \cap Q_2 \subset \mathbb{P}^4$. Geometrically, smooth Del Pezzo surfaces are obtained by blowing up $\leqslant 8$ general points in $\mathbb{P}^3$. The singular ones are blow-ups of $\mathbb{P}^2$ in special configurations of points or in infinitely near points. Over number fields, we say that a Del Pezzo surface is split if all of the exceptional curves are defined over $\mathbb{Q}$; there exist *non-split* forms, some of which are not birational to $\mathbb{P}^2$ over the ground field.

From now on, we work over $\mathbb{Q}$. Manin's conjecture in the special case of Del Pezzo surfaces can be formulated as follows.

CONJECTURE 1. — *Let S be a Del Pezzo surface with at most rational double points over $\mathbb{Q}$. Then there exists a dense Zariski open subset $S^\circ \subset S$ such that*

$$N(S^\circ, -K_S, B) \sim c_{S,H} \cdot B(\log B)^{r-1}, \tag{1.3}$$

as $B \to \infty$, where r is the rank of the Picard group of the minimal desingularization $\widetilde{S}$ of S, over $\mathbb{Q}$.

The constant $c_{S,H}$ has been defined by Peyre [**Pey95**]; it should be nonzero if $S(\mathbb{Q}) \neq \emptyset$. Note that a $\mathbb{Q}$-rational line on a Del Pezzo surface such as S_3 or S_4 contributes $\sim B^2$ rational points to the counting function. Thus it is expected that S° is the complement to all $\mathbb{Q}$-rational lines (exceptional curves).

Table 1 gives an overview of current results towards Conjecture 1 for Del Pezzo surfaces. In Column 4 ("type of result"), "asymptotic" means that the analog of (1.3) is established, including the predicted value of the constant; "bounds" means that only upper and lower bounds of the expected order of magnitude with unknown constants are proved.

The paper [**BT98**] contains a proof of Manin's conjecture for toric Fano varieties, including all smooth Del Pezzo surfaces of degree $\geqslant 6$ and the unique $3\mathbf{A}_2$ cubic surface[(1)]. This result also covers:

– all singular surfaces of degree $\geqslant 7$ (i.e., $\mathbf{A}_1$ in degree 7 and 8),
– $\mathbf{A}_1$, $2\mathbf{A}_1$, $\mathbf{A}_2 + \mathbf{A}_1$ in degree 6,
– $2\mathbf{A}_1$, $\mathbf{A}_2 + \mathbf{A}_1$ in degree 5,
– $4\mathbf{A}_1$, $\mathbf{A}_2 + 2\mathbf{A}_1$, $\mathbf{A}_3 + 2\mathbf{A}_1$ in degree 4.

Figure 1 shows all points of height $\leqslant 50$ on the Cayley cubic surface (Example 14), which has four singularities of type $\mathbf{A}_1$ and was considered in [**HB03**]. In Figure 2, we see points of height $\leqslant 1000$ on the $\mathbf{E}_6$ cubic surface ([**Der05**] and [**dlBBD05**]).

The proofs of Manin's conjecture proceed either via the height zeta function

$$Z(s) := \sum_{\mathbf{x} \in X^\circ(\mathbb{Q})} H_{-K_X}(\mathbf{x})^{-s},$$

whose analytic properties are related to the asymptotic (1.3) by Tauberian theorems, or via the lifting of the counting problem to the *universal torsor* – an auxiliary variety parametrizing rational points. The torsor approach has been developed by Colliot-Thélène and Sansuc in the context of the Brauer-Manin obstruction [**CTS87**] and applied to Manin's conjecture by Peyre [**Pey98**] and Salberger [**Sal98**].

(1) Singular Del Pezzo surfaces will be labeled by the type (in the ADE-classification) and number of their singularities.

degree	singularities	(non-)split	type of result	reference
$\geqslant 6$	–	split	asymptotic	[**BT98**]
5	–	split	asymptotic	[**dlB02**]
5	–	non-split	asymptotic	[**dlBF04**]
4	$\mathbf{D}_5$	split	asymptotic	[**CLT02**], [**dlBB04**]
4	$\mathbf{D}_4$	non-split	asymptotic	[**dlBB05**]
4	$\mathbf{D}_4$	split	asymptotic	this paper
4	$3\mathbf{A}_1$	split	bounds	[**Bro05**]
3	$3\mathbf{A}_2$	split	asymptotic	[**BT98**], [**dlB98**], ...
3	$4\mathbf{A}_1$	split	bounds	[**HB03**]
3	$\mathbf{D}_4$	split	bounds	[**Bro04**]
3	$\mathbf{E}_6$	split	asymptotic	[**Der05**], [**dlBBD05**]

Table 1. Results for Del Pezzo surfaces

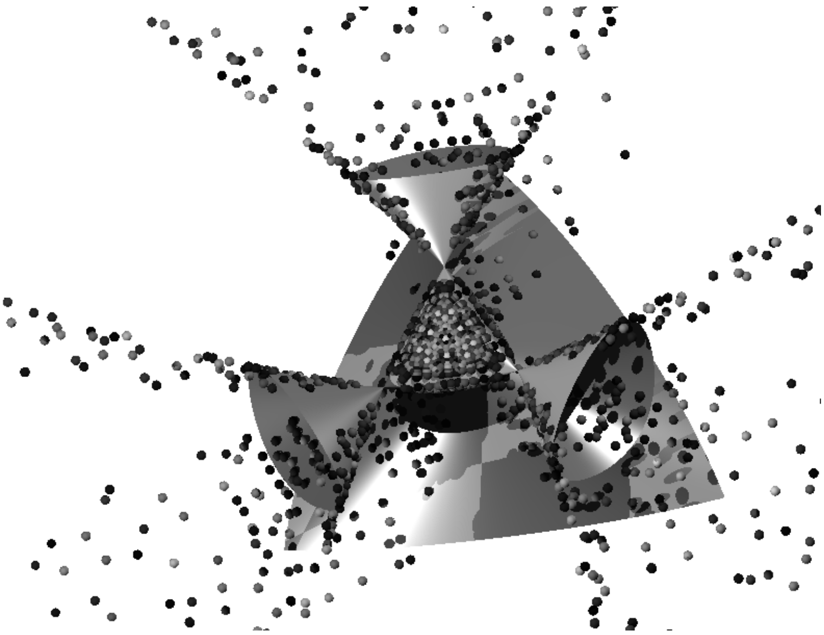

Figure 1. Points of height $\leqslant 50$ on the Cayley cubic surface $x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$.

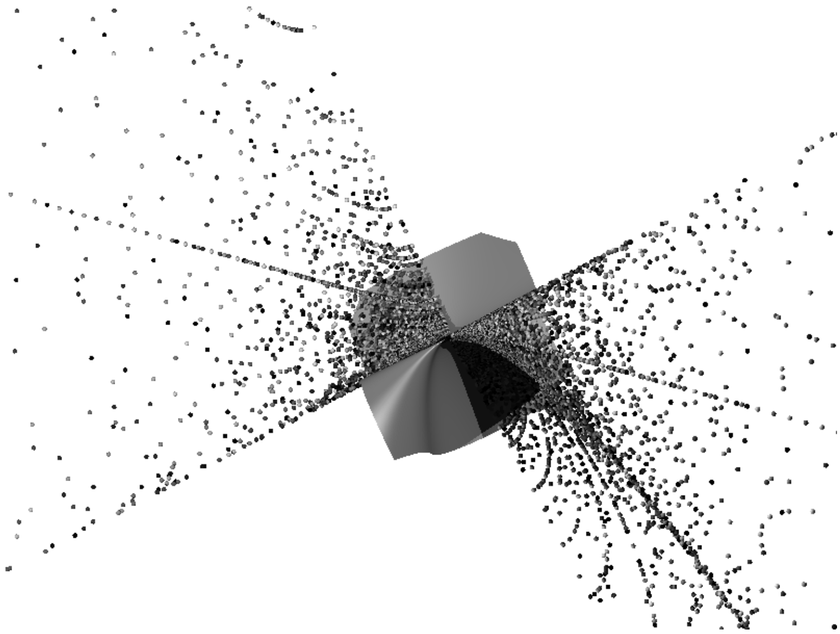

FIGURE 2. Points of height $\leqslant 1000$ on the $\mathbf{E}_6$ singular cubic surface $x_1x_2^2 + x_2x_0^2 + x_3^3 = 0$ with $x_0, x_2 > 0$.

In the simplest case of hypersurfaces $X = X_f \subset \mathbb{P}^n$ over $\mathbb{Q}$, with $n \geqslant 4$, this is exactly the passage from rational vectors $\mathbf{x} = (x_0, \ldots, x_n)$, modulo the diagonal action of $\mathbb{Q}^*$, to primitive lattice points $(\mathbb{Z}^{n+1}_{\text{prim}} \setminus 0)/\pm$. Geometrically, we have

$$\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\mathbb{G}_{\text{m}}} \mathbb{P}^n \qquad \text{and} \qquad \mathcal{T}_X \xrightarrow{\mathbb{G}_{\text{m}}} X.$$

Here, $\mathcal{T}_X$ is the hypersurface in $\mathbb{A}^{n+1} \setminus 0$ defined by the form f, the torus $\mathbb{G}_{\text{m}}$ is interpreted as the Néron-Severi torus T_{NS}, i.e., an algebraic torus whose characters are isomorphic to the Néron-Severi group (lattice) of $\mathbb{P}^n$, resp. X, and the map is the natural quotient by its (diagonal) action. Rational points on the base are lifted to integral points on the torsor, modulo the action of the group of units $T_{\text{NS}}(\mathbb{Z}) = \{\pm 1\}$. The height inequality on the base $H(\mathbf{x}) \leqslant B$ translates into the usual height inequality on the torsor (1.2).

