RATIONAL CURVES AND POINTS ON K3 SURFACES

by

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Abstract. — We study the distribution of algebraic points on K3 surfaces.

Contents

1. Introduction

Let k be a field and $\bar{k}$ a fixed algebraic closure of k. We are interested in connections between geometric properties of algebraic varieties and their arithmetic properties over k, over its finite extensions k'/k or over $\bar{k}$. Here we study certain varieties of intermediate type, namely K3 surfaces and their higher dimensional generalizations, Calabi-Yau varieties.

To motivate the following discussion, let S be a K3 surface over k. In positive characteristic, S may be unirational and covered by rational curves. Examples are *supersingular* K3 surfaces over fields of characteristic two or the surface

$$x^4 + y^4 + z^4 + t^4 = 0$$

over fields of characteristic three. If k has characteristic zero, then S contains at most finitely many rational curves in each homology class of S (the counting of which is an interesting problem in enumerative geometry, see [4], [6], [7], [26]). Over uncountable fields, there may, of course, exist k-rational points on S not contained in any rational curve defined over $\bar{k}$. The following extremal statement, proposed by the first author in 1981, is however still a logical possibility:

Let k be either a finite field or a number field. Let S be a K3 surface defined over k. Then every $\bar{k}$-rational point on S lies on some rational curve $C \subset S$, defined over $\bar{k}$.

In this note we collect several representative examples illustrating this statement. One of our results is:

THEOREM 1.1. — *Let S be a Kummer surface over a finite field k. Then every $s \in S(\bar{k})$ lies on a rational curve $C \subset S$ defined over $\bar{k}$.*

Actually, such surfaces S are rationally connected in a very strong sense: there is a Zariski open subset $S^0 \subset S$ such that for every finite set of points $\{s_1, \ldots, s_n\} \subset S^0(\bar{k})$ there is a (singular) irreducible rational curve $C \subset S$ defined over $\bar{k}$ which contains s_j, for all j. If S is not supersingular, then S is not uniruled. This resolves a problem raised by Katsura in [9], Question 12, and a question of Kollár in [2], Remark 12.

Using this theorem we produce examples of non-uniruled surfaces of general type (with nontrivial unramified Brauer groups) over finite fields which are "rationally chain connected" (any two algebraic points can be joined by a chain of rational curves).

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2. Preliminaries: abelian varieties

In this section we collect some facts concerning abelian varieties. Our basic reference is [14].

Let A be an abelian variety over $\bar{k}$. Let $A[n] \subset A(\bar{k})$ be the set of the n-torsion points of A. If k is finite, then every point in $A(\bar{k})$ is a torsion point. For every torsion point $x \in A(\bar{k})$ let

$$\mathrm{ord}(x) := \min\{n \in \mathbb{Z}_{>0} \mid nx = 0\}$$

be the order of x. Let $\mathrm{End}_{\bar{k}}(A)$ be the ring of $\bar{k}$-endomorphisms of A. Every abelian variety A defined over $\bar{k}$ is isogenous to a product of *simple* abelian varieties (over $\bar{k}$).

An elliptic curve over a field k of characteristic $p > 0$ is called *supersingular* if its p-rank is zero, and an abelian variety over k is called supersingular if it is $\bar{k}$-isogenous to a product of supersingular elliptic curves.

Remark 2.1. — In our applications, we will use hyperelliptic curves contained in abelian varieties. Over an algebraically closed field, every (principally polarized) abelian surface is the Jacobian of a (possibly reducible) hyperelliptic curve (see [25]). This fails in higher dimensions: a generic principally polarized abelian variety of dimension ≥ 3 over $\mathbb{C}$ does not contain hyperelliptic curves [17]. A similar result holds over *large* fields of positive characteristic, such as an algebraic closure of $\bar{\mathbb{F}}_q(t)$ [16]. It could still be that over an algebraic closure of a *finite* field, every abelian variety of dimension ≥ 2 contains a hyperelliptic curve.

