Integral points and effective cones of moduli spaces of stable maps

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May 8, 2002

Abstract

We compute the cones of effective divisors of certain moduli spaces of stable maps. This yields a geometric interpretation of known asymptotic formulas for the number of integral points of bounded height on compactifications of SL_2 in the space of binary forms of degree $n \geq 3$.

Contents

*Partially supported by NSF grant 0196187

†Partially supported by NSF grant 0100277 and the Clay Foundation

0 Introduction

In this paper we compute the $\mathfrak{S}_n$-invariant cone of effective divisors of the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ of stable maps of degree one from genus zero curves with n marked points to $\mathbb{P}^1$. We also compute the effective cone of the generic fiber of the natural map

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)/\mathfrak{S}_n \to \overline{\mathcal{M}}_{0,n}/\mathfrak{S}_n.$$

Our motivation is to provide a geometric explanation of a formula, obtained by Duke, Rudnick and Sarnak, giving the asymptotic behavior of the number of binary forms of degree n with fixed discriminant and bounded integral coefficients. This fits into a larger program to predict and prove asymptotic formulas for the number of rational and integral points on algebraic varieties.

We introduce a counting function for integral points on an algebraic variety as follows: given a variety U over a ring of integers $\mathfrak{o}$ and functions $g_1, ..., g_n$, regular on U, define

$$N(U, B) := \{x \in U(\mathfrak{o}) \mid \max_j(\|g_j(x)\|) \leq B\},$$

where $\|\cdot\|$ is a valuation on $\mathfrak{o}$. This is finite only when the functions g_j give an embedding of U.

The most natural way to interpret the functions g_j is as *sections* of a line bundle L on a projective compactification $X \supset U$ defined over the fraction field F of $\mathfrak{o}$. The fact that the sections embed U implies that L is *big*, i.e., is contained in the interior of the effective cone $\Lambda_{\mathrm{eff}}(X)$ of X. Therefore, in order to describe all natural counting functions on open subsets of X we need to compute its effective cone. Furthermore, it turns out that the asymptotic properties of $N(U, B)$ are intimately related to the structure of this cone.

Let $P(\mathbf{x}) = P(x_0, ..., x_r)$ be a polynomial of degree n in $r+1$ variables. A standard heuristic in number theory predicts that the number

$$N_P(B) := \{\mathbf{x} \mid \max(|x_j|) \leq B,\ P(\mathbf{x}) = 0,\ \text{and } \mathbf{x} \in \mathbb{Z}^{r+1}\}$$

of integral solutions of the equation $P(\mathbf{x}) = 0$ of "height" $\leq B$ grows asymptotically like B^{r+1-n} as $B \to \infty$. When the number of variables is $\gg 2^n$, the affine variety V_P defined by $P = 0$ is smooth and there are no local obstructions, an asymptotic formula can be established using the classical circle method in analytic number theory (see [3], [24] and the references therein). Of course, there may be difficulties when the number of variables is small or the variety V_P is singular.

The following example appeared in the paper by Duke, Rudnick and Sarnak [7]. Consider the vector space of binary forms of degree n

$$x_n z^n + x_{n-1} z^{n-1} w + \ldots + x_0 w^n.$$

The group SL_2 acts on this space by coordinate substitutions. When $n = 3$, the *discriminant form*

$$\mathrm{disc}(x_0, ..., x_3) := 27x_0^2x_3^2 - 18x_0x_1x_2x_3 + 4x_0x_2^3 + 4x_1^3x_3 - x_1^2x_2^2$$

generates the ring of SL_2-invariants. Then there exists a constant $c > 0$ so that

$$N_{\mathrm{disc}-1}(B) = cB^{2/3}(1 + o(1))$$

as $B \to \infty$. Note that the exponent $2/3$ is larger than what is predicted by the standard heuristic.

More generally, one has the

Theorem 0.1 *[7] Fix a generic binary form f of degree $n \geq 3$ with integral coefficients. Let $N(B)$ be the number of binary forms $\mathrm{SL}_2(\mathbb{Z})$-equivalent to f with coefficients bounded by B. Then there exists a $c > 0$ such that*

$$N(B) = cB^{2/n}(1 + o(1)),$$

as $B \to \infty$.

We give a geometric interpretation of the exponent $2/n$ in Theorem 0.1. To this end, we refine the heuristics for counting integral points to take into account singularities of the relevant varieties (see Conjecture 1.6). We verify that Conjecture 1.6 is consistent with Theorem 0.1 in Theorem 2.1. Its proof involves the computations of effective cones alluded to above.

Acknowledgments: We are grateful to the Alfréd Rényi Institute of the Hungarian Academy of Sciences for organizing the conference at which much of this work was done.

1 Generalities

1.1 Singularities of pairs and effective cones

We work over a field of characteristic zero. Let X be a normal projective variety and D a reduced effective Weil divisor of X.

Definition 1.1 A *good* pair (X, D) consists of a smooth projective variety X, defined over a number field, and a strict normal crossings divisor D in X. This means that all irreducible components of D are smooth and intersect transversally.

Let (X, D) be a good pair and L a big line bundle on X. Denote by $\Lambda_{\mathrm{eff}}(X)$ the closed cone of effective divisors of X and by K_X the canonical class of X. Define

$$a(L, D) := \inf\{a \in \mathbb{R} \,|\, aL + (K_X + D) \in \Lambda_{\mathrm{eff}}(X)\},$$

where we identify line bundles and their classes in the Picard group of X. By definition, big line bundles are in the interior of $\Lambda_{\mathrm{eff}}(X)$ which implies that $a(L, D)$ is a positive real number. The constant $-a(L, D)$ is called the *log-Kodaira energy* of L (see [11]).