In general, a torsor under an algebraic torus T is determined by a homomorpism $\chi : \mathfrak{X}^*(T) \to \text{NS}(X)$ to the Néron-Severi group of the underlying variety X; the term *universal* is applied when χ is an isomorphism.

However, for hypersurfaces in $\mathbb{P}^3$, or more generally for complete intersection surfaces, the Néron-Severi group may have higher rank. For example, for split smooth cubic surfaces $S = S_3 \subset \mathbb{P}^3$ the rank is 7, so that the dimension of the corresponding universal torsor $\mathcal{T}_S$ is 9; for quartic Del Pezzo surfaces these are 6 and 8, respectively.

It is expected that the passage to universal torsors, which can be considered as natural *descent varieties*, will facilitate the proof of Manin's conjecture (Conjecture 1), at least for Del Pezzo surfaces. Rational points on S are lifted to certain integral points on $\mathcal{T}_S$, modulo the action of $T_{\mathrm{NS}}(\mathbb{Z}) = (\pm 1)^r$, where r is the rank of $\mathrm{NS}(S)$, and the height inequality on S translates into appropriate inequalities on $\mathcal{T}_S$. This explains the interest in the projective geometry of torsors, and expecially, in their equations. The explicit determination of these equations is an interesting algebro-geometric problem, involving tools from invariant theory and toric geometry.

In this note, we illustrate the torsor approach to asymptotics of rational points in the case of a particular singular surface $S \subset \mathbb{P}^4$ of degree 4 given by:

$$x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0. \tag{1.4}$$

This is a split Del Pezzo surface, with a singularity of type $\mathbf{D}_4$.

Theorem 2. — *The number of $\mathbb{Q}$-rational points of anticanonical height bounded by B on the complement S° of the $\mathbb{Q}$-rational lines on S as in* (1.4) *satisfies*

$$N(S^\circ, -K_S, B) = c_{S,H} \cdot B \cdot Q(\log B) + O(B(\log B)^3) \qquad \text{as } B \to \infty,$$

where Q is a monic polynomial of degree 5, *and*

$$c_{S,H} = \frac{1}{34560} \cdot \omega_\infty \cdot \prod_p (1 - 1/p)^6(1 + 6/p + 1/p^2)$$

with

$$\omega_\infty = 3 \int\int\int_{\{(t,u,v)\in\mathbb{R}^3 \mid 0\leqslant v\leqslant 1, |tv^2|, |v^2u|, |v(tv+u^2)|, |t(tv+u^2)|\leqslant 1\}} 1 \,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v,$$

is the constant predicted by Peyre [**Pey95**].

In [**dlBB05**], Manin's conjecture is proved for a non-split surface with a singularity of the same type. However, these results do not follow from each other.

In Section 2, we collect some facts about the geometric structure of S. In Section 3, we calculate the expected value of $c_{S,H}$ and show that Theorem 2 agrees with Manin's conjecture.

In our case, the universal torsor is an affine hypersurface. In Section 4, we calculate its equation, stressing the relation with the geometry of S. We make explicit the coprimality and the height conditions. The method is more systematic than the derivation of torsor equations in [**dlBB04**] and [**dlBBD05**], and should bootstrap to more complicated cases, e.g., other split Del Pezzo surfaces.

Note that our method gives coprimality conditions which are different from the ones in [**dlBB04**] and [**dlBBD05**], but which are in a certain sense more natural: They are related to the set of points on $\mathcal{T}_S$ which are *stable* with respect to the action of the Néron-Severi torus (in the sense of geometric invariant theory). Our conditions involve only coprimality of certain pairs of variables; these might be easier to handle than for example a mix of square-free variables and coprimalities produced by the other method.

In Section 5, we estimate the number of integral points on the universal torsor by iterating summations over the torsor variables and using results of elementary analytic number theory. Finally we arrive at Lemma 10, which is very similar to [**dlBB04**, Lemma 10] and [**Der05**, Lemma 12]. In Section 6 we use familiar methods of height zeta functions to derive the exact asymptotic. We isolate the expected constant $c_{S,H}$ and finish the proof of Theorem 2. In Section 7 we write down examples of universal torsors for other Del Pezzo surfaces and discuss their geometry.

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2. Geometric background

In this section, we collect some geometric facts concerning the surface S. We show that Manin's conjecture for S is not a special case of available more general results for Del Pezzo surfaces.

LEMMA 3. — *The surface S has the following properties:*

(1) *It has exactly one singularity of type $\mathbf{D}_4$ at the $q = (0:0:0:0:1)$.*

(2) *S contains exactly two lines:*

$$E_5 = \{x_0 = x_1 = x_2 = 0\} \quad \text{and} \quad E_6 = \{x_1 = x_2 = x_3 = 0\},$$

which intersect in q.

(3) *The projection from the line E_5 is a birational map*

$$\begin{array}{rccc} \phi: & S & \dashrightarrow & \mathbb{P}^2 \\ & \mathbf{x} & \mapsto & (x_0 : x_2 : x_1) \end{array}$$

which is defined outside E_5. It restricts to an isomorphism between

$$S^\circ = S \backslash (E_5 \cup E_6) = \{\mathbf{x} \in S \mid x_1 \neq 0\} \quad \text{and} \quad \mathbb{A}^2 \cong \{(t:u:v) \mid v \neq 0\} \subset \mathbb{P}^2,$$

whose inverse is the restriction of

$$\begin{array}{rccc} \psi: & \mathbb{P}^2 & \dashrightarrow & S, \\ & (t:u:v) & \mapsto & (tv^2 : v^3 : v^2u : -v(tv+u^2) : -t(tv+u^2)) \end{array}$$

Similar results hold for the projection from E_6.

(4) *The process of resolving the singularity q gives four exceptional divisors $E_1, \dots, E_4$ and produces the minimal desingularization $\widetilde{S}$, which is also the blow-up of $\mathbb{P}^2$ in five points.*

Proof. — Direct computations. □

It will be important to know the details of the sequence of five blow-ups of $\mathbb{P}^2$ giving $\widetilde{S}$ as in Lemma 3(4):

In order to describe the points in $\mathbb{P}^2$, we need the lines

$$E_3 = \{v = 0\}, \qquad A_1 = \{u = 0\}, \qquad A_2 = \{t = 0\}$$

and the curve $A_3 = \{tv + u^2 = 0\}$.

LEMMA 4. — *The following five blow-ups of $\mathbb{P}^2$ result in $\widetilde{S}$:*

- *Blow up the intersection of E_3, A_1, A_3, giving E_2.*
- *Blow up the intersection of E_2, E_3, A_3, giving E_1.*

– *Blow up the intersection of E_1 and A_3, giving E_4.*
– *Blow up the intersection of E_4 and A_3, giving E_6.*
– *Blow up the intersection of E_3 and A_2, giving E_5.*

Here, the order of the first four blow-ups is fixed, and the fifth blow-up can be done at any time.

The Dynkin diagram in Figure 3 describes the final configuration of divisors $E_1, \dots, E_6, A_1, A_2, A_3$. Here, A_1, A_2, A_3 intersect at one point.

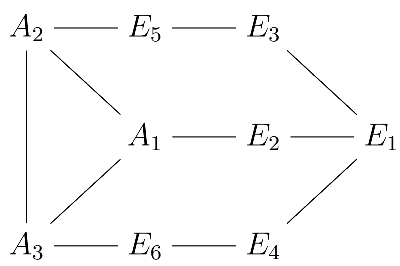

FIGURE 3. Extended Dynkin diagram

The quartic Del Pezzo surface with a singularity of type $\mathbf{D}_4$ is not toric, and Manin's conjecture does not follow from the results of [**BT98**]. The $\mathbf{D}_5$ example of [**dlBB04**] is an equivariant compactification of $\mathbb{G}_{\mathrm{a}}^2$, and thus a special case of [**CLT02**].

LEMMA 5. — *The quartic Del Pezzo surface with a singularity of type $\mathbf{D}_4$ is a compactification of $\mathbb{A}^2$, but not an equivariant compactification of $\mathbb{G}_{\mathrm{a}}^2$.*

Proof. — We follow the strategy of [**HT04**, Remark 3.3].

Consider the maps ϕ, ψ as in Lemma 3(3). As ψ restricts to an isomorphism between $\mathbb{A}^2$ and the open set $S^\circ \subset S$, the surface S is a compactification of $\mathbb{A}^2$.