Let C be a smooth projective geometrically connected curve of genus $\mathsf{g} = \mathsf{g}(C) \geq 2$ over a field k. Let $J = J_C$ be the Jacobian of C. Throughout, we assume that $C(k) \neq \emptyset$ and choose a point $c_0 \in C(k)$ which we use to identify the degree n Jacobian $J^{(n)}$ with J and to embed C in J. Consider the maps

$$C^n \xrightarrow{\phi_n} \mathrm{Sym}^{(n)}(C) \xrightarrow{\varphi_n} J^{(n)}(C),$$

$$c = (c_1, \dots, c_n) \longrightarrow (c_1 + \cdots + c_n) \longrightarrow [c],$$

Here $(c_1 + \cdots + c_n)$ denotes the zero-cycle. The map ϕ_n is a finite cover of degree $n!$. For all $n \geq 2\mathsf{g} + 1$, the map φ_n is a $\mathbb{P}^{n-\mathsf{g}}$-bundle and the map $C^n \to J^{(n)}(C)$ is surjective with geometrically irreducible fibers (see [13], Corollary 9.1.4, for example). For $x \in J(k) = J^{(n)}(k)$ put $\mathbb{P}_x := \varphi_n^{-1}(x) \subset \mathrm{Sym}^{(n)}(C)$.

Lemma 2.2. — *Let C be a smooth projective geometrically connected curve over $\mathbb{F}_q$ of genus $\mathsf{g} = \mathsf{g}(C) \geq 2$, with Jacobian J. For every point $x \in J(\mathbb{F}_q)$ and every $n \geq 2\mathsf{g} + 1$ there exist a finite extension $k/\mathbb{F}_q$ and a point $y \in \mathbb{P}_x(k)$ such that the degree n zero-cycle $c_1 + \cdots + c_n$ on C corresponding to y is k-irreducible.*

Proof. — Let $x \in J(\mathbb{F}_q)$ be a point and $\mathbb{P}_x = \varphi_n^{-1}(x)$ the fiber over x. The restriction $\phi_{n,x}$ of ϕ_n to $\mathbb{P}_x$ is a cover of degree $n!$.

We apply an equidistribution theorem of Deligne as in [13]. Let $k/\mathbb{F}_q$ be a finite extension. In the terminology of [13], Theorem 9.4.4, let $T = \mathrm{Spec}(\mathbb{F}_q)$ and put $t = \mathrm{Spec}(k)$, (t is a k-valued point of T). Let E/k be the (unique) degree n extension and $X_{t,\mathrm{prime}}(E)$ the subset of E-valued points of a natural $\mathbb{G}_m$-bundle X_t over a Zariski open subvariety of the fiber $(\varphi_n \circ \phi_n)^{-1}(x) \subset C^n$, defined in [13], p. 189. The image of $X_{t,\mathrm{prime}}(E)$ in $C^n(E)$ consists of n-tuples of distinct points $(c_1, \dots, c_n)$ such that the Galois group $\mathrm{Gal}(E/k)$ acts transitively on the set $\{c_1, \dots, c_n\}$. By Theorem 9.4.4 in [13], there exist constants $a(x) = a(x, T)$ and $c(x) = c(x, T)$ such that for any k with $\mathrm{Card}(k) \geq a(x)$ one has

$$|\#X_{t,\mathrm{prime}}(E)/\#X_t(E) - 1/n| \leq c(x)n!/(\#E)^{1/2}. \tag{2.1}$$

Note that $\#E = \mathrm{Card}(k)^n$. By effective Weil estimates as in Theorem 9.1.2 of [13], $X_t(E) \neq \emptyset$; combining this with the inequality (2.1) we find that for k sufficiently large $\#X_{t,\mathrm{prime}}(E) \neq \emptyset$, as claimed. □

Remark 2.3. — A similar result has been used in [18], Lemma 5.

