SPECIAL ELLIPTIC FIBRATIONS

by

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Abstract. — We construct examples of elliptic fibrations of orbifold general type (in the sense of Campana) which have no étale covers dominating a variety of general type.

Contents

To the memory of our friend and colleague Andrey Tyurin.

1. Introduction

Consider the following two classes of varieties:

– admitting an étale cover which dominates a (positive dimensional) variety of general type;
– admitting a nonconstant map with target an *orbifold* of general type (defined by taking into account possible multiple fibers of the map, see Section 2 for details).

In this note we construct examples of complex three-dimensional varieties in the second class which are not in the first class, answering a question of Campana (see [2]).

Acknowledgments. We have benefited from conversations with F. Campana and J. Kollár. Both authors were partially supported by the NSF.

2. Generalities

Throughout, let X be a smooth projective algebraic variety over $\mathbb{C}$ with function field $\mathbb{C}(X)$, $\mathrm{Pic}(X)$ its Picard group and K_X its canonical class. For $D \in \mathrm{Pic}(X)_{\mathbb{Q}}$ we let $\kappa(D)$ be the Kodaira dimension of D, $\kappa(X) = \kappa(K_X)$ the Kodaira dimension of X and $\kappa(X, D) := \kappa(K_X + D)$ the corresponding log-Kodaira dimension. We denote by Ω^n_X the sheaf of differential n-forms, by $\mathcal{T}_X$ the tangent bundle and by $\pi_1(X)$ the fundamental group.

We now recall some notions concerning fibrations following [2]. Let $\varphi : X \to B$ be a morphism between smooth algebraic varieties, such that the locus $D := \cup_j D_j \subset B$ over which the scheme-theoretic fibers of φ are not smooth is a (strict) normal crossing divisor (with irreducible components D_j). For each j, let n_j be the minimal (scheme-theoretic) multiplicity of a fiber-component over D_j and

$$D(\varphi) := \sum_j (1 - 1/n_j) D_j \in \mathrm{Pic}(B)_{\mathbb{Q}}$$

the multiplicity divisor of φ. The pair $(B, D(\varphi))$ will be called an *orbifold* associated to φ. It is called of an orbifold of *general type* if

$$\kappa(B, D(\varphi)) = \dim(B) > 0.$$

Example 2.1. — Let $\varphi : X \to B = \mathbb{P}^1$ be an elliptic fibration such that $D(\varphi) \neq \emptyset$. The degenerate fibers with $n_j \geq 2$ are *multiple* fibers. The associated orbifold $(B, D(\varphi))$ is of general type provided

$$\sum (1 - 1/n_j) > 2. \tag{2.1}$$

This condition implies that there exists a finite cover $\tilde{B} \to B$ ramified with multiplicity n_j at points $D_j \subset D(\varphi)$ and of genus ≥ 2. Let $\tilde{X}$ be the pullback of the elliptic fibration to $\tilde{B}$. Then $\tilde{X} \to X$ is étale and has a surjective map $\tilde{X} \to \tilde{B}$.

THEOREM 2.2. — *There exist smooth projective algebraic threefolds X admitting an elliptic fibration $\varphi : X \to B$ such that*

- $\pi_1(X) = 0$*;*
- *B is a smooth elliptic surface with $\kappa(B) = 1$;*
- *$D(\varphi) \subset B$ is a smooth irreducible divisor;*
- *the orbifold $(B, D(\varphi))$ is of general type.*

3. Logarithmic transforms

We recall the construction of logarithmic transforms of elliptic fibrations due to Kodaira [4] (for more details see [3], Section 1.6).