If S were an equivariant compactification of $\mathbb{G}_{\mathrm{a}}^2$ then the projection ϕ from E_5 would be a $\mathbb{G}_{\mathrm{a}}^2$-equivariant map, giving a $\mathbb{G}_{\mathrm{a}}^2$-action on $\mathbb{P}^2$. The line $\{v = 0\}$ would be invariant under this action. The only such action is the standard translation action

$$\begin{array}{rccc} \tau : & \mathbb{P}^2 & \to & \mathbb{P}^2, \\ & (t : u : v) & \mapsto & (t + \alpha v : u + \beta v : v). \end{array}$$

However, this action does not leave the linear series

$$(tv^2 : v^3 : v^2u : -v(tv+u^2) : -t(tv+u^2))$$

invariant, which can be seen after calculating

$$\begin{aligned} t(tv+u^2) \mapsto & (t+\alpha v)((t+\alpha v)v + (u+\beta v)^2) \\ = & t(tv+u^2) + 2\beta tuv + (\beta^2+\alpha)tv^2 + \alpha v(tv+u^2) \\ & + 2\alpha\beta v^2 u + (\alpha\beta^2+\alpha^2)v^3, \end{aligned}$$

since the term tuv does not appear in the original linear series. □

3. Manin's conjecture

LEMMA 6. — *Let S be the surface* (1.4). *Manin's conjecture for S states that the number of rational points of height $\leqslant B$ outside the two lines is given by*

$$N(S^\circ, -K_S, B) \sim c_{S,H} \cdot B(\log B)^5,$$

where $c_{S,H} = \alpha(S) \cdot \beta(S) \cdot \omega_H(S)$ with

$$\begin{aligned} \alpha(S) &= (5! \cdot 4 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2)^{-1} = (34560)^{-1} \\ \beta(S) &= 1 \\ \omega_H(S) &= \omega_\infty \cdot \prod_p (1-1/p)^6(1+6/p+1/p^2) \end{aligned}$$

and

$$\omega_\infty = 3 \int\int\int_{\{(t,u,v)\in\mathbb{R}^3 \mid 0\leqslant v\leqslant 1, |tv^2|, |v^2u|, |v(tv+u^2)|, |t(tv+u^2)|\leqslant 1\}} 1 \, \mathrm{d}t \, \mathrm{d}u \, \mathrm{d}v$$

Proof. — Since S is split over $\mathbb{Q}$, we have $\mathrm{rk}(\mathrm{NS}(\widetilde{S})) = 6$, and the expected exponent of $\log B$ is 5. Further, $\beta(S) = 1$. The computation of $c_{S,H}$ is done on the desingularization $\widetilde{S}$. For the computation of $\alpha(S)$, observe that the effective cone of $\widetilde{S}$ is simplicial, and

$$-K_{\widetilde{S}} = 4E_1 + 2E_2 + 3E_3 + 3E_4 + 2E_5 + 2E_6.$$

The calculation is analog to [**Der05**, Lemma 2]. The constant $\omega_H(S)$ is computed as in [**dlBB04**, Lemma 1] and [**Der05**, Lemma 2]. □

4. The universal torsor

As explained above, the problem of counting rational points of bounded height on the surface S translates into a counting problem for certain integral points on the universal torsor, subject to coprimality and height inequalities. In the first part of this section, we describe these conditions in detail. They are obtained by a process of introducing new variables which are the greatest common divisors of other variables. Geometrically, this corresponds to the realization of $\widetilde{S}$ as a blow-up of $\mathbb{P}^2$ in five points.

In the second part, we prove our claims.

The universal torsor $\mathcal{T}_S$ of S is an open subset of the hypersurface in $\mathbb{A}^9 = \operatorname{Spec} \mathbb{Z}[\eta_1, \dots, \eta_6, \alpha_1, \alpha_2, \alpha_3]$ defined by the equation

$$T(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \alpha_1^2\eta_2 + \alpha_2\eta_3\eta_5^2 + \alpha_3\eta_4\eta_6^2 = 0. \tag{4.1}$$

The projection $\Psi : \mathcal{T}_S \to S$ is defined by

(4.2)

$$(\Psi^*(x_i)) = (\eta^{(2,1,2,1,2,0)}\alpha_2, \eta^{(4,2,3,3,2,2)}, \eta^{(3,2,2,2,1,1)}\alpha_1, \eta^{(2,1,1,2,0,2)}\alpha_3, \alpha_2\alpha_3),$$

where we use the notation $\eta^{(n_1,n_2,n_3,n_4,n_5,n_6)} = \eta_1^{n_1}\eta_2^{n_2}\eta_3^{n_3}\eta_4^{n_4}\eta_5^{n_5}\eta_6^{n_6}$.

The coprimality conditions can be derived from the extended Dynkin diagram (see Figure 3). Two variables are allowed to have a common factor if and only if the corresponding divisors (E_i for η_i and A_i for α_i) intersect (i.e., are connected by an edge in the diagram). Furthermore, $\gcd(\alpha_1, \alpha_2, \alpha_3) > 1$ is allowed (corresponding to the fact that A_1, A_2, A_3 intersect in one point).

We will show below that there is a bijection between rational points on $S^\circ \subset S$ and integral points on an open subset of $\mathcal{T}_S$, subject to these coprimality conditions.

We will later refer to

(4.3) coprimality between η_i as in Figure 3,

$$\gcd(\alpha_1, \eta_1\eta_3\eta_4\eta_5\eta_6) = 1, \tag{4.4}$$

$$\gcd(\alpha_2, \eta_1\eta_2\eta_3\eta_4\eta_6) = 1, \tag{4.5}$$

$$\gcd(\alpha_3, \eta_1\eta_2\eta_3\eta_4\eta_5) = 1. \tag{4.6}$$

To count the number of $\mathbf{x} \in S(\mathbb{Q})$ such that $H(\mathbf{x}) \leqslant B$, we must lift this condition to the universal torsor, i.e., $H(\Psi(\boldsymbol{\eta}, \boldsymbol{\alpha})) \leqslant B$. This is the same as

$$|\eta^{(2,1,2,1,2,0)}\alpha_2| \leqslant B, \qquad \dots, \qquad |\alpha_2\alpha_3| \leqslant B,$$

using the five monomials occuring in (4.2). These have no common factors, provided the coprimality conditions are fulfilled (direct verification).

It will be useful to write the height conditions as follows: Let

$$X_0 = (\frac{\eta^{(4,2,3,3,2,2)}}{B})^{1/3},\ X_1 = (B\eta^{(-1,-2,0,0,1,1)})^{1/3},\ X_2 = (B\eta^{(2,1,0,3,-2,4)})^{1/3}.$$

Then

$$(4.7) \qquad |X_0^3| \leqslant 1$$

$$(4.8) \qquad |X_0^2(\alpha_1/X_1)| \leqslant 1$$

$$(4.9) \qquad \begin{aligned} &|X_0^2(\alpha_2/X_2)| \leqslant 1, \qquad |X_0(X_0(\alpha_2/X_2) + (\alpha_1/X_1)^2)| \leqslant 1,\\ &|(\alpha_2/X_2)(X_0(\alpha_2/X_2) + (\alpha_1/X_1)^2)| \leqslant 1\end{aligned}$$

are equivalent to the five height conditions. Here we have used the torsor equation to eliminate α_3 because in our counting argument we will also use that α_3 is determined by the other variables.

We now prove the above claims.

Lemma 7. — *The map Ψ gives a bijection between the set of points $\mathbf{x}$ of $S^\circ(\mathbb{Q})$ such that $H(\mathbf{x}) \leqslant B$ and the set*

$$\mathcal{T}_1 := \left\{ (\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}^3 \,\middle|\, \begin{array}{l} \textit{equation } (4.1), \\ \textit{coprimality } (4.3), (4.4), (4.5), (4.6), \\ \textit{inequalities } (4.7), (4.8), (4.6) \textit{ hold} \end{array} \right\}$$

Proof. — The map ψ of Lemma 3(3) induces a bijection

$$\psi_0 : (\eta_3, \alpha_1, \alpha_2) \mapsto (\eta_3^2\alpha_2, \eta_3^3, \eta_3^2\alpha_1, \eta_3\alpha_3, \alpha_2\alpha_3),$$

where $\alpha_3 := -(\eta_3\alpha_2 + \alpha_1^2)$, i.e.,

$$T_0 := \alpha_1^2 + \eta_3\alpha_2 + \alpha_3 = 0,$$

between

$$\{(\eta_3, \alpha_1, \alpha_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}^2 \mid \gcd(\eta_3, \alpha_1, \alpha_2) = 1\} \quad \text{and} \quad S^\circ(\mathbb{Q}) \subset S(\mathbb{Q}).$$

The height function on $S^\circ(\mathbb{Q})$ is given by

$$H(\psi_0(\eta_3, \alpha_1, \alpha_2)) = \frac{\max(|\eta_3^2\alpha_2|, |\eta_3^3|, |\eta_3^2\alpha_1|, |\eta_3\alpha_3|, |\alpha_2\alpha_3|)}{\gcd(\eta_3^2\alpha_2, \eta_3^3, \eta_3^2\alpha_1, \eta_3\alpha_3, \alpha_2\alpha_3)}.$$

The derivation of the torsor equation from the map ψ_0 together with the coprimality conditions and the lifted height function is parallel to the blow-up process described in Lemma 4. More precisely, each line E_3, A_1, A_2 in $\mathbb{P}^2$ corresponds to a coordinate function $\eta_3, \alpha_1, \alpha_2$ vanishing in one of the lines; the blow-up of the intersection of two divisors gives an exceptional divisor E_i, corresponding to the introduction of a new variable η_i as the greatest common divisor of two old variables. Two divisors are disjoint if and only if the corresponding variables are coprime. This is summarized in Table 2.