Corollary 2.4. — *Let C be a curve of genus $\mathsf{g}(C) \geq 2$ over a (sufficiently large) finite field k, let J be its Jacobian and $x \in J(k)$ a point.*

Choose a point c_0 *on* $C(k)$ *and use it to identify* J *with* $J^{(n)}$*, for all* n*, and to embed* C *in* J*. For every* $n \geq 2\mathsf{g}+1$ *there exist a point* $c \in C(E)$*, where* E/k *is the (unique) extension of* k *of degree* n*, and an endomorphism* $\Phi = \Phi_n \in \mathrm{End}_k(J)$ *such that* $\Phi(c) = x$*.*

Proof. — For any $n \geq 2\mathsf{g}(C)+1$ consider the surjective map φ_n. Let $x \in J^{(n)}(k)$ be a point and let $\mathbb{P}_x$ be the projective space over x. Extending k, if necessary, we find a $y \in \mathbb{P}_x(k)$ such that the zero-cycle $(c_1+\cdots+c_n)$ corresponding to y is irreducible over k, by Lemma 2.2.

We have $y = \sum_{g\in G} c^g$, with $c := c_1 \in C(E)$, where E/k is the unique extension of k of degree n and $G := \mathrm{Gal}(E/k)$. The group G is cyclic, generated by the Frobenius automorphism, which we denote by Fr. Thus

$$y = \sum_{j=0}^{n-1} \mathrm{Fr}^j(c).$$

The Frobenius morphism "lifts" to an endomorphism of J, that is, there exists an endomorphism $\tilde{\mathrm{Fr}} \in \mathrm{End}_k(J)$ which acts on $J(E)$ in the same way as the Galois automorphism $\mathrm{Fr} \in \mathrm{Gal}(E/k)$. Put

$$\Phi := \sum_{j=0}^{n-1} \tilde{\mathrm{Fr}}^j,$$

as an element of $\mathrm{End}_k(J)$. □

Remark 2.5. — In particular, Corollary 2.4 implies that if $\mathrm{ord}(x) = m$ then there exist infinitely many points in $C(\bar{k}) \subset J(\bar{k})$ whose order is divisible by m. Indeed, notice that $\mathrm{ord}(c) = \mathrm{ord}(c^g)$, for all $g \in \mathrm{Gal}(E/k)$. Since the order of x is m the order $\mathrm{ord}(c)$ is divisible by m.

A related result has been proved in [1]: Let ℓ be a prime, C a curve (defined over a finite field k), J its Jacobian, $C \subset J$ an Albanese embedding and $\lambda : J(\bar{k}) \to J(\bar{k})_\ell$ the projection onto the ℓ-primary part. Then the map $\lambda : C(\bar{k}) \to J(\bar{k})_\ell$ is surjective. It was noticed in [19], p. 112, that the method of [1] can be used to prove that any positive-dimensional subvariety of a geometrically simple abelian variety (over a finite field) contains infinitely many points of pairwise prime orders.

The argument in the proof of Corollary 2.4 gives a statement very much in the spirit of [12]:

COROLLARY 2.6. — *Let C be a curve of genus g over a sufficiently large finite field k and J its Jacobian. Then there exist a morphism $\lambda : C \to J$ (depending on k) and a field E/k such that $J(k) \subset \lambda(C(E))$.*

LEMMA 2.7. — *Let K be a number field (or any field where Hilbert's irreducibility holds). Let C be a curve of genus $\mathsf{g} = \mathsf{g}(C) \geq 2$ over K and J its Jacobian. Assume that C has a point $c_0 \in C(K)$ and use this point to identify $J^{(n)} = J$ and the embedding $C \to J$. For any point $x \in J(K)$ and any $n \geq 2\mathsf{g}+1$ there exist an extension K'/K of degree n and a point $c \in C(K')$ such that the cycle $\mathrm{Tr}_{K'/K}(c)$ equals $x \in J^{(n)}(K) = J(K)$.*

Proof. — The inverse image of $\mathbb{P}_x$ under $C^n \to \mathrm{Sym}^{(n)}(C)$ is a geometrically irreducible, generically Galois cover of $\mathbb{P}_x$ (see Corollary 9.1.4 in [13], for example). Hilbert's irreducibility theorem (as in [22], Proposition 2 in Section 9.2 and "Hilbert's theorem" in Section 9.6) implies the claim. □

3. Preliminaries: K3 surfaces

In this section we assume that the ground field k is algebraically closed. A good general reference for the following material is [21] and [8].