Let C be a smooth curve and $\eta : \mathcal{E} \to C$ a *nonisotrivial* elliptic fibration. Let $\Delta \subset C$ be a unit disc with center p_0 and *smooth* central fiber E_0 over p_0. For every $m \in \mathbb{N}$ consider the diagram

$$\begin{CD} \tilde{\mathcal{J}} @>>> \mathcal{J} \\ @V{\tilde{\eta}}VV @VV{\eta}V \\ \tilde{\Delta} @>>{\iota_m}> \Delta \end{CD}$$

where $\mathcal{J}$ is the restriction of $\mathcal{E}$ to Δ, ι_m is a cyclic cover of degree m given by

$$\tilde{z} \mapsto \tilde{z}^m = z$$

(with z a local analytic coordinate at p_0) and $\tilde{\mathcal{J}}$ the pullback of $\mathcal{J}$ to $\tilde{\Delta}$. After appropriate choices one has

$$\mathcal{J} = (\mathbb{C} \times \Delta)/\Lambda(z), \quad \tilde{\mathcal{J}} = (\mathbb{C} \times \Delta)/\Lambda(\tilde{z}^m)$$

(where $\Lambda(z) \subset \mathbb{C}$ is a family of lattices) and

$$\mathcal{J} = \tilde{\mathcal{J}}/\mathfrak{C}_m,$$

where $\mathfrak{C}_m$ is a finite cyclic group generated by

$$(s, \tilde{z}) \mapsto (s, \zeta_m \tilde{z}) \mod \Lambda(\tilde{z}^m)$$

(and ζ_m is an m-th root of 1). Let $\omega_m(z)/m$ be a local m-torsion section of $\mathcal{J}$ and define

$$\mathcal{J}' := \tilde{\mathcal{J}}/\mathfrak{C}'_m,$$

where $\mathfrak{C}'_m$ is a cyclic group generated by

$$(s, \tilde{z}) \mapsto (s + \frac{\omega_m(\tilde{z}^m)}{m}, \zeta_m \tilde{z}) \quad \mod \Lambda(\tilde{z}^m).$$

We have an isomorphism

$$\mathcal{J}' \setminus (E_0/\mathfrak{C}_m) \simeq \mathcal{J} \setminus E_0$$

and we can extend $\mathcal{J}'$ to an elliptic fibration $\eta' : \mathcal{E}' \to C$, called the *logarithmic transform* (twist) of $\mathcal{E}$. In $\mathcal{E}'$ a cycle (circle) $\mathbb{S}$ which was bounding a holomorphic section over a disc in $\mathcal{E}$ is homologous to a nontrivial cycle $\mathbb{S}' \in E_0$.

Proposition 3.1. — *Assume that $\mathcal{E}$ is locally Jacobian and not locally isotrivial and that $\mathcal{E}'$ is obtained from $\mathcal{E}$ by a logarithmic transform at exactly one point $p_0 \in C$. Then*
- $H^0(\mathcal{E}, K_{\mathcal{E}}) \simeq H^0(\mathcal{E}', K_{\mathcal{E}'})$;
- $\pi_1(\mathcal{E}) = 0 \Rightarrow \pi_1(\mathcal{E}') = 0$;
- *$\mathcal{E}'$ is Kähler.*

Proof. — Every form $w \in H^{2,0}(\mathcal{E})$ has a local representation as

$$w = dh \wedge d\log(s).$$

It is visibly invariant under translation by s on $\mathcal{E} \setminus E_0$, is preserved under gluing and can be extended from $\mathcal{E} \setminus E_0$ to $\mathcal{E}'$. Moreover, on $\mathcal{E}'$ it has a zero of multiplicity $m-1$ along $E_0/\mathfrak{C}_m$. After twisting exactly one fiber, we have

$$K_{\mathcal{E}'} = K_{\mathcal{E}} + (1 - 1/m)E,$$

where E is a (generic) fiber of η. Since $\mathcal{E}$ is locally Jacobian we have $K_{\mathcal{E}} = \eta^* L$, where $L \in \mathrm{Pic}(C)$, and $H^0(\mathcal{E}, K_{\mathcal{E}}) = H^0(C, L)$. Similarly, we have an imbedding $K_{\mathcal{E}'} \hookrightarrow \eta^*(L + p_0)$ and

$$H^0(\mathcal{E}', K_{\mathcal{E}'}) \subset H^0(\mathcal{E}', \eta'^*(L + p_0)) = H^0(C, (L + p_0)).$$