Variables, Equations	Geometry
variables	divisors
initial variables	coordinate lines
$\eta_3, \alpha_1, \alpha_2$	E_3, A_1, A_2
taking gcd of two variables	blowing up intersection of divisors
new gcd-variable	exceptional divisor
$\eta_2, \eta_1, \eta_4, \eta_6, \eta_5$	E_2, E_1, E_4, E_6, E_5
extra variable	extra curve
α_3	A_3
starting relation	starting description
$\alpha_3 = -(\eta_3\alpha_2 + \alpha_1^2)$	$A_3 = \{\eta_3\alpha_2 + \alpha_1^2 = 0\}$
final relation	torsor equation
$\alpha_3\eta_4\eta_6^2 = -(\alpha_2\eta_3\eta_5^2 + \alpha_1^2\eta_2)$	$\alpha_1^2\eta_2 + \alpha_2\eta_3\eta_5^2 + \alpha_3\eta_4\eta_6^2 = 0$

TABLE 2. Dictionary between gcd-process and blow-ups

This plan will now be implemented in five steps; at each step, the map

$$\psi_i : \mathbb{Z}_{>0}^{i+1} \times \mathbb{Z}^3 \to S^\circ(\mathbb{Q})$$

gives a bijection between:

– the set of all $(\eta_j, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_{>0}^{i+} \times \mathbb{Z}^3$ satisfying certain coprimality conditions (described by the extended Dynkin diagram corresponding to the i-th blow-up of Lemma 4), an equation T_i,

$$H(\psi_i(\eta_j, \alpha_j)) = \frac{\max_k(|\psi_i(\eta_j, \alpha_j)_k|)}{\gcd(\psi_i(\eta_j, \alpha_j)_k)} \leqslant B,$$

– the set of all $\mathbf{x} \in S^\circ(\mathbb{Q})$ with $H(\mathbf{x}) \leqslant B$.

The steps are as follows:

(1) Let $\eta_2 := \gcd(\eta_3, \alpha_1) \in \mathbb{Z}_{>0}$. Then

$$\eta_3 = \eta_2 \eta_3', \qquad \alpha_1 = \eta_2 \alpha_1', \qquad \text{with } \gcd(\eta_3', \alpha_1') = 1.$$

Since $\eta_2 \mid \alpha_3$, we can write $\alpha_3 = \eta_2 \alpha_3'$. Then $\alpha_3' = -(\eta_3' \alpha_2 + \eta_2 \alpha_1'^2)$. After renaming the variables, we have

$$T_1 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \alpha_3 = 0$$

and

$$\psi_1 : (\eta_2, \eta_3, \alpha_1, \alpha_2, \alpha_3) \mapsto (\eta_2 \eta_3^2 \alpha_2 : \eta_2^2 \eta_3^3 : \eta_2^2 \eta_3^2 \alpha_1 : \eta_2 \eta_3 \alpha_3 : \alpha_2 \alpha_3).$$

Here, we have eliminated the common factor η_2 which occured in all five components of the image. Below, we repeat the corresponding transformation at each step.

(2) Let $\eta_1 := \gcd(\eta_2, \eta_3) \in \mathbb{Z}_{>0}$. Then

$$\eta_2 = \eta_1 \eta_2', \qquad \eta_3 = \eta_1 \eta_3', \qquad \text{with } \gcd(\eta_2', \eta_3') = 1.$$

As $\eta_1 \mid \alpha_3$, we write $\alpha_3 = \eta_1 \alpha_3'$, and we obtain:

$$T_2 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \alpha_3 = 0$$

and

$$\psi_2 : (\eta_1, \eta_2, \eta_3, \alpha_1, \alpha_2, \alpha_3) \mapsto (\eta_1^2 \eta_2 \eta_3^2 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 : \eta_1^3 \eta_2^2 \eta_3^2 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \alpha_3 : \alpha_2 \alpha_3).$$

(3) Let $\eta_4 := \gcd(\eta_1, \alpha_3) \in \mathbb{Z}_{>0}$. Then

$$\eta_1 = \eta_4 \eta_1', \qquad \alpha_3 = \eta_4 \alpha_3', \qquad \text{with } \gcd(\eta_1', \alpha_3') = 1.$$

We get after removing $'$ again:

$$T_3 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \eta_4 \alpha_3 = 0$$

and

$$\psi_3 : (\eta_1, \eta_2, \eta_3, \eta_4, \alpha_1, \alpha_2, \alpha_3) \mapsto (\eta_1^2 \eta_2 \eta_3^2 \eta_4 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \alpha_3 : \alpha_2 \alpha_3).$$

(4) Let $\eta_6 := \gcd(\eta_4, \alpha_3) \in \mathbb{Z}_{>0}$. Then

$$\eta_4 = \eta_6 \eta_4', \qquad \alpha_3 = \eta_6 \alpha_3', \qquad \text{with } \gcd(\eta_4', \alpha_3') = 1.$$

We obtain

$$T_4 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \eta_4 \eta_6^2 \alpha_3 = 0$$

and

$$\psi_4 : (\eta_1, \eta_2, \eta_3, \eta_4, \eta_6, \alpha_1, \alpha_2, \alpha_3) \mapsto$$
$$(\eta_1^2 \eta_2 \eta_3^2 \eta_4 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 \eta_6^2 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_6 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_6^2 \alpha_3 : \alpha_2 \alpha_3).$$

(5) The final step is $\eta_5 := \gcd(\eta_3, \alpha_2) \in \mathbb{Z}_{>0}$, we could have done it earlier (just as the blow-up of the intersection of E_3, A_2 in Lemma (4.2)). Then

$$\eta_3 = \eta_5 \eta_3', \qquad \alpha_2 = \eta_5 \alpha_2', \qquad \text{with } \gcd(\eta_3', \alpha_2') = 1.$$

We get

$$T_5 = \eta_2 \alpha_1^2 + \eta_3 \eta_5 \alpha_2 + \eta_4 \eta_6^2 \alpha_3 = 0$$

and

$$\psi_5 : (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \alpha_1, \alpha_2, \alpha_3) \mapsto$$
$$(\eta_1^2 \eta_2 \eta_3^2 \eta_4 \eta_5^2 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 \eta_5^2 \eta_6^2 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_6^2 \alpha_3 : \alpha_2 \alpha_3)$$

We observe that at each stage the coprimality conditions correspond to intersection properties of the respective divisors. The final result is summarized in Figure 3, which encodes data from (4.3), (4.4), (4.5), (4.6).

Note that ψ_5 is Ψ from (4.2). As mentioned above, $\gcd(\psi_5(\eta_j, \alpha_j)_k)$ (over all five components of the image) is trivial by the coprimality conditions of Figure 3. Therefore, $H(\psi_5(\boldsymbol{\eta}, \boldsymbol{\alpha})) \leqslant B$ is equivalent to (4.7), (4.8), (4.9).

Finally, T_5 is the torsor equation T (4.1). □

5. Summations

In the first step, we estimate the number of $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ which fulfill the torsor equation T (4.1) and the height and coprimality conditions. For fixed (α_1, α_2), the torsor equation T has a solution α_3 if and only if the congruence

$$\alpha_1^2 \eta_2 + \alpha_2 \eta_3 \eta_5^2 \equiv 0 \pmod{\eta_4 \eta_6^2}$$

holds and the conditions on the height and coprimalities are fulfilled.

We have already written the height conditions so that they do not depend on α_3. For the coprimality, we must ensure that (4.5) and (4.6) are fulfilled.

As $\gcd(\eta_3\eta_5^2, \eta_4\eta_6^2) = 1$, we can find the multiplicative inverse c_1 of $\eta_3\eta_5^2$ modulo $\eta_4\eta_6^2$, so that

$$c_1\eta_3\eta_5^2 = 1 + c_2\eta_4\eta_6^2 \tag{5.1}$$

for a suitable c_2. Choosing

$$\alpha_2 = c_3\eta_4\eta_6^2 - c_1\alpha_1^2\eta_2, \tag{5.2}$$

$$\alpha_3 = c_2\alpha_1^2\eta_2 - c_3\eta_3\eta_5^2 \tag{5.3}$$

gives a solution of (4.1) for any $c_3 \in \mathbb{Z}$.

Without the coprimality conditions, the number of pairs (α_2, α_3) satisfying T and (4.9) would differ at most by $O(1)$ from $1/\eta_4\eta_6^2$ of the length of the interval described by (4.9). However, the coprimality conditions (4.5) and (4.6) impose further restrictions on the choice of c_3. A slight complication arises from the fact that because of T, some of the conditions are fulfilled automatically once $\boldsymbol{\eta}, \alpha_1$ satisfy (4.3) and (4.4).

Conditions (4.3) imply that the possibilities for a prime p to divide more than one of the η_i are very limited. We distinguish twelve cases, listed in Column 2 of Table 3.

In Columns 4 and 5, we have denoted the relevant information for the divisibility of α_2, α_3 by primes p which are divisors of the η_i in Column 2, but of no other η_j:

– "allowed" means that α_i may be divisible by p.

– "automatically" means that the conditions on the η_i and the other α_j imply that $p \nmid \alpha_i$. These two cases do not impose conditions on c_3 modulo p.

– "restriction" means that c_3 is not allowed to be in a certain congruence class modulo p in order to fulfill the condition that p must not divide α_i.