DEFINITION 3.1. — *A smooth connected simply-connected projective algebraic surface with trivial canonical class is called a K3 surface. A K3 surface S with* $\mathrm{rk}\,\mathrm{Pic}(S) = 22$ *is called* supersingular.

EXAMPLE 3.2. — Examples of K3 surfaces are double covers of $\mathbb{P}^2$ ramified in a smooth curve of degree 6, smooth quartic hypersurfaces in $\mathbb{P}^3$ or smooth intersections of 3 quadrics in $\mathbb{P}^5$.

Another interesting series of examples is given by (generalized) Kummer surfaces: desingularizations of quotients of abelian surfaces by certain finite group actions (see Proposition 4.4).

REMARK 3.3. — If S is a K3 surface over a field of characteristic zero, then $\mathrm{rk}\,\mathrm{Pic}(S) \leq 20$. An example of a supersingular S over a field of positive characteristic is given by a desingularization of A/σ, where A is a supersingular abelian variety and σ the standard involution (multiplication by -1 map).

Remark 3.4. — If S is uniruled then the Brauer group of S has trivial transcendental part, and all cycles are algebraic. This implies that $\mathrm{rk}\,\mathrm{Pic}(S) = 22$ (i.e., S is supersingular). In particular, this is possible only in positive characteristic [21], [3].

In characteristic 2, every supersingular K3 surface is unirational [20]. It is conjectured that all supersingular K3 surfaces are unirational (see [21], Section 5, or [9], Problem 12). A generalized Kummer surface $S \sim A/G$ is uniruled iff it is unirational iff the corresponding abelian surface A is supersingular [23], [11].

4. Construction

Unless stated otherwise, the ground field k is algebraically closed of characteristic $\neq 2$. We recall the classical construction of special K3 surfaces, called *Kummer* surfaces. Let A be an abelian surface,

$$\begin{array}{rcl} \sigma : A & \to & A \\ a & \mapsto & -a \end{array}$$

the standard involution. The set of fixed points of σ is exactly $A[2]$. The blowup $S := \widehat{A/\sigma}$ of the image of $A[2]$ in the quotient A/σ is a smooth K3 surface S, called a *Kummer* surface:

$$A/\sigma \to S, \qquad \widehat{A/\sigma} \to S.$$

Lemma 4.1. — *Rational curves C in A/σ correspond to hyperelliptic curves $\tilde{C} \subset A$ containing a point $P \in A[2]$ and such that the hyperelliptic involution on $\tilde{C}$ coincides with σ.*

Proof. — The hyperelliptic involution on $\tilde{C}$ acts as an involution $\sigma : x \to -x$ on the Jacobian $J = J_{\tilde{C}}$ and hence also on the abelian subvariety which is the image of J in A. In particular, the involution σ on A induces the standard hyperelliptic involution on C. Hence C/σ is rational and defines a rational curve in A/σ. Conversely, if $C \in A/\sigma$ is rational then the preimage of C in A is irreducible (since A doesn't contain rational curves). Thus $C = \tilde{C}/\sigma$ and $\tilde{C}$ is hyperelliptic and all ramification points of the map $\tilde{C} \to C$ are contained among the two-torsion points $A[2] \cap \tilde{C}$. $\square$

THEOREM 4.2. — *Let S be a Kummer surface over a finite field k, C a curve of genus 2 defined over k, J its Jacobian and $S \sim J/\sigma$ the associated Kummer surface. Then every algebraic point $s \in S(\bar{k})$ lies on some rational curve, defined over $\bar{k}$.*