If (X, D) is not good then resolution of singularities implies the existence of a *good resolution* $\rho : (\tilde{X}, \tilde{D}) \to (X, D)$. Precisely, $(\tilde{X}, \tilde{D})$ is a good pair, ρ a birational projective morphism, and $\tilde{D}$ is the union of the exceptional divisors of ρ and the proper transform of D. We say that (X, D) is *log-canonical* if $K_X + D$ is $\mathbb{Q}$-Cartier and

$$K_{\tilde{X}} + \tilde{D} = \rho^*(K_X + D) + \sum d_j E_j,$$

where the E_j are the exceptional divisors of ρ and $d_j \geq 0$ for all j.

Example 1.2 When X is a smooth surface, (X, D) is log-canonical only when the curve D is smooth or nodal. If X is smooth of arbitrary dimension, D must have nodes in codimension one.

If L is a line bundle on X put

$$a(L, D) := a(\rho^* L, D^t),$$

where $D^t \subset \tilde{X}$ is the total transform of D. Note that everything is computed on $\tilde{X}$.

Proposition 1.3 *Let (X, D) be a log-canonical pair and assume that $X - D$ has canonical singularities. If L is a big line bundle on X then*

$$a(L, D) = \inf\{a \in \mathbb{R} \,|\, aL + (K_X + D) \in \Lambda_{\text{eff}}(X)\}.$$

In particular, $a(L, D)$ does not depend on the choice of a desingularization.

Proof. Choose a good resolution $\rho : (\tilde{X}, \tilde{D}) \to (X, D)$, so that

$$\alpha\rho^*(L) + K_{\tilde{X}} + \tilde{D} - \sum d_j E_j = \rho^*(K_X + D + \alpha L)$$

where $d_j \geq 0$, and $d_j \geq 1$ if $\rho(E_j) \not\subset D$. In particular, each exceptional divisor not contained in the total transform D^t has log discrepancy ≥ 1. Therefore, we have

$$\alpha\rho^*(L) + K_{\tilde{X}} + D^t - \sum d'_j E_j = \rho^*(K_X + D + \alpha L),$$

with each $d'_j \geq 0$. For any $\mathbb{Q}$-Cartier divisor M on X and effective divisor $\sum d'_j E_j$ supported in the exceptional locus of ρ, M is effective iff $\rho^*(M) + \sum d'_j E_j$ is effective. $\square$

Proposition 1.4 *Let (X_1, D_1) and (X_2, D_2) be log-canonical pairs, so that $X_1 - D_1$ and $X_2 - D_2$ have canonical singularities. Assume that $\pi : X_1 \to X_2$ is a finite dominant morphism so that*

$$\pi^*(K_{X_2} + D_2) = K_{X_1} + D_1.$$

Let L be a big divisor on X_2. Then $a(L, D_2) = a(\pi^*(L), D_1)$.

In fact, it suffices to assume that either (X_1, D_1) or (X_2, D_2) satisfies the singularity condition [17] 20.3.
Proof. Given a finite dominant morphism $\pi : X_1 \to X_2$ and a $\mathbb{Q}$-Cartier divisor M on X_2, M is effective iff $\pi^*(M)$ is effective. Indeed, the divisor $\pi_*\pi^*M$ is defined and equal to $\deg(\pi)M$. Combining this with Proposition 1.3 gives the result. $\square$

Remark 1.5 Let (X, D) be a log terminal pair so that $X - D$ has singularities which are *not* canonical. Then our definition of the Kodaira energy differs slightly from Fujita's [11]. In applications to integral points, we are interested in invariants of the open variety $X - D$. In Fujita's definition, on passing from (X, D) to a good resolution, any exceptional divisors over $X - D$ with negative discrepancy must be added to the boundary. This changes the open variety.

1.2 Integral points

Retain the notation from the previous section and assume that X and D are defined over a number field F. Let $\mathfrak{o}_S$ denote the ring of integers of F, where S is a finite set of nonarchimedean places of F. Fix models $\mathcal{X}$ and $\mathcal{D}$ flat and proper over the ring of integers $\mathfrak{o}_S$. A (D, S)-integral point is a point in $(\mathcal{X} - \mathcal{D})(\mathfrak{o}_S)$. In particular, if $D = \emptyset$ an integral point is the same as a rational point on X.

Let $\mathcal{L}$ be a very ample metrized line bundle on X, $U \subset X$ a Zariski open subset and $\mathcal{U}$ a model of U over $\mathfrak{o}_S$. Let S be a finite set of places in F, including the archimedean places. Denote by

$$N(\mathcal{U}, \mathcal{L}, B) := \#\{x \in \mathcal{U}(\mathfrak{o}_S) \mid H_{\mathcal{L}}(x) \leq B\}$$

the number of (D, S) integral points on U of $\mathcal{L}$-height bounded by B. A natural extrapolation of Vojta's conjecture about integral and rational points on varieties of (log)-general type ([26]) and Batyrev-Manin conjectures about rational points of bounded height on Fano varieties ([10], [1]) would be:

Conjecture 1.6 *For any $\epsilon > 0$, there exists a dense Zariski open subset $U \subset X$ such that*

$$N(\mathcal{U}, \mathcal{L}, B) \ll B^{a(L,D)+\epsilon}$$

as $B \to \infty$. If $-(K_X + D)$ is big then

$$N(\mathcal{U}, \mathcal{L}, B) \gg B^{a(L,D)-\epsilon},$$

as $B \to \infty$, at least after a suitable finite extension of F and S.

The statement is independent of the choice of S and the choice of a metrization on L.

Many precise results about asymptotics of rational and integral points are currently available (see, for example, [10], [2], [4], [20], [21], [7], [8], [9] and the references therein). As far as we know, Conjecture 1.6 is compatible with all of them. However, to actually check this compatibility one has to compute the geometric invariants of (some resolution of) the pair (X, D). In particular, one has to determine the effective cone. This can be a formidable task even for rational varieties, e.g., like the moduli space of pointed rational curves $\overline{\mathcal{M}}_{0,n}$ (see [14]).