We have $h^0(C, (L+p_0)) \le h^0(C, L) + 1$. An $f \in H^0(C, (L+p_0))$ which is not in the image of $H^0(C, L)$ is nonzero at p_0. The corresponding element in $H^0(\mathcal{E}', \eta'^*(L+p_0))$ is also nonzero on the fiber over p_0. However, every global section of $K_{\mathcal{E}'}$ vanishes on the central fiber. Thus $f \notin H^0(\mathcal{E}', K_{\mathcal{E}'})$ so that every section of $K_{\mathcal{E}'}$ is an extension of a section of $K_{\mathcal{E}}$ (restricted to $\mathcal{E} \setminus E_0$):

$$H^0(\mathcal{E}, K_{\mathcal{E}}) = H^0(\mathcal{E}', K_{\mathcal{E}'}).$$

Since $\pi_1(\mathcal{E}) = 0$ we have $C = \mathbb{P}^1$. We claim that $\pi_1(\mathcal{E} \setminus E_0) = 0$. Indeed, the fundamental group of the elliptic fibration $(\mathcal{E} \setminus E_0) \to (C \setminus p_0)$ lies in the image of $\pi_1(E)$, which is a *finite* abelian group since there are nontrivial vanishing cycles which are homotopic to zero. Since the global monodromy has finite index in $\mathrm{SL}_2(\mathbb{Z})$ the group $\pi_1(\mathcal{E} \setminus E_0)$ is also finite abelian and the corresponding covering is fiberwise. Thus it extends as a finite étale covering of $\mathcal{E}$, contradicting the assumption that $\pi_1(\mathcal{E}) = 0$.

Consider the (topological) quotient spaces $\mathcal{E}/E_0$ and $\mathcal{E}'/E_0'$. They are naturally isomorphic and we have two exact homology sequences

$$\ldots \to H_3(\mathcal{E}, \mathbb{Q}) \to H_3(\mathcal{E}/E_0, \mathbb{Q}) \xrightarrow{d} H_2(E_0, \mathbb{Q}) \to H_2(\mathcal{E}, \mathbb{Q})$$

and

$$\ldots \to H_3(\mathcal{E}', \mathbb{Q}) \to H_3(\mathcal{E}'/E_0', \mathbb{Q}) \xrightarrow{d'} H_2(E_0', \mathbb{Q}) \to H_2(\mathcal{E}', \mathbb{Q}).$$

Since $\mathcal{E}$ is Kähler

$$\mathbb{Q} = H_2(E_0, \mathbb{Q}) \hookrightarrow H_2(\mathcal{E}, \mathbb{Q})$$

and

$$H_3(\mathcal{E}, \mathbb{Q}) = H_3(\mathcal{E}/E_0, \mathbb{Q}) = H_1(C, \mathbb{Q})^*.$$

Here we used that $H_1(\mathcal{E}, \mathbb{Q}) = H_1(C, \mathbb{Q})$ which follows from the local nonisotriviality of $\mathcal{E}$. Geometrically it means that every 3-cycle on $\mathcal{E}$ and $\mathcal{E}/E_0$ can be realized as a product of a 1-cycle on C and an elliptic fiber.

Since $\mathcal{E}/E_0 = \mathcal{E}'/E_0'$ the differential d' is also zero and

$$H_2(E_0', \mathbb{Q}) \hookrightarrow H_2(\mathcal{E}', \mathbb{Q}).$$

Thus the class of the generic fiber E is nontrivial. This implies the existence of a Kähler metric (see [5]). Therefore, if $\mathcal{E}$ is Kähler then so is $\mathcal{E}'$. □

COROLLARY 3.2. — *If $\mathcal{E}$ is algebraic and rational then $\mathcal{E}'$ is algebraic.*

Proof. — A smooth surface S is projective iff there is a class $x \in H_2(S, \mathbb{Q})$ with $x^2 > 0$ which is orthogonal to $H^{2,0}(S) \subset H^2(S, \mathbb{C})$. Since $\mathcal{E}'$ is Kähler and $H^{2,0}(\mathcal{E}) = H^{2,0}(\mathcal{E}') = 0$ there is such a class in $H_2(\mathcal{E}, \mathbb{Q})$. □