Proof. — Let $s \in S(\bar{k})$ be an algebraic point (on the complement to the 16 exceptional curves) and $x \in J(\bar{k})$ one of its preimages. We have proved in Corollary 2.4 that for every $x \in J(\bar{k})$ (and any Albanese embedding $C \to J$) there is an endomorphism $\Phi \in \mathrm{End}_{\bar{k}}(J)$ such that $\Phi \cdot C(\bar{k})$ contains x (note that Φ commutes with the involution σ). The image of the curve $\Phi \cdot C$ in S contains s. $\square$

Combining Theorem 4.2 with Corollary 2.4 we obtain

COROLLARY 4.3. — *Let S be a Kummer surface over a finite field k. There are infinitely many rational curves (defined over $\bar{k}$) through every point in the complement of the 16 exceptional curves in $S(\bar{k})$. If S is non-uniruled, these curves do not form an algebraic family.*

In addition to quotients A/σ, there exist generalized Kummer K3 surfaces obtained as desingularizations of abelian surfaces under actions of other finite groups. Such actions have been classified:

PROPOSITION 4.4 (see [11]). — *Let A be an abelian surface over a field k and G a finite group acting on A such that the quotient A/G is birational to a K3 surface. If $\mathrm{char}(k) > 0$ then G is one of the following:*

- *a cyclic group of order $2, 3, 4, 5, 6, 8, 10, 12$;*
- *a binary dihedral group $(2, 2, n)$ with $n = 2, 3, 4, 5, 6$;*
- *a binary tetrahedral group $(2, 3, 3)$;*
- *a binary octahedral group $(2, 3, 4)$;*
- *a binary icosahedral group $(2, 3, 5)$.*

If $\mathrm{char}(k) = 0$ then G is one of the following:

- *a cyclic group of order $2, 3, 4, 6$;*
- *a binary dihedral group $(2, 2, n)$ with $n = 2, 3$;*
- *a binary tetrahedral group $(2, 3, 3)$.*

The groups listed above do indeed occur.

Corollary 4.5. — *If $S \sim A/G$ is a generalized Kummer K3 surface over a finite field k (of characteristic ≥ 7) then every algebraic point on S lies on infinitely many rational curves, defined over $\bar{k}$.*

Proof. — By Remark 3.4, a supersingular generalized Kummer K3 surface is uniruled and the claim follows. By Lemma 6.2 in [11], if S is not supersingular and G is divisible by two then G has a unique element of order two, acting as the standard involution. An argument as in the proof of Theorem 4.2 applies to show that every algebraic point lies on a rational curve. The generalized Kummer K3 surfaces with $G = \mathbb{Z}/5$ are supersingular [11].

It remains to consider $G = \mathbb{Z}/3$. In this case, the abelian variety A is isogenous to $E \times E$ with an action of $\mathbb{Z}/3$ which is obtained from the cyclic permutation action on E^3 divided by the diagonal. The quotient surface $A/\mathbb{Z}/3$ is birationally equivalent to a $K3$-surface (it is simply-connected, has a nontrivial holomorphic $(2,0)$-form and Kodaira dimension 0). In order to apply our general argument we need to find a generating curve $C \in A$ with a rational quotient $C/\mathbb{Z}/3$. Consider the action of $\mathfrak{S}_3$ on $\mathbb{P}^1$ with $\mathbb{Z}/3$-invariant points $0, \infty$. Let S be an $\mathfrak{S}_3$-orbit in $\mathbb{P}^1$ and C_S the double cover of $\mathbb{P}^1$ ramified in S. Then $\mathtt{g}(C_S) = 2$ and $\mathfrak{S}_3$ acts on the hyperelliptic curve C_S. The automorphism group of C_S, for a generic orbit S, is equal to $\mathfrak{S}_3 \times \mathbb{Z}/2$. There is an action of $\mathfrak{S}_3$ on $J = J_{C_S}$, note that J is isogenous to $E \times E$. For any subgroup $\mathbb{Z}/2 \subset \mathfrak{S}_3$ the quotient $C_S/\mathbb{Z}/2$ is an elliptic curve. Since all such subgroups are conjugated it is the same elliptic curve. Any elliptic curve (over a field of characteristic $\neq 2$) can be obtained in this way: realize it as the double cover of $\mathbb{P}^1$ ramified in

$$\{1, (x + 1/x)/2, (\zeta x + 1/\zeta x)/2, (\zeta^2 x + 1/\zeta^2 x)/2\}$$

corresponding to the $\mathfrak{S}_3$-orbit

$$\{x, \zeta x, \zeta^2 x, 1/x, \zeta/x, \zeta^2/x\},$$

where ζ is a third root of 1 and $x \in \mathbb{P}^1$ is an arbitrary point not equal to $0, \infty$ and any cubic root of 1 or -1. The quotient $C_S/\mathbb{Z}/3$ is rational.