1.3 Computing effective cones

Let X be a nonsingular projective variety, perhaps with an action by a finite group G. We review strategies for computing the G-invariant effective cone $\Lambda_{\mathrm{eff}}(X)^G$ and thus the effective cone of the quotient X/G (cf. [16]).

A curve class $[C] \in N_1(X)$ is said to be *nef* if $[C].D \geq 0$ for each $D \in \Lambda_{\mathrm{eff}}(X)$. A family of curves passing through the generic point of X is automatically nef. Indeed, consider a family $\mathcal{C} \to B$ of integral projective curves in X and an irreducible codimension-one subvariety $D \subset X$. If, for generic $b \in B$, the fiber $C_b \not\subset D$, we have $[C_b].D \geq 0$.

Fix a collection of effective divisors

$$\Gamma = \{A_1, \dots, A_m\}$$

which we expect to generate $\Lambda_{\mathrm{eff}}(X)^G$. To prove that Γ generates the (G-invariant) effective cone, it suffices to find a collection of nef (G-invariant) curve classes

$$\Xi = \{C_1, \dots, C_\ell\}$$

so that the cone generated by Γ contains the dual to the cone generated by Ξ.

In section 4, we shall use a refinement of this method (see [5], [23]). A divisor $D \in \Lambda_{\mathrm{eff}}(X)$ is *moving relative to* Γ if some multiple of D contains no element of Γ as a fixed component. Every effective divisor is a sum

$$M + \sum_{i=1}^{m} A_i d_i, \quad d_i \geq 0$$

where M is moving relative to Γ. To prove that Γ generates the effective cone, it suffices to show that M is an effective sum of the A_i.

A curve class is *nef relative to* Γ if $[C].M \geq 0$ for each M which is moving relative to Γ. Any family of curves passing through the generic point of *some* A_i is nef relative to Γ. Consequently, to show that Γ generates the effective cone, it suffices to find a collection Ξ of curve classes, nef relative to Γ, so that the cone generated by Γ contains the dual to the cone generated by Ξ.

2 Construction of resolutions

2.1 Binary forms and SL_2-orbit closures

Let V be a two-dimensional vector space with coordinates z and w, equipped with the standard SL_2-action. Let $\mathrm{Sym}^n V^*$ be the space of binary forms of degree n

$$f = x_0 z^n + x_1 z^{n-1} w + \ldots + x_n w^n.$$

It carries an induced action of SL_2 by substitution.

Associating to each form $f \neq 0$ its roots $\alpha_1, \ldots, \alpha_n$ yields a map

$$(\mathrm{Sym}^n V^* - 0) \to \mathbb{P}(V)^n / \mathfrak{S}_n$$

and an identification $\mathbb{P}(\mathrm{Sym}^n V^*) \simeq \mathbb{P}(V)^n / \mathfrak{S}_n$. The *discriminant* of a polynomial f is a homogeneous form in its coefficients $x_0, \ldots, x_n$ and defines a divisor $D \subset X = \mathbb{P}(\mathrm{Sym}^n V^*)$.

Now we may state our main result:

Theorem 2.1 (Computation of Kodaira Energy) *Let f be a generic bilinear form of degree n, $X_f \subset \mathbb{P}(\mathrm{Sym}^n V^*)$ the closure of the SL_2-orbit through f, D_f the intersection of the discriminant with X_f, and L the restriction of the standard polarization to X_f. Then we have*

$$a(L, D_f) = 2/n.$$

In particular, Conjecture 1.6 is consistent with Theorem 0.1.

To prove this, we require a resolution (i.e., a partial desingularization) of (X_f, D_f) on which we may evaluate $a(L, D_f)$ using Proposition 1.3. This resolution will be induced by a natural resolution of (X, D).

Remark 2.2 Example 1.2 shows that (X, D) is far from being log-canonical. When $n = 3$, the discriminant has cusps in codimension one: a transverse slice

$$z^3 + bzw^2 + cw^3$$

intersects the discriminant in the cuspidal curve

$$4b^3 + 27c^2 = 0.$$

Our resolution of (X, D) will be a $\mathfrak{S}_n$-quotient of a natural desingularization for $(\mathbb{P}(V)^n, \Delta)$, where Δ is the diagonal, i.e., the points lying over the discriminant. Both admit interpretations as moduli spaces of stable maps.

2.2 Moduli spaces

Fix an integer $n \geq 3$. We recall the definition and basic properties of the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable curves of genus zero with n marked points [6],[18],[19]. A stable curve is

1. a nodal connected curve C of arithmetic genus zero, i.e., a tree of $\mathbb{P}^1$'s;
2. distinct smooth points $p_1, \ldots, p_n \in C$;

subject to the stability condition that

$$K_C + p_1 + \ldots + p_n$$

is ample.

Next, we recall the definition of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ of stable maps of degree one from genus zero curves with n marked points to $\mathbb{P}^1$. This is naturally isomorphic to the Fulton-MacPherson [12] configuration space $\mathbb{P}^1[n]$ for n points in $\mathbb{P}^1$. A stable map is

1. a nodal connected curve C of arithmetic genus zero, i.e., a tree of $\mathbb{P}^1$'s;
2. distinct smooth points $p_1, \ldots, p_n \in C$;
3. a projective morphism $\mu : C \to \mathbb{P}^1$ of degree one, i.e., μ is an isomorphism on restriction to one of the irreducible components and maps the other components to points;

subject to the stability condition that

$$\mu^*\mathcal{O}(+1) + K_C + p_1 + \ldots + p_n$$

is ample.

