SIMPLE EXAMPLES OF SYMPLECTIC FOUR-MANIFOLDS WITH EXOTIC PROPERTIES

by

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Abstract. — We construct examples of simply connected nonalgebraic symplectic fourfolds with a prescribed number of nonintersecting symplectic curves with positive self-intersections.

Contents

1. Introduction

Projective varieties of complex dimension 2 are basic examples of symplectic fourfolds. Of course, the class of algebraic surfaces is much smaller. There are many examples of symplectic nonalgebraic varieties with various distinguishing properties (see [8], [5],[6], [1]). In this note we focus on embedded curves in the algebraic versus symplectic category. It is known that an algebraic surface X satisfies the following properties:

- the intersection form on the subspace of homology generated by complex curves in X is hyperbolic;

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– the fundamental group of any smooth ample curve in X surjects onto the fundamental group of the surface.

We construct simple examples of symplectic fourfolds violating both of these properties.

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2. Construction

Let C be an orientable compact Riemann surface of genus $g = g(C) > 0$. Consider a smooth fibration $X \to C$ such that every fiber X_c (with $c \in C$) is a smooth orientable compact Riemann surface of genus $g(X_c) > 2$. We assume that the monodromy along every loop in the base is represented by an orientable automorphism and that $X \to C$ admits a smooth section.

Lemma 2.1. — *Such a fibration is symplectic.*

Proof. — This is a well known fact, but we sketch a proof for completeness. We build a fiberwise nondegenerate form as follows: choose a standard basis of loops a_i, b_i of the fundamental group of the base C. Each loop defines a monodromy diffeomorphism of X (modulo isotopy), denoted by the same letter. The group $\mathrm{Diff}_g(C)$ of orientation preserving diffeomorphisms of C admits a contraction onto the group of volume-preserving diffeomorphisms. Therefore, we can find representatives for a_i, b_i preserving the volume. By invariance, this defines a vertical nondegenerate *closed* form ω^0 on the preimage of a neighborhood of the basic loops. The complement is a disc and the boundary is isomorphic in a standard way (by the existence of a section) to $\mathbb{S}^1 \times X$. The monodromy diffeomorphism on the fiber is isotopic to the identity. We can choose the isotopy to be volume-preserving. This defines a closed 2-form ω and a closed 2-form

$$\omega_X := \omega_C + \lambda\omega \tag{2.1}$$

(here ω_C is a 2-form on the base C and λ is an arbitrary nonzero constant). The form ω_X defines the symplectic structure on the fibration $X \to C$. □

Remark 2.2. — The proof gives a recipe how to construct symplectic fibrations (see [4]). Any relation in the mapping class group Map_g of the form

$$\prod_{i=1}^{g}[g_i, g_i'] = 1 \tag{2.2}$$

defines a symplectic fibration of the above type over a Riemann surface C.

We will need standard extensions of the mapping class group Map_g. Denote by $\mathrm{Map}_g(n)$ the mapping class group of a Riemann surface of genus g with n distinct labeled points. Let $\mathrm{Map}_g\langle n\rangle$ be the mapping class group preserving a small disc at each labeled point. It is a standard central $\mathbb{Z}^n$-extension of $\mathrm{Map}_g(n)$: the kernel is generated by Dehn twists in the neighborhood of each labeled point. More precisely, a vector

$$(\ell_1, ..., \ell_n) \in \mathrm{Ker}(\mathrm{Map}_g\langle n\rangle \to \mathrm{Map}_g(n))$$

has the following geometric interpretation. Lift the relation (2.2) into $\mathrm{Map}_g(n)$. The obtained family of Riemann surfaces has n nonintersecting sections s_i. We can compute the squares of these sections by considering the product $\prod_{i=1}^{g(C)}[\tilde{g}_i, \tilde{g}_i']$ (where $\tilde{g}$ is an arbitrary lifting of g into $\mathrm{Map}_g, \langle n\rangle$); it is an element of the center of $\mathrm{Map}_g\langle n\rangle$ and thus a vector $(\ell_1, ..., \ell_n) \in \mathbb{Z}^n$. We have $s_i^2 = \ell_i$. Notice that for $g > 2$ the groups $\mathrm{Map}_g, \mathrm{Map}_g(n)$ and $\mathrm{Map}_g\langle n\rangle$ are equal to their commutator subgroup.

Remark 2.3. — The group $H_2(\mathrm{Map}_g(0), \mathbb{Z})$ is equal to $\mathbb{Z}$ for $g \geq 3$. There exists a linear lower bound for the genus of smooth curves realizing a given class in this H_2 (see [4], for example). The bound follows from the relation between this class and the signature of the corresponding symplectic fourfold. Bounds of such type appeared previously in the context of nilpotent groups in [2].