Applying the argument of Corollary 2.4 and endomorphisms (sums of powers of the Frobenius, they commute with the $\mathbb{Z}/3$-action) we obtain our claim. □

Remark 4.6. — There exist K3 surfaces that are not generalized Kummer K3 surfaces but are dominated by such. Clearly, they satisfy the conclusion of Corollary 4.5.

Remark 4.7. — We do not know whether or not *every* algebraic K3 surface contains infinitely many rational curves (elliptic K3 surfaces do, see [5]). It is known that primitive classes in $\mathrm{Pic}(S)$ of a *general* K3 surface S over $\mathbb{C}$ are represented by rational curves with at worst nodal singularities (see [26], [7], for example). In particular, a general polarized S with $\mathrm{rk}\,\mathrm{Pic}(S) \geq 2$ has infinitely many rational curves. See, however, [10] for examples of surfaces with $\mathrm{rk}\,\mathrm{Pic}(S_{\bar{\mathbb{Q}}}) = 1$.

Remark 4.8. — Theorem 4.2 can fail if $k = \bar{\mathbb{F}}_q(t)$ as we now show. Let S_0 be a non-supersingular Kummer surface over $\bar{\mathbb{F}}_q$ (and therefore not uniruled, by Remark 3.4). Let S be a base extension of S_0 to k. Choose a non-rational curve C_0 in S_0. View the function field $k_0 = \bar{\mathbb{F}}_q(C_0)$ as a finite extension of k. Restricting the diagonal map $C_0 \to S_0 \times C_0$ to the generic point gives a point $s \in S(k_0)$. If the conclusion of Theorem 4.2 were valid for S, then over some finite extension k_0' of k_0, there would be a non-constant rational curve through s and hence a dominant rational map $\mathbb{P}^1 \times C_0' \to S_0$, where C_0' is a curve over $\bar{\mathbb{F}}_q$ with function field k_0'. Therefore, S_0 is uniruled – contradiction.

5. Surfaces of general type

Using similar ideas we can construct non-uniruled surfaces S of general type over finite fields k with nontrivial Brauer group of finite height [3] such that every algebraic point $s \in S(\bar{k})$ lies on a rational curve and any two points can be connected by a chain of rational curves. (However, the degrees of these curves cannot be bounded, *a priori*).

For simplicity, let us assume that $p := \mathrm{char}(k) \geq 5$. Let S_0 be a unirational surface of general type over k, for example

$$x^{p+1} + y^{p+1} + z^{p+1} + t^{p+1} = 0$$

([21], Section 5). Let $\mathbb{P}^2 \to S_0$ be the corresponding (purely inseparable) covering of degree a power of p.

Let S_1 be a non-supersingular, and therefore, non-uniruled, Kummer K3 surface admitting an abelian cover onto $\mathbb{P}^2$ of degree prime to p with

Galois group G for example, a double cover (here we may have to enlarge the ground field k).