We have the following natural maps:

1. the point map

$$\begin{array}{rcl} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) & \longrightarrow & (\mathbb{P}^1)^n; \\ (C, p_1, \ldots, p_n, \mu) & \mapsto & (\mu(p_1), \ldots \mu(p_n)); \end{array}$$

2. forgetting the point p_j

$$\begin{array}{rcl} \phi_j : \overline{\mathcal{M}}_{0,n} & \longrightarrow & \overline{\mathcal{M}}_{0,n-1} \\ (C, p_1, \ldots, p_n) & \mapsto & (C', p_1, \ldots, \hat{p}_j, \ldots, p_n); \\ \phi_j : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) & \longrightarrow & \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^1, 1) \\ (C, p_1, \ldots, p_n, \mu) & \mapsto & (C', p_1, \ldots, \hat{p}_j, \ldots, p_n, \mu'), \end{array}$$

where C' is obtained from C by 'collapsing' the irreducible components which are destabilized when p_j is removed;

3. taking projective equivalence classes

$$\begin{array}{rcl} \psi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) & \longrightarrow & \overline{\mathcal{M}}_{0,n} \\ (C, p_1, \ldots, p_n, \mu) & \mapsto & (C', p_1, \ldots, p_n) \end{array}$$

where C' is obtained from C by 'collapsing' the irreducible components which are destabilized when the polarization is removed.

Finally, we enumerate the boundary divisors of these moduli spaces. For each partition

$$\{1, \ldots, n\} = S \cup S', \quad 2 \leq |S| \leq |S'| \leq n - 2,$$

consider stable curves

$$C = (\mathbb{P}^1, p_j, j \in S) \cup (\mathbb{P}^1, p_j, j \in S'),$$

which form a divisor $\delta_{S,S'} \subset \overline{\mathcal{M}}_{0,n}$. The union of these is denoted δ. Note that the $\mathfrak{S}_n$-orbits of $\{\delta_{S,S'}\}$ correspond to the integers

$$|S| = 2, \ldots , \lfloor n/2 \rfloor.$$

For each subset

$$S \subset \{1, \ldots , n\}, \quad 2 \leq S$$

consider stable maps

$$\mu : C = (\mathbb{P}^1, p_j, j \in S) \cup (\mathbb{P}^1, p_j, j \in S') \longrightarrow \mathbb{P}^1$$

collapsing the first component and mapping the second isomorphically onto $\mathbb{P}^1$. These form a divisor $B_S \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$. The $\mathfrak{S}_n$-orbits of $\{B_S\}$ correspond to integers

$$s = |S| = 2, \ldots , n$$

and we define

$$B[s] := \sum_{|S|=s} B_S \text{ and } B := \sum_{s=2}^{n} B[s]$$

Theorem 2.3 *The moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ and $\overline{\mathcal{M}}_{0,n}$ are smooth projective algebraic varieties. Moreover, the boundary is a divisor with strict normal crossings.*

Remark 2.4 In particular, the pair $(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1), B)$ is log-canonical.

2.3 Resolution for the full moduli space

We obtain a good resolution $(\tilde{X}, \tilde{D})$ of (X, D) using the above formalism. Consider the quotient map

$$q : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \tilde{X} := \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)/\mathfrak{S}_n.$$

Let $\tilde{D}[s]$ and $\tilde{D}$ be the images of $B[s]$ and B under this map.

Proposition 2.5 *The map q is ramified only along the boundary B. At the generic points of $B[2]$ the ramification has order 2. For all $s = 3, .., n$, the map q is unramified at the generic points of $B[s]$. We have the formula*

$$q^*(K_{\tilde{X}} + \tilde{D}) = K_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1)} + B$$

and $(\tilde{X}, \tilde{D})$ is log-canonical.

Proof. The map q ramifies at points corresponding to stable maps

$$(C, p_1, ..., p_n, \mu)$$

that admit an automorphism permuting the marked points. The ramification order is the order of this automorphism group. If the images of the n points under μ are distinct then there is no automorphism of μ permuting them. This proves the first assertion. If marked points coincide there is an irreducible component $\mathbb{P}^1 \subset C$ which is collapsed by μ and which contains these points. If there are two such points this component admits an automorphism of order two exchanging the points and fixing the point of intersection with the rest of C. This proves the second assertion. If there are more than two marked points then there is generally no such automorphism. This proves the third assertion.

The ramification formula and the fact that the pair $(\tilde{X}, \tilde{D})$ is log-canonical follow from an easy local computation combined with Remark 2.4 (see Proposition 20.2 and 20.3 of [17]). □

Take $\mathfrak{S}_n$-quotients of the point map to obtain a birational map

$$\varrho \,:\, \tilde{X} \to \mathbb{P}(\mathrm{Sym}^n V^*),$$

assigning to $p_1, ..., p_n \in \mathbb{P}^1$ a polynomial vanishing at these points. The boundary divisor $\tilde{D}[2]$ is the proper transform of the discriminant D under ϱ. The boundary divisors $\tilde{D}[s]$ (for $s \geq 3$) are the exceptional divisors for ϱ.

2.4 Resolution of the generic orbit

Let $\alpha := (\alpha_1, ..., \alpha_n)$ be a set of distinct complex numbers and $f = f_\alpha$ the binary form of degree n with roots α_j. Let $C_\alpha \in \mathcal{M}_{0,n}$ be the corresponding pointed rational curve and $\mu_\alpha \in \mathcal{M}_{0,n}(\mathbb{P}^1)$ the corresponding map. The fiber

$$Y_\alpha := \psi^{-1}(C_\alpha) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$$

contains μ_α. Let $\tilde{X}_f$ be the image of Y_α under the quotient map q and $\tilde{D}_f$ its intersection with the boundary $\tilde{D}$. This coincides with the generic fiber of the map

$$\psi' : \tilde{X} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,n}/\mathfrak{S}_n.$$

The map ϱ induces a resolution

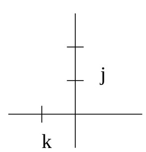
$$\varrho_f : \tilde{X}_f \to X_f,$$

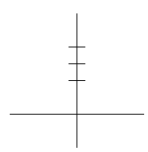
with $\varrho_f(\tilde{D}_f) = D_f$.