The information in the table is derived as follows:

– If $p \mid \eta_3$, then $p \nmid c_2$ from (5.1), and $p \nmid \alpha_1\eta_2$ because of (4.3), (4.4), so by (5.3), $p \nmid \alpha_3$ independently of the choice of c_3. Since $p \nmid \eta_4\eta_6^2$, we

case	$p \mid \dots$	$p \mid \alpha_1$	$p \mid \alpha_2$	$p \mid \alpha_3$
0	–	allowed	allowed	allowed
i	η_1	restriction	restriction	restriction
ii	η_2	allowed	restriction	automatically
iii	η_3	restriction	restriction	automatically
iv	η_4	restriction	automatically	restriction
v	η_5	restriction	allowed	automatically
vi	η_6	restriction	automatically	allowed
vii	η_1, η_2	restriction	restriction	automatically
$viii$	η_1, η_3	restriction	restriction	automatically
ix	η_1, η_4	restriction	automatically	restriction
x	η_3, η_5	restriction	restriction	automatically
xi	η_4, η_6	restriction	automatically	restriction

Table 3. Coprimality conditions

see from (5.2) that $p \mid \alpha_2$ for one in p subsequent choices of c_3 which we must therefore exclude. This explains cases *iii* and *viii*.

– In case *vii*, the same is true for α_2. More precisely, we see that we must exclude $c_3 \equiv 0 \pmod{p}$. By (5.3), $p \nmid c_3$ implies that $p \nmid \alpha_3$, so we do not need another condition on c_3.

– In case *i*, we see that $p \mid \alpha_2$ for one in p subsequent choices of c_3, and the same holds for α_3. However, in this case, p cannot divide α_2, α_3 for the same choice of c_3, as we can see by considering T: since $p \nmid \alpha_1^2\eta_2$, it is impossible that $p \mid \alpha_2, \alpha_3$. Therefore, we must exclude two out of p subsequent choices of p in order to fulfill $p \nmid \alpha_2, \alpha_3$.

– In the other cases, the arguments are similar.

The number of $(\alpha_2, \alpha_3) \in \mathbb{Z}^2$ subject to T, (4.5), (4.6), (4.9) equals the number of c_3 such that α_2, α_3 as in (5.2), (5.3) satisfy these conditions. This can be estimated as $1/\eta_4\eta_6^2$ of the interval described by (4.9), multiplied by a product of local factors whose value can be read off from Columns 2, 4, 5 of Table 3: The divisibility properties of η_i by p determine whether zero, one or two out of p subsequent values of c_3 have to be excluded. Different primes can be considered separately, and we define

$$\vartheta_{1,p} := \begin{cases} 1 - 2/p, & \text{case } i, \\ 1 - 1/p, & \text{cases } ii - iv, vi - xi, \\ 1, & \text{case } 0, v. \end{cases}$$

Let

$$\vartheta_1(\boldsymbol{\eta}) = \prod_p \vartheta_{1,p}$$

be the product of these local factors, and

$$g_1(u,v) = \int_{\{t \in \mathbb{R} \mid |tv^2|, |t(tv+u^2)|, |v(tv+u^2)| \leqslant 1\}} 1 \, \mathrm{d}t. \tag{5.4}$$

Let $\omega(n)$ denote the number of primes dividing n.

Lemma 8. — *For fixed* $(\boldsymbol{\eta}, \alpha_1) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}$ *as in* (4.3), (4.4), (4.7), (4.8), *the number of* $(\alpha_2, \alpha_3) \in \mathbb{Z}^2$ *satisfying* T, (4.5), (4.6), (4.9) *is*

$$\mathcal{N}_1(\boldsymbol{\eta}, \alpha_1) = \frac{\vartheta_1(\boldsymbol{\eta}) X_2}{\eta_4 \eta_6^2} g_1(\alpha_1/X_1, X_0) + O(2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)}).$$

The sum of error terms for all possible values of $(\boldsymbol{\eta}, \alpha_1)$ *is* $\ll B(\log B)^3$.

Proof. — The number of c_3 such that the resulting α_2, α_3 satisfy (4.9) differs from $\frac{X_2}{\eta_4\eta_6^2} g_1(\alpha_1/X_1, X_0)$ by at most $O(1)$.

Each $\vartheta_{1,p} \neq 1$ corresponds to a congruence condition on c_3 imposed by one of the cases $i - iv, vi - xi$. For each congruence condition, the actual ratio of allowed c_3 can differ at most by $O(1)$ from the $\vartheta_{1,p}$. The total number of these primes p is

$$\omega(\eta_1\eta_2\eta_3\eta_4\eta_6) \ll 2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)},$$

which is independent of η_5 since any prime dividing only η_5 contributes a trivial factor (see case v).

Using the estimate (4.8) for α_1 in the first step and ignoring (4.3) (4.4), which can only increase the error term, we obtain:

$$\sum_{\boldsymbol{\eta}} \sum_{\alpha_1} 2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)} \leqslant \sum_{\boldsymbol{\eta}} \frac{B \cdot 2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)}}{\eta^{(3,2,2,2,1,1)}} \ll B(\log B)^3.$$

Here, we use $2^{\omega(n)} \ll_\epsilon n^\epsilon$ for the summations over $\eta_1, \eta_2, \eta_3, \eta_4$. For η_6, we employ

$$\sum_{n \leqslant x} 2^{\omega(n)} \ll x(\log x)$$

together with partial summation, contributing a factor $(\log B)^2$, while the summation over η_5 gives another factor $\log B$. □

Next, we sum over all α_1 subject to the coprimality condition (4.4) and the height condition (4.8). Let

$$g_2(v) = \int_{\{u \in \mathbb{R} \mid |v^2 u| \leqslant 1\}} g_1(u, v)\, \mathrm{d}u \tag{5.5}$$

Similar to our discussion for α_2, α_3, the number of possible values for α_1 as in (4.8), while ignoring (4.4) for the moment, is $X_1 g_2(X_0) + O(1)$.

None of the coprimality conditions are fulfilled automatically, and only common factors with η_2 are allowed (see Column 3 of Table 3). Therefore, each prime factor of $\eta_1\eta_3\eta_4\eta_5\eta_6$ reduces the number of allowed α_1 by a factor of $\vartheta_{2,p} = 1 - 1/p$ with an error of at most $O(1)$. For all other primes p, let $\vartheta_{2,p} = 1$, and let

$$\vartheta_2(\boldsymbol{\eta}) = \prod_p \vartheta_{2,p} \qquad \text{and} \qquad \vartheta(\boldsymbol{\eta}) = \begin{cases} \vartheta_1(\boldsymbol{\eta}) \cdot \vartheta_2(\boldsymbol{\eta}), & \text{(4.3) holds} \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 9. — *For fixed $\eta \in \mathbb{Z}^6_{>0}$ as in* (4.3), (4.7), *the sum of $\mathcal{N}_1(\boldsymbol{\eta}, \alpha_1)$ over all $\alpha_1 \in \mathbb{Z}$ satisfying* (4.4), (4.8) *is*

$$\mathcal{N}_2(\boldsymbol{\eta}) := \frac{\vartheta(\boldsymbol{\eta}) X_1 X_2}{\eta_4 \eta_6^2} g_2(X_0) + \mathcal{R}_2(\boldsymbol{\eta}),$$

where the sum of error terms $\mathcal{R}_2(\boldsymbol{\eta})$ over all possible $\boldsymbol{\eta}$ is $\ll B \log B$.

Proof. — Let

$$\mathcal{N}(b_1, b_2) = \vartheta_1(\boldsymbol{\eta}) \cdot \#\{\alpha_1 \in [b_1, b_2] \mid \gcd(\alpha_1, \eta_1\eta_3\eta_4\eta_5\eta_6) = 1\}.$$

Using Möbius inversion, this is estimated as

$$\mathcal{N}(b_1, b_2) = \vartheta_1(\boldsymbol{\eta}) \cdot \vartheta_2(\boldsymbol{\eta}) \cdot (b_2 - b_1) + \mathcal{R}(b_1, b_2)$$

with $\mathcal{R}(b_1, b_2) = O(2^{\omega(\eta_1\eta_3\eta_4\eta_5\eta_6)})$. By partial summation,

$$\mathcal{N}_2(\boldsymbol{\eta}) = \frac{\vartheta(\boldsymbol{\eta})X_1X_2}{\eta_4\eta_6^2} g_2(X_0) + \mathcal{R}_2(\boldsymbol{\eta})$$

with

$$\mathcal{R}_2(\boldsymbol{\eta}) = \frac{-X_2}{\eta_4\eta_6^2} \int_{\{u \mid |X_0^2 u| \leqslant 1\}} (D_1 g_1)(u, X_0)\mathcal{R}(-X_1/X_0^2, X_1 u)\,\mathrm{d}u$$

where $D_1 g_1$ is the partial derivative of g_1 with respect to the first variable. Using the above bound for $\mathcal{R}(b_1, b_2)$, we obtain:

$$\mathcal{R}_2(\boldsymbol{\eta}) \ll \frac{X_2}{\eta_4\eta_6^2} 2^{\omega(\eta_1\eta_3\eta_4\eta_5\eta_6)}.$$

Summing this over all $\boldsymbol{\eta}$ as in (4.7) while ignoring (4.3) which can only enlarge the sum, we obtain:

$$\sum_{\boldsymbol{\eta}} \mathcal{R}_2(\boldsymbol{\eta}) \ll \sum_{\boldsymbol{\eta}} \frac{X_2 \cdot 2^{\omega(\eta_1\eta_3\eta_4\eta_5\eta_6)}}{\eta_4\eta_6^2 X_0^2} = \sum_{\boldsymbol{\eta}} \frac{B \cdot 2^{\omega(\eta_1\eta_3\eta_4\eta_5\eta_6)}}{\eta^{(2,1,2,2,2,2)}} \ll B \log B$$

In the first step, we use $X_0 \leqslant 1$. □

Let

$$\Delta(n) = B^{-2/3} \sum_{\eta_i, \eta^{(4,2,3,3,2,2)} = n} \frac{\vartheta(\boldsymbol{\eta})X_1X_2}{\eta_4\eta_6^2} = \sum_{\eta_i, \eta^{(4,2,3,3,2,2)} = n} \frac{\vartheta(\boldsymbol{\eta})(\eta^{(4,2,3,3,2,2)})^{1/3}}{\eta^{(1,1,1,1,1,1)}}.$$

In view of Lemma 7, the number of rational points of bounded height on S° can be estimated by summing the result of Lemma 9 over all suitable $\boldsymbol{\eta}$. The error term is the combination of the error terms in Lemmas 8 and 9.