Lemma 5.1. — *For any n coprime to p, and any finite purely inseparable extension L/K we have a natural isomorphism, induced by inclusion $K \hookrightarrow L$,*

$$K^*/(K^*)^n = L^*/(L^*)^n.$$

Proof. — Indeed, there exists an $m \in \mathbb{N}$ such that K^* contains the p^m-powers of all elements of L^*. Since p^m and n are coprime the claimed isomorphism follows. □

Let $L = \bar{k}(\mathbb{P}^2)$. By Kummer theory, the extension of function fields $\bar{k}(S_1)$ over L is obtained by adjoining the n-th roots of the elements of a finite subset T of L^*, for some positive integer n prime to p. By Lemma 5.1, we may multiply each element of T by an element of $(L^*)^n$ in order to assume that $T \subset K^*$. Adjoining the n-th roots of the elements of T to $\bar{k}(S_0)$ gives the function field of a surface S over $\bar{k}$. In particular, we have rational maps:

$$\begin{array}{ccc} S_1 & \rightarrow & S \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \rightarrow & S_0, \end{array}$$

where S is a surface of general type (since the corresponding function field is a separable abelian extension of degree coprime to p). At the same time there is a surjective purely inseparable map $S_1 \to S$. Surjectivity implies that there is a rational curve (defined over $\bar{k}$) passing through every algebraic point of S, to get *every* point we may need to pass to a blowup $\tilde{S}_1$ of S_1 resolving the indeterminacy of the dominant map $S_1 \to S$ (exceptional curves are rational over $\bar{k}$). By pure inseparability, if we had a dominant map $C \times \mathbb{P}^1 \to S$ then we would also have a dominant map $C \times \mathbb{P}^1 \to S_1$ (seen on the level of function fields), contradicting the assumption that S_1 is not uniruled.

6. Higher dimensions

Arguments as in the proof of Theorem 4.2 give us the following result: Let k be a finite field, C a hyperelliptic curve of genus ≥ 2 over k, J its Jacobian, σ the standard involution on J and $S = J/\sigma$ the associated

Kummer variety. Then every rational point $s \in S(\bar{k})$ lies on some rational curve defined over $\bar{k}$. Similar results hold for some other classes of non-uniruled higher-dimensional varieties.

Definition 6.1. — *A smooth projective variety V is called Calabi-Yau if its canonical class is trivial and $h^0(\Omega^i_V) = 0$ for all $i = 1, \ldots, \dim X - 1$.*

Example 6.2. — Let E an elliptic curve over k with an automorphism ρ of order 3 and $A := E^3$. The quotient A/ρ (diagonal action) admits an desingularization V which is a Calabi-Yau variety.

There are many embeddings $\iota : E \hookrightarrow A$ and, in particular, every torsion point in A lies on some $\iota(E)$.

If k is finite then every point in $V(\bar{k})$ lies on some $\bar{k}$-rational curve in V. Moreover, E^2/ρ (diagonal action) is a rational surface. Hence every point in $V(\bar{k})$ lies in fact on a rational surface defined over $\bar{k}$.

Example 6.3. — Let C be the Klein quartic curve and J its Jacobian. Then the quotient of J/σ, where σ is an automorphism of order 7, admits a desingularization V which is a Calabi-Yau threefold (see [15], for example). Again, over finite fields, one can show that every algebraic point of V lies on a rational curve.

Example 6.4. — The following varieties have been considered in [24]: Let S be a K3 surface with an involution σ and E an elliptic curve with the standard involution τ. There exists a nonsingular model V of $E \times S/(\tau \times \sigma)$, which is a Calabi-Yau threefold. If we choose S and E, defined over a finite field, so that every algebraic point of S lies on a rational curve, then the same property holds for V.

Conjecture 6.5. — Let X be any smooth projective variety over a finite field k. Assume that X has trivial canonical class and that $X_{\bar{k}}$ has trivial algebraic fundamental group. Then every algebraic point of X lies on a rational curve $C \subset X$, defined over $\bar{k}$.

Remark 6.6. — If A is a general abelian variety of dimension $n \geq 3$ (over $\mathbb{C}$ or over an algebraic closure of $\bar{\mathbb{F}}_q(x)$) and σ is the standard involution, then A/σ contains no rational curves, has trivial fundamental group and has Kodaira dimension zero (see Remark 2.1). However, the canonical class of a desingularization is nontrivial, for $n \geq 3$. This also shows that the presence of rational curves is highly unstable under deformations.

An interesting test of Conjecture 6.5 would be the case of a smooth quintic in $\mathbb{P}^4$.

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