Lemma 2.4. — *For any vector* $(\ell_1, ..., \ell_n)$ *there exists a curve* C*, a smooth fibration* $X \to C$ *into curves* X_c *of genus* $g(X_c) \geq 3$ *and a set of smooth sections* $s_1, ..., s_n$ *of this fibration such that* $s_i^2 = \ell_i$.

Proof. — Every element in $\mathrm{Map}_g\langle n\rangle$ is representable as a product of commutators. It suffices to represent the central element $(\ell_1, ..., \ell_n)$. ☐

Corollary 2.5. — *There exists a symplectic fourfold* X *and a smooth symplectic curve* $D \subset X$ *with* $D^2 > 0$ *such that the image of* $\pi_1(D)$ *in* $\pi_1(X)$ *has infinite index. In particular,* X *has no topological Lefschetz pencils containing* D *(or its multiples* rD*).*

Proof. — Assume that $\ell_i > 0$ for all $i = 1, ..., n$. Take X as in Lemma 2.4. Changing the symplectic form ω_X in 2.1 (by making λ sufficiently large) we can insure that all sections s_i are symplectic. Take D to be one of these sections. ☐

Remark 2.6. — This corollary corrects the argument in Section 4 of [3].

Corollary 2.7. — *There exists a symplectic fourfold* X *such that the intersection form on symplectic curves* $D \subset X$ *is not hyperbolic.*

Proof. — For any $n > 1$ and any vector $(\ell_1, ..., \ell_n)$ with positive ℓ_i we choose X as in Lemma 2.4. The restriction of the intersection form to the sections s_i is positive, contradicting hyperbolicity. ☐

Remark 2.8. — Surfaces of such type can also be obtained from complex Kodaira fibrations by reversing the orientation. Then the smooth complex curves which have negative normal bundle are turned into curves with positive self-intersection.

It is well understood that symplectic geometry is, in a sense, more flexible or closer to differential geometry and topology than to algebraic geometry if we allow large fundamental groups. Now we show how to modify the above examples to obtain simply connected symplectic varieties with the same interesting properties.

PROPOSITION 2.9. — *For any $n > 0$ there exists a simply connected symplectic fourfold containing n smooth symplectic nonintersecting curves with positive self-intersection.*

Proof. — Choose a vector $(\ell_1, ..., \ell_n) \in \mathbb{Z}^n$ such that all $\ell > 1$. Choose $X \to C$ as in Lemma 2.4. By construction, every fiber of $X \to C$ is a symplectic subvariety. Blow up (symplectically) the intersection points of a fiber X_0 with the sections s_i. The obtained symplectic variety $\hat{X} \to X$ has a collection of nonintersecting symplectic subvarieties: proper transforms $\hat{s}_i$ of the sections and $\hat{X}_0$ of the fiber X_0. We have $\hat{s}_i^2 = \ell_i - 1 > 0$ and $\hat{X}_0^2 = -n$. Choose an algebraic simply connected surface V_0 containing a curve Z_0 (a symplectic surface) with self-intersection $Z_0^2 = n$. Choose a rational (algebraic) surface V_1 containing a smooth algebraic curve Z_1 of genus $g(C)$ with self-intersection $Z_1^2 = -\hat{s}_1^2$. We glue V_0 and V_1 to $\hat{X}$ along Z_0 and $\hat{X}_0$, resp. Z_1 and $\hat{s}_1$. We denote the obtained fourfold by $\tilde{X}^{\mathrm{sing}}$. By results of Gromov [9] and Gompf [7], we can smooth symplectically $\tilde{X}^{\mathrm{sing}}$ without changing the symplectic form outside of a small neighborhood of $\hat{X}_0$ and $\hat{s}_1$. The resulting smooth symplectic variety $\tilde{X}$ still contains $n-1$ nonintersecting symplectic curves $\tilde{s}_i$ with positive self-intersection.

We claim that $\tilde{X}$ is simply connected. Clearly, the singular variety $\tilde{X}^{\mathrm{sing}}$ is simply connected (the rational surface V_1 kills the $\pi_1(\hat{s}_1)$, V_0 kills $\pi_1(\hat{X}_0)$ and $\pi_1(\tilde{X}^{\mathrm{sing}})$ is generated by these subgroups). Finally, the smoothing doesn't change the fundamental group. ☐

REMARK 2.10. — The surfaces in Proposition 2.9 and Lemmas 2.5 and 2.7 were constructed in response to a question of Vik. Kulikov. He pointed out that small symplectic quasi-complex deformations of (the graphs) of algebraic surfaces with normal intersections as in [7] or, more generally, in [3], fail to produce quasi-complex embedded symplectic curves of the above type.

Our approach is similar to [4], though the precise result appears to be new.

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