Lemma 10. — *We have*

$$N(S^\circ, -K_S, B) = B^{2/3} \sum_{n \leqslant B} \Delta(n) g_2((n/B)^{1/3}) + O(B(\log B)^3).$$

6. Completion of the proof

We need an estimate for

$$M(t) := \sum_{n \leqslant t} \Delta(t).$$

Consider the Dirichlet series $F(s) := \sum_{n=1}^{\infty} \Delta(n) n^{-s}$. Using

$$F(s+1/3) = \sum_{\boldsymbol{\eta}} \frac{\vartheta(\boldsymbol{\eta})}{\eta_1^{4s+1}\eta_2^{2s+1}\eta_3^{3s+1}\eta_4^{3s+1}\eta_5^{2s+1}\eta_6^{2s+1}},$$

we write $F(s+1/3) = \prod_p F_p(s+1/3)$ as its Euler product. To obtain $F_p(s+1/3)$ for a prime p, we need to restrict this sum to the terms in which all η_i are powers of p. Note that $\vartheta(\boldsymbol{\eta})$ is non-zero if and only if the divisibility of η_i by p falls into one of the twelve cases described in Table 3. The value of $\vartheta(\boldsymbol{\eta})$ only depends on these cases.

Writing $F_p(s+1/3) = \sum_{i=1}^{11} F_{p,i}(s+1/3)$, we have for example:

$$F_{p,0}(s+1/3) = 1,$$

$$F_{p,1}(s+1/3) = \sum_{j=1}^{\infty} \frac{(1-1/p)(1-2/p)}{p^{j(4s+1)}} = \frac{(1-1/p)(1-2/p)}{p^{4s+1}-1},$$

$$F_{p,7}(s+1/3) = \sum_{j,k=1}^{\infty} \frac{(1-1/p)^2}{p^{j(4s+1)}p^{k(2s+1)}} = \frac{(1-1/p)^2}{(p^{4s+1}-1)(p^{2s+1}-1)}.$$

The other cases are similiar, giving

$$\begin{aligned} F_p(s+1/3) =& 1 + \frac{1-1/p}{p^{4s+1}-1}\left((1-2/p) + \frac{1-1/p}{p^{2s+1}-1} + 2\frac{1-1/p}{p^{3s+1}-1}\right) \\ &+ \frac{1-1/p}{p^{2s+1}-1} + 2\frac{(1-1/p)^2}{p^{3s+1}-1} + 2\frac{1-1/p}{p^{2s+1}-1} + 2\frac{(1-1/p)^2}{(p^{2s+1}-1)^2}. \end{aligned}$$

Defining

$$E(s) := \zeta(4s+1)\zeta(3s+1)^2\zeta(2s+1)^3 \qquad \text{and} \qquad G(s) := F(s+1/3)/E(s),$$

we see as in [**Der05**] that the residue of $F(s)t^s/s$ at $s = 1/3$ is

$$\mathrm{Res}(t) = \frac{3G(0)t^{1/3}Q_1(\log t)}{5! \cdot 4 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2}$$

for a monic $Q_1 \in \mathbb{R}[x]$ of degree 5. By Lemma 6, $\alpha(S) = \frac{1}{5!\cdot 4\cdot 2\cdot 3\cdot 3\cdot 2\cdot 2}$. By a Tauberian argument as in [**Der05**, Lemma 13]:

Lemma 11. — *$M(t) = \mathrm{Res}(t) + O(t^{1/3-\delta})$ for some $\delta > 0$.*

By partial summation,

$$\sum_{n \leqslant B} \Delta(n) g_2((n/B)^{1/3}) = \alpha(S) \cdot G(0) \cdot B^{1/3} Q(\log B) \cdot 3 \int_0^1 g_2(v)\, \mathrm{d}v + O(B^{\frac{1}{3}-\delta})$$

for a monic polynomial Q of degree 5. We identify $\omega_H(S)$ from

$$G(0) = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right), \quad \text{and } \omega_\infty = 3 \int_0^1 g_2(v)\, \mathrm{d}v.$$

Together with Lemma 10, this completes the proof of Theorem 2.

7. Equations of universal torsors

The simplest universal torsors are those which can be realized as Zariski open subsets of the affine space. This happens iff the Del Pezzo surface is toric.

Example 12. — There are 20 types of singular Del Pezzo surfaces of degree $d \geqslant 3$ whose universal torsor is an open subset of a hypersurface in $\mathbb{A}^{13-d}$. They are listed in the following table.

degree	singularities	# of lines	defining equation
6	$\mathbf{A}_1$	3	$\eta_2\alpha_1 + \eta_3\alpha_2 + \eta_4\alpha_3$
6	$\mathbf{A}_2$	2	$\eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3$
5	$\mathbf{A}_1$	7	$\eta_2\eta_6 + \eta_3\eta_7 + \eta_4\eta_8$
5	$\mathbf{A}_2$	4	$\eta_3\alpha_1 + \eta_4\alpha_2 + \eta_2\eta_5^2\eta_6$
5	$\mathbf{A}_3$	2	$\eta_1\alpha_1^2 + \eta_3\eta_4^2\alpha_2 + \eta_5\alpha_3$
5	$\mathbf{A}_4$	1	$\eta_1^2\eta_2\alpha_1^3 + \eta_4\alpha_2^2 + \eta_5\alpha_3$
4	$3\mathbf{A}_1$	6	$\eta_4\eta_5 + \eta_1\eta_6\eta_7 + \eta_8\eta_9$
4	$\mathbf{A}_2 + \mathbf{A}_1$	6	$\eta_5\eta_7 + \eta_6\eta_8 + \eta_1\eta_3\eta_9^2$
4	$\mathbf{A}_3$	5	$\eta_5\alpha + \eta_1\eta_4^2\eta_7 + \eta_3\eta_6^2\eta_8$
4	$\mathbf{A}_3 + \mathbf{A}_1$	3	$\eta_6\alpha_2 + \eta_7\alpha_1 + \eta_1\eta_3\eta_4^2\eta_5^3$
4	$\mathbf{A}_4$	3	$\eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7$
4	$\mathbf{D}_4$	2	$\eta_3\eta_5^2\alpha_2 + \eta_4\eta_6^2\alpha_3 + \eta_2\alpha_1^2$
4	$\mathbf{D}_5$	1	$\eta_3\alpha_1^2 + \eta_2\eta_6^2\alpha_3 + \eta_4\eta_5^2\alpha_2^3$
3	$\mathbf{D}_4$	6	$\eta_2\eta_5^2\eta_8 + \eta_3\eta_6^2\eta_9 + \eta_4\eta_7^2\eta_{10}$
3	$\mathbf{A}_3 + 2\mathbf{A}_1$	5	$\eta_1\eta_2\eta_6^2 + \eta_4\eta_7^2\eta_{10} + \eta_8\eta_9$
3	$2\mathbf{A}_2 + \mathbf{A}_1$	5	$\eta_3\eta_5\eta_7^2 + \eta_1\eta_6\eta_8 + \eta_9\eta_{10}$
3	$\mathbf{A}_4 + \mathbf{A}_1$	4	$\eta_1\eta_5\eta_8^2 + \eta_3\eta_4^2\eta_6^3\eta_9 + \eta_7\alpha$
3	$\mathbf{D}_5$	3	$\eta_2\eta_6^2\alpha_2 + \eta_4\eta_5^2\eta_7^3\eta_8 + \eta_3\alpha_1^2$
3	$\mathbf{A}_5 + \mathbf{A}_1$	2	$\eta_1^3\eta_2^2\eta_3\eta_7\eta_8 + \eta_5\alpha_1^2 + \eta_6\alpha_2$
3	$\mathbf{E}_6$	1	$\eta_4^2\eta_5\eta_7^3\alpha_3 + \eta_2\alpha_2^2 + \eta_1^2\eta_3\alpha_1^3$

Example 13 (Cubic surface with $\mathbf{A}_1 + \mathbf{A}_3$ singularities)

This surface has 7 lines, 4 additional variables correspond to exceptional divisors of the desingularization. Its 9-dimensional universal torsor is a Zariski open subset of a complete intersection in

$$\mathbb{A}^{11} = \operatorname{Spec} \mathbb{Z}[\eta_0, \dots, \eta_3, \mu_0, \dots, \mu_6]$$

given by

$$\eta_1\eta_2\mu_1\mu_2 + \mu_4\mu_6 + \mu_3\mu_5 = 0 \qquad \text{and} \qquad \eta_0\eta_1\mu_2^2 + \eta_3\mu_5\mu_6 + \mu_0\mu_1 = 0.$$