To describe the Y_α explicitly, we use the tower

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) & \xrightarrow{\psi} & \overline{\mathcal{M}}_{0,n} \\
\downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_n} \\
\overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^1, 1) & \xrightarrow{\psi} & \overline{\mathcal{M}}_{0,n-1} \\
\vdots & & \vdots \\
\downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_4} \\
\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1) & \xrightarrow{\psi} & \overline{\mathcal{M}}_{0,3}.
\end{array}$$

When $n = 3$, $\overline{\mathcal{M}}_{0,3}$ = point and $Y_\alpha \simeq \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$, which is isomorphic to the product $(\mathbb{P}^1)^3$ blown up along the small diagonal Δ_{small}. The boundary divisors correspond to the following stable maps

$B[2] =$ 

, $B[3] =$  .

In the above pictures the collapsed components are represented by vertical lines. Note that the normal bundle

$$\mathcal{N}_\Delta = \mathcal{O}(+2) \oplus \mathcal{O}(+2),$$

so that the exceptional divisor $E = B[3] \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let

$$\pi_1 : E \to \mathbb{P}^1$$

be the projection to the cross ratio of the marked points and the node and

$$\pi_2 : E \to \mathbb{P}^1$$

the projection onto the image of the collapsed curve.

The divisor $B[2]$ is the proper transform of Δ, the large diagonal.

For the arbitrary degree case, we analyze the failure of the block squares in the tower to be fiber products. Given a generic

$$C_\alpha = (\mathbb{P}^1, \alpha_1, \dots, \alpha_n) \in \mathcal{M}_{0,n}, \quad \alpha_i \neq \alpha_j,$$

we compare the fibers

$$Y_{\alpha_1,\dots,\alpha_n} = \psi^{-1}(C_\alpha) \text{ and } Y_{\alpha_1,\dots,\alpha_{n-1}} = \psi^{-1}(\phi_n(C_\alpha)) = \psi^{-1}(\mathbb{P}^1, \alpha_1, \dots, \alpha_{n-1})$$

using the forgetting map

$$\phi_n : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^1, 1).$$

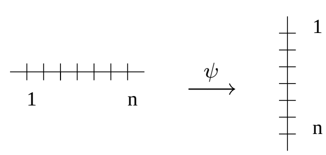
Given a stable map

$$(C, \alpha_1, \dots, \alpha_n, \mu) \in \psi^{-1}(C_\alpha),$$

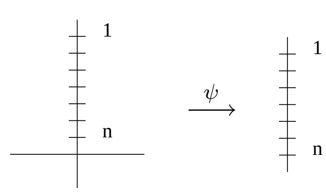
there are three cases to consider:

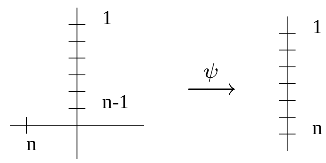
1. $C = \mathbb{P}^1$;
2. $C = \mathbb{P}^1 \cup \mathbb{P}^1$ with the collapsed component containing $\alpha_1, ..., \alpha_n$;
3. $C = \mathbb{P}^1 \cup \mathbb{P}^1$ with the collapsed component containing $\alpha_1, ..., \alpha_{n-1}$ but not α_n.

Case 1:

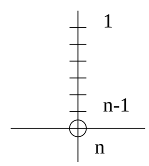


Case 2:



Case 3: 

Over the open subset of $Y_{\alpha_1,\ldots,\alpha_n}$ corresponding to the first two cases, ϕ_n induces an isomorphism between $Y_{\alpha_1,\ldots,\alpha_n}$ and $Y_{\alpha_1,\ldots,\alpha_{n-1}}$. In the third case, we forget the image of the n-th marked point. The map ϕ_n blows up the locus in $Y_{\alpha_1,\ldots,\alpha_{n-1}}$ where $\alpha_1, \ldots, \alpha_{n-1}$ are on the collapsed component and α_n coincides with the node (of attachment). This is a curve isomorphic to $\mathbb{P}^1 \subset (B[n-1] \cap Y_{\alpha_1,\ldots,\alpha_{n-1}})$: The generic map takes the form:



We summarize the above discussion in the following

Proposition 2.6 *Let $\alpha_1, \ldots \alpha_n$ be distinct complex numbers. The forgetting maps induce a sequence of birational morphisms*

$$Y_{\alpha_1,\ldots,\alpha_n} \xrightarrow{\phi_n} Y_{\alpha_1,\ldots,\alpha_{n-1}} \ldots \xrightarrow{\phi_4} Y_{\alpha_1,\alpha_2,\alpha_3} \simeq \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1).$$

The moduli space of stable maps $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$ is isomorphic to $(\mathbb{P}^1)^3$ blown up along the small diagonal with exceptional divisor $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The map ϕ_j blows up the proper transform of $\pi_1^{-1}(\alpha_j)$. In particular, $Y_{\alpha_1,\ldots,\alpha_n}$ is smooth and its boundary has strict normal crossings, contained in $B[n-1] \cup B[n]$.

Remark 2.7 We are blowing up along disjoint curves, so the order of the blow-up does not matter.