EXAMPLE 3.3. — Let $\xi : \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial map of degree $n \geq 2$ which is cyclically n-ramified over ∞. Let $\bar{\varphi} : \bar{\mathcal{E}} \to \mathbb{P}^1$ be a rational elliptic surface and $\bar{\mathcal{E}}'$ its logarithmic nm-twist over ∞. Consider the diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\eta} & \mathbb{P}^1 & \xleftarrow{\eta'} & \mathcal{E}' \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ \bar{\mathcal{E}} & \xrightarrow[\bar{\eta}]{} & \mathbb{P}^1 & \xleftarrow[\bar{\eta}']{} & \bar{\mathcal{E}}' \end{array}$$

The surface $\mathcal{E}'$ (induced by ξ) is a logarithmic m-twist at ∞ of $\mathcal{E}$ (induced from $\bar{\mathcal{E}}$). We have $h^0(\mathcal{E}, K_{\mathcal{E}}) = n - 1$. Since $\bar{\mathcal{E}}'$ is algebraic (by Corollary 3.2), $\mathcal{E}'$ is also algebraic.

For more details concerning algebraicity of elliptic fibrations obtained by logarithmic transformations we refer to [3], Section 1.6.2.

4. Construction

Consider the following diagram

$$\begin{array}{ccccc} X & & & & \mathcal{E}' \\ \downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \eta'} \\ B & \xrightarrow[\beta]{} & S & \xrightarrow[\psi]{} & C \\ & & \downarrow{\scriptstyle \varphi_1} & & \\ & & C_1 & & \end{array}$$

where

- C_1, C are $\mathbb{P}^1$;
- S is a nonisotrivial locally Jacobian elliptic surface with irreducible fibers, $\pi_1(S) = 0$ and $\kappa(S) = 1$;
- $\psi : S \to C$ is a rational map with connected fibers defined by a generic line $\mathbb{P}^1_\psi \subset \mathbb{P}(H^0(S, L))$, where L is a polarization on S;
- $\beta : B \to S$ is a minimal blowup so that $\gamma := \psi \circ \beta : B \to C$ is a fibration with irreducible fibers (it exists since L is very ample and ψ is generic, i.e., all singularities of ψ are simple and lie in different smooth fibers of φ_1);

– $\eta' : \mathcal{E}' \to C$ is the fibration from Example 3.3;
– $\varphi : X \to B$ is the pullback of η' via γ.

LEMMA 4.1. — *Let B be the surface above, $p \in C$ a generic point and $D = \gamma^{-1}(p)$. Then*
– *$\varphi_1 \circ \beta : B \to C_1$ is an elliptic fibration and $\kappa(B) = 1$;*
– *$\pi_1(B \setminus D) = 0$.*

Proof. — The genericity of L and ψ implies that all fibers are irreducible and that B is a blowup of S in a finite number of distinct points in which the divisors from $\mathbb{P}^1_\psi \subset \mathbb{P}(H^0(S, L))$ intersect transversally. Since $\pi_1(S) = 0$ we have

$$\pi_1(S \setminus D) = \mathfrak{C}_m$$

(by Lefschetz theorem), where m is the largest integer dividing L in $\mathrm{Pic}(S)$. The corresponding cyclic covering of S is m-ramified along D. We have

$$(B \setminus D) = (S \setminus D) \cup \bigcup_{i \in I} \ell_i$$

where I is a finite set and ℓ_i are affine lines. A cycle generating $\pi_1(B \setminus D)$ is contracted inside one of these lines, which implies that the image of $\pi_1(S \setminus D)$ in $\pi_1(B \setminus D)$ is trivial. □

Proof of Theorem 2.2. — The elliptic fibration $\varphi : X \to B$ satisfies the claimed properties.

First observe that D intersects all components of the fibers E of

$$\varphi_1 \circ \beta : B \to C_1.$$

Indeed, by genericity every such E is either irreducible or a union of a smooth elliptic curve and a rational (-1) curve P. If E is irreducible the claim follows from the ampleness of L. For the same reason we have $\deg(D_{|E}) \geq 2$. Since $D \cdot P = 1$ there is a nontrivial intersection with another component.