There are examples of universal torsors which are not complete intersections, but have still been successfully used in the context of Manin's conjecture:

Example 14 (Cayley cubic). — The Cayley cubic surface

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$

(Figure 1) is a split singular cubic surface with four singularities $q_1, \dots, q_4$ of type $\mathbf{A}_1$ and nine lines. It is the blow-up of $\mathbb{P}^2$ in the 6 intersection points of 4 lines in general position. The universal torsor is an open subvariety of the variety in

$$\mathbb{A}^{13} = \operatorname{Spec} \mathbb{Z}[v_{12}, v_{13}, v_{14}, y_1, y_2, y_3, y_4, z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{25}]$$

defined by six equations of the form

$$z_{ik}z_{il}y_j + z_{jk}z_{jl}y_i = z_{ij}v_{ij}$$

and three equations of the form

$$v_{ij}v_{ik} = z_{il}^2 y_j y_k - z_{jk}^2 y_i y_l,$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and

$$z_{ij} = z_{ji}, \qquad v_{ij} = v_{ji}, \qquad \text{and} \qquad v_{ij} = -v_{kl}.$$

The variables y_i correspond to the four exceptional divisors E_i obtained by blowing up q_i, z_{ij} correspond to the six lines m_{ij} through two of the singularities, and v_{ij} correspond to the other three lines ℓ_{ij}. The first six equations can be interpreted in connection with the projection from m_{ij}, and the other three equations are connected to the projection from ℓ_{ij}.

Upper and lower bounds of the expected order of magnitude have been established in [**HB03**].

Example 15 (Smooth degree 5 Del Pezzo surface)

The blow-up of $\mathbb{P}^2$ in

$$(1:0:0), \qquad (0:1:0), \qquad (0:0:1), \qquad (1:1:1)$$

is a split smooth Del Pezzo surface of degree 5. Its universal torsor is an open subset of the variety defined by the following five equations in ten variables:

$$\begin{aligned}
\lambda_{13}\eta_1 - \lambda_{23}\eta_2 + \lambda_{34}\eta_4 &= 0\\
\lambda_{14}\eta_4 - \lambda_{13}\eta_3 + \lambda_{12}\eta_2 &= 0\\
\lambda_{34}\eta_3 - \lambda_{24}\eta_4 + \lambda_{14}\eta_1 &= 0\\
\lambda_{24}\eta_4 - \lambda_{23}\eta_3 + \lambda_{12}\eta_1 &= 0\\
\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{23}\lambda_{14} &= 0
\end{aligned}$$

The asymptotic formula (1.3) has been established in [**dlB02**].

To illustrate some of the difficulties in proving Conjecture 1 for a smooth split cubic surface, we now write down equations for its universal torsor (up to radical).

Example 16 (Smooth cubic surfaces). — Let S be the blow-up of $\mathbb{P}^2$ in

$$(1:0:0),\ (0:1:0),\ (0:0:1),\ (1:1:1),\ (1:a:b),\ (1:c:d),$$

in general position. Conjecturally, the universal torsor is an open subset of the intersection of 81 quadrics in 27-dimensional space $\operatorname{Spec}\mathbb{Z}[\eta_i, \mu_{i,j}, \lambda_i]$, where

– $\eta_1, \dots, \eta_6$ correspond to the preimages of the points,
– $\mu_{i,j}$ ($i < j \in \{1, \dots, 6\}$) correspond to the 15 lines $m_{i,j}$ through two of the points,
– $\lambda_1, \dots, \lambda_6$ correspond to the conics Q_i through five of the six points,

and relations arise from conic bundle structures on S. Batyrev and Popov proved that the above variables are indeed generators and that the relations give the universal torsor, up to radical [**BP04**].

We now write down these equations explicitly. The 81 defining quadrics occur in sets of three. These 27 triples correspond to projections from the 27 lines on S. We use

$$E := (b-1)(c-1) - (a-1)(d-1) \quad\text{and}\quad F := bc - ad$$

to simplify the equations.

$$\begin{aligned}
q_{Q_1,1} &= -\eta_2\mu_{1,2} - \eta_3\mu_{1,3} + \eta_4\mu_{1,4}\\
q_{Q_1,2} &= -a\eta_2\mu_{1,2} - b\eta_3\mu_{1,3} + \eta_5\mu_{1,5}\\
q_{Q_1,2} &= -c\eta_2\mu_{1,2} - d\eta_3\mu_{1,3} + \eta_6\mu_{1,6}
\end{aligned}$$

$$q_{Q_2,1} = \eta_1\mu_{1,2} - \eta_3\mu_{2,3} + \eta_4\mu_{2,4}$$
$$q_{Q_2,2} = \eta_1\mu_{1,2} - b\eta_3\mu_{2,3} + \eta_5\mu_{2,5}$$
$$q_{Q_2,3} = \eta_1\mu_{1,2} - d\eta_3\mu_{2,3} + \eta_6\mu_{2,6}$$

$$q_{Q_3,1} = \eta_1\mu_{1,3} + \eta_2\mu_{2,3} + \eta_4\mu_{3,4}$$
$$q_{Q_3,2} = \eta_1\mu_{1,3} + a\eta_2\mu_{2,3} + \eta_5\mu_{3,5}$$
$$q_{Q_3,3} = \eta_1\mu_{1,3} + c\eta_2\mu_{2,3} + \eta_6\mu_{3,6}$$

$$q_{Q_4,1} = \eta_1\mu_{1,4} + \eta_2\mu_{2,4} + \eta_3\mu_{3,4}$$
$$q_{Q_4,2} = (1-b)\eta_1\mu_{1,4} + (a-b)\eta_2\mu_{2,4} + \eta_5\mu_{4,5}$$
$$q_{Q_4,3} = (1-d)\eta_1\mu_{1,4} + (c-d)\eta_2\mu_{2,4} + \eta_6\mu_{4,6}$$

$$q_{Q_5,1} = 1/b\eta_1\mu_{1,5} + a/b\eta_2\mu_{2,5} + \eta_3\mu_{3,5}$$
$$q_{Q_5,2} = (1-b)/b\eta_1\mu_{1,5} + (a-b)/b\eta_2\mu_{2,5} + \eta_4\mu_{4,5}$$
$$q_{Q_5,3} = (b-d)/b\eta_1\mu_{1,5} + F/b\eta_2\mu_{2,5} + \eta_6\mu_{5,6}$$

$$q_{Q_6,1} = 1/d\eta_1\mu_{1,6} + c/d\eta_2\mu_{2,6} + \eta_3\mu_{3,6}$$
$$q_{Q_6,2} = (1-d)/d\eta_1\mu_{1,6} + (c-d)/d\eta_2\mu_{2,6} + \eta_4\mu_{4,6}$$
$$q_{Q_6,3} = (b-d)/d\eta_1\mu_{1,6} + F/d\eta_2\mu_{2,6} + \eta_5\mu_{5,6}$$

$$q_{m_{1,2},1} = \mu_{4,5}\mu_{3,6} - \mu_{3,5}\mu_{4,6} + \mu_{3,4}\mu_{5,6}$$
$$q_{m_{1,2},2} = (b-d)\mu_{3,5}\mu_{4,6} + (d-1)\mu_{3,4}\mu_{5,6} + \eta_2\lambda_1$$
$$q_{m_{1,2},3} = F\mu_{3,5}\mu_{4,6} + a(d-c)\mu_{3,4}\mu_{5,6} + \eta_1\lambda_2$$

$$q_{m_{1,3},1} = \mu_{4,5}\mu_{2,6} - \mu_{2,5}\mu_{4,6} + \mu_{2,4}\mu_{5,6}$$
$$q_{m_{1,3},2} = (c-a)\mu_{2,5}\mu_{4,6} + (1-c)\mu_{2,4}\mu_{5,6} + \eta_3\lambda_1$$
$$q_{m_{1,3},3} = -F\mu_{2,5}\mu_{4,6} + b(c-d)\mu_{2,4}\mu_{5,6} + \eta_1\lambda_3$$

$$q_{m_{2,3},1} = \mu_{4,5}\mu_{1,6} - \mu_{1,5}\mu_{4,6} + \mu_{1,4}\mu_{5,6}$$
$$q_{m_{2,3},2} = (a-c)\mu_{1,5}\mu_{4,6} + a(c-1)\mu_{1,4}\mu_{5,6} + \eta_3\lambda_2$$
$$q_{m_{2,3},3} = (b-d)\mu_{1,5}\mu_{4,6} + b(d-1)\mu_{1,4}\mu_{5,6} + \eta_2\lambda_3$$

$$q_{m_{1,4},1} = \mu_{3,5}\mu_{2,6} - \mu_{2,5}\mu_{3,6} + \mu_{2,3}\mu_{5,6}$$
$$q_{m_{1,4},2} = -E\mu_{2,5}\mu_{3,6} + (b-1)(c-1)\mu_{2,3}\mu_{5,6} + \eta_4\lambda_1$$
$$q_{m_{1,4},3} = -F\mu_{2,5}\mu_{3,6} + bc\mu_{2,3}\mu_{5,6} + \eta_1\lambda_4$$