Proposition 2.8 *Let f be a generic binary form of degree $n \geq 3$ with roots $\alpha_1, \ldots, \alpha_n$. Then the restriction of q to X_f is ramified only along the boundary $B \cap X_f$. At generic points of $(B[n] \cup B[n-1]) \cap Y_\alpha$, the restriction of q is unramified. We have the formula*

$$q^*(K_{\tilde{X}_f} + \tilde{D}_f) = K_{Y_\alpha} + [Y_\alpha \cap B]$$

and $(\tilde{X}_f, \tilde{D}_f)$ is log-canonical.

Proof. The argument is similar to the one in Proposition 2.5. □

3 Verification of exponents

3.1 Explicit basis of $\mathrm{Pic}(Y_\alpha)$

Write

$$\mathrm{Pic}((\mathbb{P}^1)^3) = \mathbb{Z}g_1 + \mathbb{Z}g_2 + \mathbb{Z}g_3, \quad g_i = \mathrm{pr}_i^*(c_1(\mathcal{O}_{\mathbb{P}^1}(+1))),$$

with large diagonals

$$\Delta_{ij} = g_i + g_j - E,\ B[2] = 2(g_1 + g_2 + g_3) - 3E.$$

By Proposition 2.6, Y_α is obtained by blowing up the $(n-3)$ sections of

$$\pi_2 : B[3] \to \mathbb{P}^1.$$

Let $F_4, ..., F_n$ denote the corresponding exceptional divisors and identify E and its proper transform. Relabel

$$\begin{aligned} F_k &= \Delta_{ij},\ \{i,j,k\} = [1,2,3] \\ &= g_i + g_j - E - F_4 - \cdots - F_n, \end{aligned}$$

so that $\mathfrak{S}_n$ acts on the F_k, $k = 1, ..., n$, in the obvious way. Note that E and the F_k generate $\mathrm{Pic}(Y_\alpha)$.

Proposition 3.1 *The $\mathfrak{S}_n$-stable boundary divisors*

$$\begin{aligned} A[n-1] &= F_1 + \cdots + F_n \\ A[n] &= E, \end{aligned}$$

generate the $\mathfrak{S}_n$-invariant Picard group of Y_α, and $A[j] = B[j] \cap Y_\alpha$. The canonical class of Y_α is

$$\begin{aligned} K &= -2(g_1 + g_2 + g_3) + E + 2(F_4 + \cdots + F_n) \\ &= -A[n-1] - 2A[n] \end{aligned}$$

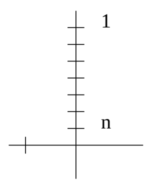
3.2 Computation of the effective cone

Lemma 3.2 *The $\mathfrak{S}_n$-invariant effective cone of Y_α is generated by the classes $A[n]$ and $A[n-1]$.*

Proof. We apply the method of §1.3. The class $A[n]$ is exceptional and thus a generator of the effective cone. To show that $A[n-1]$ is the second generator, we exhibit a nef curve not intersecting $A[n-1]$. Consider the $\mathbb{G}_m$-action on $\mathbb{P}^1$:

$$\rho_t \,:\, (z,w) \mapsto (tz, w).$$

We may assume that the points $\alpha_1, ..., \alpha_n$ are not contained in the fixed point locus $\mathrm{Fix}(\rho_t)$. Any singular element in the orbit closure is:



where the point of attachment is 0 (or ∞) and the other labelled point is ∞ (resp. 0). This is disjoint from $A[n-1]$. □

3.3 Proof of Theorem 2.1

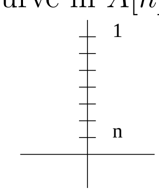
Proof. The Kodaira energy for (X_f, D_f) can be computed on $(\tilde{X}_f, \tilde{D}_f)$, by Propositions 1.3 and 2.8. By Propositions 1.4 and 2.6, it suffices to compute the Kodaira energy for $(Y_\alpha, A[n] + A[n-1])$. Recall there is a composed morphism

$$\beta : Y_\alpha \xrightarrow{q} X_f \hookrightarrow \mathbb{P}(\mathrm{Sym}^n V^*) \simeq \mathbb{P}^n.$$

Lemma 3.3 *The pull-back of the hyperplane class takes the form*

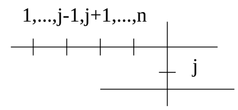
$$L =: [\beta^* \mathcal{O}_{\mathbb{P}^n}(+1)] = \frac{1}{2}\left((n-2)A[n-1] + nA[n]\right).$$

Proof. Let R be the class of a curve in $A[n]$ corresponding to



with varying point of attachment on the *collapsed* component. This is the proper transform of the generic fiber of the map $\pi_1 : E \to \mathbb{P}^1$. Then

$$A[n-1]|_R = n = \# \left\{ \quad , j = 1, \dots, n \right\}$$



$$A[n]|_R = -1-(n-3) = 2-n.$$

For the second intersection number, note that $A[n] = E$ and apply the blow-up description of Proposition 2.6. In $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$ we have

$$E = \mathbb{P}(\mathcal{N}_{\Delta_{\mathrm{small}}}) = \mathbb{P}^1 \times \mathbb{P}^1$$

and $\mathcal{N}_E = \mathcal{O}(-1)$. After blowing up $(n-3)$ further sections of

$$E \to \Delta_{\mathrm{small}}$$

the normal bundle is reduced to $\mathcal{O}(-1-(n-3))$.