Put $F := K_B + (1 - 1/m)D$. Since $\kappa(B) = 1$ and $K_B \cdot D > 0$ a subspace of sections in $H^0(B, amF)$ (for some $a \in \mathbb{N}$) gives a surjection $B \to C_1 \times C$, so that $\kappa(F) = 2$. Moreover, F intersects positively every divisor in B (except finitely many rational curves P_i obtained by blowing up S). It follows that $F = H + \sum m_i P_i$, where H is a polarization on B and $m_i \geq 0$. Thus $(B, D(\varphi))$ is an orbifold of general type. In particular,

$$\varphi^* K_B \subset \varphi^* F \subset \Omega^2_X,$$

where $\varphi^* F$ is saturated, and $\kappa(\varphi^* F) = 2$. Notice that $\kappa(X) = 2$ since

$$\varphi^* F \times \gamma^* K_{\mathcal{E}'/C} \subset K_X \quad \text{and} \quad \kappa(K_{\mathcal{E}'/C}) = 1.$$

The pullback H' (to X) of a polarization on $\mathcal{E}'$ is positive on the fibers of φ. Since B is projective it has a polarization H such that for some $a \in \mathbb{N}$ the divisor $a\varphi^* H + H'$ is positive on every curve in X and is represented by a positive definite Kähler form. This implies that X is projective.

We claim that $\pi_1(X) = 0$. We know that $\pi_1(B) = \pi_1(B \setminus D) = 0$. Hence $\pi_1(X)$ is in the image of $\pi_1(E)$ of a smooth (elliptic) fiber of φ. Since the monodromy of φ is large it kills the fundamental group of the fiber. Indeed, the restriction of φ to a $\mathbb{P}^1$ is isomorphic to $\mathcal{E}'$. Since the complement of a multiple fiber in $\mathcal{E}'$ has trivial fundamental group the same holds for X.

Thus X admits a map onto an orbifold of general type but does not dominate a variety of general type nor has (any) étale covers. □

REMARK 4.2. — In fact, we have proved that $\pi_1(X \setminus \varphi^{-1}(D)) = 0$ so that no modification can yield an étale cover.

5. Holomorphic differentials

One of the features of the construction in Section 4 was the use of a 1-dimensional subsheaf of holomorphic forms with many sections. We have seen that such sheaves impose strong restrictions on the global geometry of the variety. Generalizing several results in [1], we now give an alternative proof of Campana's theorem on the correspondence between such sheaves and maps onto orbifolds of general type (see [2]).

Let X be a smooth Kähler manifold and $\omega \in \Omega^i_X$ a form. The *kernel* of ω is the subsheaf of $\mathcal{T}_X$ generated (locally) by sections t such that for all $x \in \Lambda^{i-1}\mathcal{T}_X$

$$\omega(t \wedge x) = 0.$$

The kernel doesn't change under multiplication of ω by a nonzero (local) holomorphic section of the structure sheaf. This defines, for every subsheaf $\mathcal{F} \subset \Omega^i_X$, its kernel $\mathrm{Ker}(\mathcal{F})$ (a special case of the notion of support of a differential ideal).

Definition 5.1. — *We say that $\mathcal{F} \subset \Omega^i_X$ is k-monomial if at the generic point of X a nonzero local section f of $\mathcal{F}$ is a product of local holomorphic 1-forms:*

$$f = q_1 \wedge \ldots \wedge q_k \wedge \omega,$$

where $1 \leq k \leq i$ and ω is a local $(i-k)$-form. We call $\mathcal{F}$ monomial if $k = i$.