$$q_{m_{2,4},1} = \mu_{3,5}\mu_{1,6} - \mu_{1,5}\mu_{3,6} + \mu_{1,3}\mu_{5,6}$$
$$q_{m_{2,4},2} = E\mu_{1,5}\mu_{3,6} + (a-b)(c-1)\mu_{1,3}\mu_{5,6} + \eta_4\lambda_2$$
$$q_{m_{2,4},3} = (b-d)\mu_{1,5}\mu_{3,6} - b\mu_{1,3}\mu_{5,6} + \eta_2\lambda_4$$

$$q_{m_{3,4},1} = \mu_{2,5}\mu_{1,6} - \mu_{1,5}\mu_{2,6} + \mu_{1,2}\mu_{5,6}$$
$$q_{m_{3,4},2} = -E\mu_{1,5}\mu_{2,6} + (a-b)(1-d)\mu_{1,2}\mu_{5,6} + \eta_4\lambda_3$$
$$q_{m_{3,4},3} = (c-a)\mu_{1,5}\mu_{2,6} + a\mu_{1,2}\mu_{5,6} + \eta_3\lambda_4$$

$$q_{m_{1,5},1} = \mu_{3,4}\mu_{2,6} - \mu_{2,4}\mu_{3,6} + \mu_{2,3}\mu_{4,6}$$
$$q_{m_{1,5},2} = -E\mu_{2,4}\mu_{3,6} + (a-c)(1-b)\mu_{2,3}\mu_{4,6} + \eta_5\lambda_1$$
$$q_{m_{1,5},3} = (d-c)\mu_{2,4}\mu_{3,6} + c\mu_{2,3}\mu_{4,6} + \eta_1\lambda_5$$

$$q_{m_{2,5},1} = \mu_{3,4}\mu_{1,6} - \mu_{1,4}\mu_{3,6} + \mu_{1,3}\mu_{4,6}$$
$$q_{m_{2,5},2} = aE\mu_{1,4}\mu_{3,6} + (a-b)(c-a)\mu_{1,3}\mu_{4,6} + \eta_5\lambda_2$$
$$q_{m_{2,5},3} = (1-d)\mu_{1,4}\mu_{3,6} - \mu_{1,3}\mu_{4,6} + \eta_2\lambda_5$$

$$q_{m_{3,5},1} = \mu_{2,4}\mu_{1,6} - \mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6}$$
$$q_{m_{3,5},2} = -bE\mu_{1,4}\mu_{2,6} + (a-b)(b-d)\mu_{1,2}\mu_{4,6} + \eta_5\lambda_3$$
$$q_{m_{3,5},3} = (c-1)\mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} + \eta_3\lambda_5$$

$$
\begin{aligned}
q_{m_{4,5},1} &= \mu_{2,3}\mu_{1,6} - \mu_{1,3}\mu_{2,6} + \mu_{1,2}\mu_{3,6} \\
q_{m_{4,5},2} &= b(c-a)\mu_{1,3}\mu_{2,6} + a(b-d)\mu_{1,2}\mu_{3,6} + \eta_5\lambda_4 \\
q_{m_{4,5},3} &= (c-1)\mu_{1,3}\mu_{2,6} + (1-d)\mu_{1,2}\mu_{3,6} + \eta_4\lambda_5
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,6},1} &= \mu_{3,4}\mu_{2,5} - \mu_{2,4}\mu_{3,5} + \mu_{2,3}\mu_{4,5} \\
q_{m_{1,6},2} &= -E\mu_{2,4}\mu_{3,5} + (a-c)(1-d)\mu_{2,3}\mu_{4,5} + \eta_6\lambda_1 \\
q_{m_{1,6},3} &= (b-a)\mu_{2,4}\mu_{3,5} + a\mu_{2,3}\mu_{4,5} + \eta_1\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{2,6},1} &= \mu_{3,4}\mu_{1,5} - \mu_{1,4}\mu_{3,5} + \mu_{1,3}\mu_{4,5} \\
q_{m_{2,6},2} &= cE\mu_{1,4}\mu_{3,5} + (a-c)(d-c)\mu_{1,3}\mu_{4,5} + \eta_6\lambda_2 \\
q_{m_{2,6},3} &= (1-b)\mu_{1,4}\mu_{3,5} - \mu_{1,3}\mu_{4,5} + \eta_2\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{3,6},1} &= \mu_{2,4}\mu_{1,5} - \mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} \\
q_{m_{3,6},2} &= -dE\mu_{1,4}\mu_{2,5} + (d-b)(d-c)\mu_{1,2}\mu_{4,5} + \eta_6\lambda_3 \\
q_{m_{3,6},3} &= (a-1)\mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} + \eta_3\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{4,6},1} &= \mu_{2,3}\mu_{1,5} - \mu_{1,3}\mu_{2,5} + \mu_{1,2}\mu_{3,5} \\
q_{m_{4,6},2} &= d(c-a)\mu_{1,3}\mu_{2,5} + c(b-d)\mu_{1,2}\mu_{3,5} + \eta_6\lambda_4 \\
q_{m_{4,6},3} &= (a-1)\mu_{1,3}\mu_{2,5} + (1-b)\mu_{1,2}\mu_{3,5} + \eta_4\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{5,6},1} &= \mu_{2,3}\mu_{1,4} - \mu_{1,3}\mu_{2,4} + \mu_{1,2}\mu_{3,4} \\
q_{m_{5,6},2} &= d(c-1)\mu_{1,3}\mu_{2,4} + c(1-d)\mu_{1,2}\mu_{3,4} + \eta_6\lambda_5 \\
q_{m_{5,6},3} &= b(a-1)\mu_{1,3}\mu_{2,4} + a(1-b)\mu_{1,2}\mu_{3,4} + \eta_5\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{E_1,1} &= (d-b)/E\mu_{1,2}\lambda_2 + (c-a)/E\mu_{1,3}\lambda_3 + \mu_{1,4}\lambda_4 \\
q_{E_1,2} &= (d-1)/E\mu_{1,2}\lambda_2 + (c-1)/E\mu_{1,3}\lambda_3 + \mu_{1,5}\lambda_5 \\
q_{E_1,3} &= (b-1)/E\mu_{1,2}\lambda_2 + (a-1)/E\mu_{1,3}\lambda_3 + \mu_{1,6}\lambda_6
\end{aligned}
$$

$$q_{E_2,1} = F/E\mu_{1,2}\lambda_1 + (c-a)/E\mu_{2,3}\lambda_3 + \mu_{2,4}\lambda_4$$
$$q_{E_2,2} = (c-d)/E\mu_{1,2}\lambda_1 + (c-1)/E\mu_{2,3}\lambda_3 + \mu_{2,5}\lambda_5$$
$$q_{E_2,3} = (a-b)/E\mu_{1,2}\lambda_1 + (a-1)/E\mu_{2,3}\lambda_3 + \mu_{2,6}\lambda_6$$

$$q_{E_3,1} = F/E\mu_{1,3}\lambda_1 + (b-d)/E\mu_{2,3}\lambda_2 + \mu_{3,4}\lambda_4$$
$$q_{E_3,2} = (c-d)/E\mu_{1,3}\lambda_1 + (1-d)/E\mu_{2,3}\lambda_2 + \mu_{3,5}\lambda_5$$
$$q_{E_3,3} = (a-b)/E\mu_{1,3}\lambda_1 + (1-b)/E\mu_{2,3}\lambda_2 + \mu_{3,6}\lambda_6$$

$$q_{E_4,1} = F/(a-c)\mu_{1,4}\lambda_1 + (b-d)/(a-c)\mu_{2,4}\lambda_2 + \mu_{3,4}\lambda_3$$
$$q_{E_4,2} = c/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,5}\lambda_5$$
$$q_{E_4,3} = a/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,6}\lambda_6$$

$$q_{E_5,1} = (d-c)/(c-1)\mu_{1,5}\lambda_1 + (d-1)/(c-1)\mu_{2,5}\lambda_2 + \mu_{3,5}\lambda_3$$
$$q_{E_5,2} = -c/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{4,5}\lambda_4$$
$$q_{E_5,3} = -1/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{5,6}\lambda_6$$

$$q_{E_6,1} = (b-a)/(a-1)\mu_{1,6}\lambda_1 + (b-1)/(a-1)\mu_{2,6}\lambda_2 + \mu_{3,6}\lambda_3$$
$$q_{E_6,2} = -a/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{4,6}\lambda_4$$
$$q_{E_6,3} = -1/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{5,6}\lambda_5$$

In general, the dimension k of the ambient space $\mathbb{A}^k$ of the universal torsor is at least as large as the number of lines on the surface plus the number of exceptional divisors of its desingularization, while the dimension of the universal torsor only depends on the degree of the surface, so that the number of equations must grow with k.

Heuristically, the complexity of universal torsors should be dictated by the following considerations:

– The dimension of the universal torsor of split Del Pezzo surfaces S is $12 - d$, where d is the degree of S.

– For smooth Del Pezzo surfaces, the number of lines is bigger in smaller degrees (e.g., 10 lines in degree 5, and 27 lines in degree 3).

– Singular surfaces have less lines than smooth surfaces.

– The number of lines is higher in cases with "few mild" singularities (e.g., for cubics: $\mathbf{A}_1$ with 21 lines, $\mathbf{A}_2$ with 15 lines), while it is low for "bad" singularities (e.g., 1 for the $\mathbf{E}_6$ cubic, 2 for the $\mathbf{A}_5 + \mathbf{A}_1$ cubic).

Therefore, we expect universal torsors over surfaces which have low degree, are smooth or have mild singularities to be more complex than torsors over surfaces in large degree, or with complicated singularities.

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