We know that $A[n] = B[n] \cap Y_\alpha$ is collapsed by the map β, so

$$\beta^* \mathcal{O}(1) = c\left((n-2)A[n-1] + nA[n]\right)$$

for some $c \in \mathbb{N}$. Since

$$\begin{aligned} &(n-2)A[n-1] + nA[n] \\ &\qquad = 2\left((n-2)(g_1+g_2+g_3) - (n-3)E - (n-2)(F_4 + \cdots + F_n)\right). \end{aligned}$$

the claim follows. □

We have

$$\begin{aligned} K_{Y_\alpha} + [Y_\alpha \cap B] &= K_{Y_\alpha} + A[n] + A[n-1] = -A[n] \\ K_{Y_\alpha} + [Y_\alpha \cap B] + \alpha L &= \alpha \frac{n-2}{2} A[n-1] + (\alpha \frac{n}{2} - 1) A[n], \end{aligned}$$

and by definition

$$a(\beta^* L) := \inf\{\alpha \,|\, \alpha \beta^* L + K_{Y_\alpha} + [Y_\alpha \cap B] \in \Lambda_{\mathrm{eff}}(Y_\alpha)\}.$$

Hence Lemma 3.2 yields

$$a(L, [Y_\alpha \cap B]) = 2/n.$$

Thus $a(L, D_f) = 2/n$, as expected! □

4 The $\mathfrak{S}_n$-invariant effective cone of the full moduli space

In this section, we compute the $\mathfrak{S}_n$-invariant part of the effective cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$, its canonical class, and the Kodaira energy of the line bundle $L := \beta^*\mathcal{O}_{\mathbb{P}^n}(+1)$, where

$$\beta = \rho \circ q \,:\, \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \xrightarrow{q} \tilde{X} \xrightarrow{\rho} \mathbb{P}^n.$$

We will also compute the Kodaira energy of $H := \rho^*\mathcal{O}_{\mathbb{P}^n}(+1)$.

We first recall some basic facts about $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \simeq \mathbb{P}^1[n]$, following [12]. In addition to the divisor classes B_S introduced above, we shall also consider

$$L_a := \{(C, p_1, \dots , p_n, \mu) \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) : \mu(p_a) = 0 \in \mathbb{P}^1\}, \quad a = 1, \dots , n.$$

The cohomology $H^*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1))$ is generated by the classes L_a and B_S , subject to the relations

1. $L_a^2 = 0$;
2. $B_S \cdot B_{S'} = 0$ for $S \cap S' \neq \emptyset$;
3. $(L_a - L_{a'})B_S = 0$ for $a, a' \in S$;
4. $(\sum_{S \supset \{a,a'\}} B_S) = L_a + L_{a'}$, for $1 \leq a < a' \leq n$.

The generators of the $\mathfrak{S}_n$-invariant subspace are

$$L := \sum_{a=1}^{n} L_a, \quad B[s] = \sum_{|S|=s} B_S, \quad 2 \leq s \leq n.$$

After averaging over $\mathfrak{S}_n$

$$(n-1)L = \sum_{s=2}^{n} \frac{s(s-1)}{2} B[s]. \tag{1}$$

Theorem 4.1 *The classes $D[2], ..., D[n]$ generate the effective cone of Y. The classes $B[2], ..., B[n]$ generate the $\mathfrak{S}_n$-invariant effective cone of the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$.*

Proof. We implement the strategy of §1.3 with

$$\Gamma = \{B[2], \dots, B[n]\}.$$

This entails finding curve classes that are nef relative to Γ. Let

$$M = \sum_{j=2}^{n} d_j B[j]$$

denote an $\mathfrak{S}_n$-invariant divisor class with no boundary divisors as fixed components.

Recall the description of the boundary divisor B_S:

$$\begin{array}{rcll} B_S & \simeq & \overline{\mathcal{M}}_{0,n+1-s}(\mathbb{P}^1,1) \times \overline{\mathcal{M}}_{0,s+1}, & s = |S| > 2, \\ & \simeq & \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^1,1), & s = 2. \end{array}$$

Take $s \geq 3$ and let $C_s \subset B_S$ be the class of the generic fiber of the map

$$\overline{\mathcal{M}}_{0,n+1-s}(\mathbb{P}^1,1) \times \overline{\mathcal{M}}_{0,s+1} \to \overline{\mathcal{M}}_{0,n+1-s}(\mathbb{P}^1,1) \times \overline{\mathcal{M}}_{0,s}$$

forgetting the attaching point. Since C_s passes through the generic point of B_S, averaging C_s over $\mathfrak{S}_n$ yields a curve class which is nef relative to Γ. In particular, for each $\mathfrak{S}_n$-invariant divisor $M = \sum_{j=2}^{n} d_j B[j]$, moving relative to Γ, we have $C_s \cdot M \geq 0$.

We compute intersections of C_s with the various elements of Γ. First, the map β blows down the divisors B_S for $|S| \neq 2$; the data of the collapsed component is lost completely. It follows that $L \cdot C_s = 0$. A simple combinatorial analysis gives

$$C_s \cdot B_T = \begin{cases} 1 & \text{if } T = S - \{\sigma\}; \\ 0 & \text{otherwise, unless } T = S. \end{cases}$$

which means that $C_s \cdot B[s-1] = s$. Relation 1 gives

$$0 = B_S \cdot C_s \frac{s(s-1)}{2} + s\frac{(s-1)(s-2)}{2}$$

so $B_S \cdot C_s = -(s-2)$. To summarize, we have

$$C_s \cdot B[j] = \begin{cases} s & \text{if } j = s-1; \\ -(s-2) & \text{if } j = s; \\ 0 & \text{otherwise.} \end{cases}$$

Using this information, we extract inequalities on the coefficients of M. The condition $M \cdot C_s \geq 0$ yields

$$sd_{s-1} \geq (s-2)d_s,$$

so we get a chain of inequalities:

$$d_n \leq \frac{n}{n-2}d_{n-1} \leq \frac{n(n-1)}{(n-2)(n-3)}d_{n-2} \leq \ldots \leq \frac{n(n-1)}{2}d_2. \tag{2}$$

If some $d_s < 0$ then $d_j < 0$ for each $j \geq s$.