Proposition 5.2. — *Let X be a smooth compact Kähler manifold and $\mathcal{F} \subset \Omega^i_X$ a one-dimensional subsheaf such that*

$$h^0(X, \mathcal{F}^n) \geq an^k + b,$$

where $a > 0$ and $k \geq 1$. Then

- *$k \leq i$;*
- *$\mathcal{F} \subset \Omega^i_X$ is a k-monomial subsheaf;*
- *there exist an algebraic variety Y of dimension k and a meromorphic map*

$$\varphi = \varphi_{\mathcal{F}} : X \to Y$$

with irreducible generic fibers such that the tangent space of the fiber of φ at a generic point coincides with $\mathrm{Ker}(\mathcal{F})$.

Proof. — The ratios of sections $s_l \in H^0(X, \mathcal{F}^n)$ generate a field of transcendence degree k (for some $n \geq 1$). In particular, there is an $x \in X$, with $s_0(x) \neq 0$, where the local coordinates

$$f_l = s_l(x)/s_0(x), \;\; l = 1, ..., k$$

are independent. We know that s_0 is locally equal to w_0^n, where w_0 is a local closed form nonvanishing at x (see [1]). Further, $f_l w_0$ is also a local closed form nonvanishing at x. Since

$$ds_0 = d(f_l w_0) = df_l \wedge w_0 = 0$$

we obtain

$$w_0 = df_l \wedge w'.$$

Since the forms df_l are linearly independent we see that

- $w_0 = gdf_1 \wedge df_2 \ldots \wedge df_k \wedge \omega$, so that g is algebraically dependent on f_l and $\mathcal{F}$ is a k-monomial subsheaf of Ω^i_X;
- the fibers of the map given by f_l are tangent to the kernel of w_0.

Thus we have a meromorphic map

$$\varphi : X \to Y, \ \dim(Y) = k, \tag{5.1}$$

such that s_l are locally (at a generic point of X) products of a power of a volume form induced from Y under φ and a power of ω which is nontrivial on the fiber of φ. The map φ is holomorphic outside of the zero locus of the ring $\oplus_n H^0(X, \mathcal{F}^n)$. □

Corollary 5.3. — *If $k = i$ or $k = i - 1$ then $\mathcal{F} \subset \Omega^i_X$ is monomial.*

Proof. — It suffices to consider $f \in \mathcal{F}$ at generic points. There are two cases:

- $k = i$: then
$$f = df_1 \wedge \ldots \wedge df_k,$$
(modulo multiplication by a function).
- $k = i - 1$: then
$$f = df_1 \wedge df_2 \ldots\ldots. \wedge df_k \wedge q,$$
where q is a closed 1-form.

□

Remark 5.4. — The map from (5.1) admits a bimeromorphic modification

$$\varphi : X \to Y$$

such that

- φ is holomorphic with generically smooth and irreducible fibers;
- X and Y are smooth.

Notations 5.5. — For φ as in Remark 5.4 we define its *degeneracy locus* $D = D_\varphi$ as the subset of all $y \in Y$ such that $d\varphi(x) = 0$ for all $x \in \varphi^{-1}(y)$.

Remark 5.6. — After another modification of φ we can achieve that

- $\operatorname{codim}(D) \geq 2$ or

– $D = \cup_j D_j$ and each $\tilde{D}_j := \varphi^{-1}(D_j) = \cup_i \tilde{D}_{ij}$ is a normal crossing divisor.

Assumption 5.7. — *The map φ is as in Remarks 5.4 and 5.6.*

Lemma 5.8. — *If $k = i$ then either* $\operatorname{codim}(D) \geq 2$ *and*

$$\mathcal{F} = \varphi^* K_Y$$

or there exist integers $n_j \geq 1$ such that

– *$D(\varphi) := K_Y + \sum_j (1 - 1/n_j) D_j$ is big on Y;*
– *φ has multiplicity $\geq n_j$ along every $\tilde{D}_{ij}$;*
– *$\varphi^* D(\varphi) \subset \mathcal{F}$.*

Proof. — Every x with $d\varphi(x) \neq 0$ has a neighborhood U such that the restriction of every section $s \in H^0(X, \mathcal{F}^n)$ to U is induced from a (unique) section $s_U \in H^0(\varphi(U), nK_Y)$. There is a unique holomorphic tensor $s_Y \in H^0(Y \setminus D, nK_Y)$ (where $D = D_\varphi$ is the degeneracy locus of φ) such that the restriction of $\varphi^*(s_Y)$ to $X \setminus \varphi^{-1}(D)$ coincides with s.