We consider another curve class in B_S to get inequalities in the reverse direction. Fix $s \geq 2$ and let R_s denote the class of the generic fiber of

$$\overline{\mathcal{M}}_{0,n+1-s}(\mathbb{P}^1, 1) \times \overline{\mathcal{M}}_{0,s+1} \to \overline{\mathcal{M}}_{0,n-s}(\mathbb{P}^1, 1) \times \overline{\mathcal{M}}_{0,s+1}$$

induced by forgetting τ, one of the $n+1-s$ points not contained in S. Again, R_s passes through the generic point of B_S, so averaging over $\mathfrak{S}_n$ yields a curve class such that $R_s \cdot M \geq 0$.

We compute intersections as before. The map β sends R_s to a line in $\mathbb{P}^n$, i.e., the linear forms with $n-1$ fixed roots and one varying root. It follows that $L \cdot R_s = 1$. The line R_s intersects B_T properly in the following cases

$$R_s \cdot B_T = \begin{cases} 1 & \text{if } T = S \cup \{\tau\}; \\ 1 & \text{if } T = \{\tau, \upsilon\}, \upsilon \notin S; \\ 0 & \text{otherwise, unless } T = S. \end{cases}$$

Summing over $\mathfrak{S}_n$-orbits gives

$$R_s \cdot B[j] = \begin{cases} 1 & \text{if } j = s+1; \\ n-s-1 & \text{if } j = 2; \\ 0 & \text{otherwise, unless } j = s. \end{cases}$$

Applying Relation 1, we find

$$(n-1) = s(s-1)/2R_s \cdot B[s] + (s+1)s/2 + (n-s-1),$$

so $R_S \cdot B_S = R_s \cdot B[s] = -1$.

We extract the inequalities

$$d_{s+1} - d_s \geq (n-s-1)d_2;$$

in particular, if $d_2 > 0$ then each $d_j > 0$. Adding together the inequalities

$$\begin{array}{rcl} d_n - d_{n-1} & \geq & 0 \\ d_{n-1} - d_{n-2} & \geq & d_2 \\ & \cdots & \\ d_4 - d_3 & \geq & (n-4)d_2 \\ d_3 & \geq & (n-2)d_2. \end{array}$$

gives

$$d_n \geq \frac{n^2-5n+8}{2} d_2.$$

Combining with inequality 2, we obtain

$$(n^2-1)d_2 \geq (n^2-5n+8)d_2,$$

hence $d_2 > 0$. □

Theorem 4.2 *On the moduli space of stable maps* $(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1), B)$ *and its* $\mathfrak{S}_n$*-quotient* (Y, D)*, we have*

$$a(B, L) = a(D, H) = 2/n.$$

Proof. We proceed to calculate the canonical class:

$$K_{\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1)} = -2L + \sum_{s=3}^{n} B[s](s-2)$$

This follows from the explicit blowup realization of the Fulton-MacPherson configuration space $\mathbb{P}^1[n] = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1)$ [12]. The exceptional divisors B_S (for $s = |S| \geq 3$) arise from blowing up centers in codimension $s-1$.

We compute the log-Kodaira energy of L on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1)$ with respect to the boundary B. We have

$$\begin{array}{rcl} K + B + aL & = & (a-2)L + \sum_{s=2}^{n}(s-1)B[s] \\ & = & (a-2)\left(\sum_{s=2}^{n} \frac{s(s-1)}{2(n-1)} B[s]\right) + \sum_{s=2}^{n}(s-1)B[s] \\ & = & \sum_{s=2}^{n}(s-1)\left(\frac{s(a-2)}{2(n-1)} + 1\right) B[s] \end{array}$$

which is effective if and only if

$$a \geq \frac{2(s-n+1)}{s}, \quad s = 2, \dots, n.$$

The most restrictive inequality occurs when $s = n$, where we obtain $a \geq 2/n$. Consequently, $a(L, B) = 2/n$, as expected.

The Kodaira energy for (X, D) can be computed on $(\tilde{X}, \tilde{D})$, by Propositions 1.3 and 2.5. By Proposition 1.4 and Theorem 2.3, it is equal to the Kodaira energy for $(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1), B)$. □

5 Final remarks

A) The orbit closure X_f depends on the form $[f]$. We get equivariant compactifications of PGL_2 depending on moduli. This dependence is made abundantly clear in the blow-up description of Proposition 2.6.

B) The pair $(\overline{\mathcal{M}}_{0,n}, \delta)$ is of log general type: $K_{\overline{\mathcal{M}}_{0,n}} + \delta$ is ample and log canonical (see, for example, §7.1 of [13]). The map

$$\psi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,n}$$

is a log-Fano fibration onto a log-variety of general type (or the point, when $n = 3$).

C) The pair $(\overline{\mathcal{M}}_{0,n}, \delta)$ satisfies Vojta's conjecture. We realize $\mathcal{M}_{0,n}$ as an open subset of an algebraic torus with explicit complement. Fix $n-1$ points in $\mathbb{P}^{n-3}$ in general position. Consider the set $\mathcal{H}$ of $\frac{1}{2}(n-1)(n-2)$ hyperplanes spanned by $n-3$ of the fixed points. Kapranov [15] has shown that $\mathcal{M}_{0,n} \simeq \mathbb{P}^{n-3} - \cup_{H \in \mathcal{H}} H$. The torus is obtained by excising the $n-2$ hyperplanes spanned by subsets of the first $n-2$ of the points.

D) We therefore expect that the asymptotic behavior of integral points of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ with respect to the boundary is obtained by summing the contributions of the integral points on the fibers of ψ. This explains why the Kodaira energies for the moduli space of stable maps (Theorem 4.2) and the fibers of ψ (Theorem 2.1) should coincide (for the case of rational points, see [2]).

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