If $\operatorname{codim}(D) \geq 2$ then s_Y has a unique extension to a holomorphic tensor on Y (since Y is smooth). In this case, Y is of general type. In case $\operatorname{codim}(D) = 1$ we see (using Remark 5.6) that s_Y is a well-defined tensor on Y with poles along D_j, i.e.,

$$s_Y \in H^0(Y, nK_Y + \sum_j d_j D_j),$$

for some $d_j \in \mathbb{N}$. Let n_j be the minimal multiplicity of φ on the components $\tilde{D}_{ij}$ which surject onto D_j (for all j). Since $\varphi^* s_Y$ is holomorphic on X, a local computation shows that

$$d_j \leq n(1 - 1/n_j)$$

(see, for example, [6] and [1]) and that

$$K_Y + (1 - 1/n_j) D_j$$

is big. □

Remark 5.9. — This gives an alternative proof of Campana's theorem characterizing fibrations over orbifolds of general type.

In the case $i = k$ a section of $\mathcal{F}^n$ (at a generic point of X) descends to the nth-power of a (local) volume form on Y but the correponding global form on Y may have singularities. These singularities disappear after a finite local covering which is sufficiently ramified along the singular locus. This property can be defined for arbitrary tensors.

Definition 5.10. — *A meromorphic tensor t on Y is* locally integrable *if for every point $y \in Y$ there exist a neighborhood $U = U_y$ and a (local) manifold V together with a proper finite map*

$$\lambda : \varphi^{-1}(U) \to U$$

such that $\lambda^* t$ is holomorphic on V.

For $k = i - 1$ we have an analog of Lemma 5.8:

Lemma 5.11. — *Let X be a smooth compact Kähler manifold, $\mathcal{F} \subset \Omega^i_X$ a one-dimensional subsheaf such that*

$$a'n^{i-1} + b' > h^0(X, \mathcal{F}^n) \geq an^{i-1} + b,$$

with $a > 0$, and $\varphi = \varphi_{\mathcal{F}}$ (as in Proposition 5.2). Then there exist a nontrivial fibration

$$\rho : A_Y \to Y$$

(with fibers complex tori) and a map

$$\alpha = \alpha_{\mathcal{F}} : X \to A_Y$$

with connected fibers such that

- *$\varphi = \rho \circ \alpha$;*
- *the tangent space of the fiber of α at a generic point is contained in* Ker$(\mathcal{F})$*;*
- *there is a divisor $D \subset Y$ such that every section $s \in \mathcal{F}$ is a lifting of a monomial locally integrable tensor on A_Y.*

Proof. — By Proposition 5.2, there is map $\varphi : X \to Y$, where $\dim(Y) = i - 1$. It has a natural factorization

$$\varphi : X \xrightarrow{\alpha} A \xrightarrow{\rho} Y,$$

where the fiber A_y of ρ over a generic $y \in Y$ is the Albanese variety $\mathrm{Alb}(X_y)$. Since $\mathcal{F} \subset \Omega^i_X$ any section $s \in \mathcal{F}^n$ (at a generic point of X) can be represented as

$$s = (df_1 \wedge ..... \wedge q)^n,$$

where q is a closed 1-form. The form q defines a holomorphic form on a generic fiber X_y of φ so that ρ is *nontrivial* and $\dim(\alpha(X_y)) \geq 1$. In particular, the restriction of q to X_y is induced from A_y.

It follows that there exists a sheaf $\mathcal{G} \subset \Omega^i_A$ such that $\mathcal{F}^n$ is a saturation of $\alpha^*\mathcal{G}$. Moreover, all sections of $\mathcal{F}^n$ are obtained as lifts of integrable meromorphic sections of $\mathcal{G}$. □

REMARK 5.12. — Notice that if $\dim(A_y) = 1$ (for generic $y \in Y$) then $\mathcal{G} = K_A$ (and $A \to Y$ is an elliptic fibration).

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