RATIONAL POINTS ON COMPACTIFICATIONS OF SEMI-SIMPLE GROUPS OF RANK 1

by

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ABSTRACT. — We explain our approach to the problem of counting rational points of bounded height on equivariant compactifications of semi-simple groups.

Contents

1. Introduction

Let G be a linear algebraic group, $S \subset G$ a subgroup and $S\backslash G$ the corresponding homogeneous space, all assumed to be defined over a number

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The second author was partially supported by the Clay Mathematics Institute.
The third author was partially supported by NSF grant 0100277.

field F. We consider G-equivariant embeddings

$$S\backslash G \hookrightarrow \mathbb{P}^n.$$

Such embeddings arise from a choice of an F-rational projective representation $\varrho : G \to \mathrm{PGL}_{n+1}$, together with a point $p_0 \in \mathbb{P}^n(F)$ with stabilizer S.

The standard height on F-rational points of $\mathbb{P}^n$ is defined by

$$\begin{array}{rcrcl} H : \mathbb{P}^n(F) & \to & \mathbb{R}_{>0}, & & \\ x = (x_0, ..., x_n) & \mapsto & H(x) & := & \prod_v H_v(x), \\ & & H_v(x) & := & \max_j(|x_j|_v), \end{array}$$

the product over all valuations v of F. More generally, one considers heights whose local factors coincide with the above at almost all v and differ from the above by a globally bounded function at the remaining v. For example, one could choose $H_v(x) = (\sum_j |x_j|_v^2)^{1/2}$ at a real v.

We are interested in the asymptotic distribution of the number $N(\varrho, B)$ of F-rational points on $H\backslash G$ of height $\leq B$, as $B \to \infty$. In many cases, one finds

$$N(\varrho, B) \sim \mathsf{c} \cdot B^{\mathsf{a}} \log(B)^{\mathsf{b}-1}, \tag{1.1}$$

with $\mathsf{a} \in \mathbb{Q}, \mathsf{b} \in \mathbb{N}$ and $\mathsf{c} \in \mathbb{R}_{>0}$ (see [7],[2],[17], [4]). There is a conceptual interpretation of these constants in terms of global geometric and arithmetic invariants of the associated algebraic variety X, the closure of the G-orbit through p_0 (see [7],[1], [13] and [3]).

The proofs of the above asymptotics rely on harmonic analysis on corresponding adelic groups. Note that results of type (1.1) are quite nontrivial even when unitary representations of the adeles $G(\mathbb{A}_F)$ are well-understood. For example, we still don't know how to treat general equivariant compactifications of the Heisenberg group (the case of bi-equivariant compactifications is considered in [16]). For semi-simple groups we need to appeal to rather nontrivial results from the theory of automorphic forms: multiplicity one, uniform bounds of matrix coefficients, Eisenstein series, spectral theory etc.

We now give an outline of our approach in a special case. Let G be a split semi-simple group of adjoint type over $\mathbb{Q}$. Consider the Cartan decomposition $G(\mathbb{A}) = KA^+K$, where $K = \prod_v K_v$ is a maximal compact subgroup and $A^+ = \prod_v A_v^+$ (and the product is over all valuations v of $\mathbb{Q}$).

Here A_p^+ (resp. A_∞^+) can be identified (via the logarithmic map) with the monoid $\mathfrak{a}^+$ (resp. cone $\mathfrak{a}_\infty^+$) in the Lie algebra $\mathfrak{a}$ (resp. $\mathfrak{a}_\infty := \mathfrak{a}\otimes\mathbb{R}$), dual to the monoid (resp. cone) spanned by simple roots of the corresponding maximal torus A. Fix a triangulation Σ of $\mathfrak{a}^+$ (resp. $\mathfrak{a}_\infty^+$ for archimedean v) into simplicial subcones σ and let λ be a continuous $\mathbb{R}$-valued function on $\mathfrak{a}$ (resp. $\mathfrak{a}_\infty$) which is *linear* on every cone $\sigma \in \Sigma$. For example, we may take

$$\lambda(a_v) = \langle \mathbf{s}, \overline{a}_v \rangle, \tag{1.2}$$

where $\mathbf{s} \in \mathfrak{a}_v^* \otimes \mathbb{R}$ and $\overline{a}_v = \log(a_v) \in \mathfrak{a}_v$, $a_v \in A(\mathbb{Q}_v)$. Put $q_v = p$ for $v = p$, $q_v = e$ for $v = \infty$ and define

$$H_v(\lambda, g_v) = q_v^{\lambda(a_v)} \quad \text{and} \quad H(\lambda, g) := \prod_v H_v(\lambda, g_v),$$

where $g = (g_v) \in G(\mathbb{A}_\mathbb{Q})$, $g_v = k_v a_v k_v'$ with $k_v, k_v' \in K_v$, $a_v \in A_v^+$.

Problem 1.1. — Study the analytic properties of the *zeta function*

$$\mathcal{Z}(\lambda, s, g) := \sum_{\gamma\in G(F)} H(\lambda, \gamma g)^{-s}. \tag{1.3}$$

For λ chosen as in (1.2), the zeta function (1.3) encodes information about the distribution of rational points of bounded height on "wonderful" compactifications of G studied by de Concini and Procesi in [5]. The study of arbitrary bi-equivariant compactifications of G can be reduced to other λ.

The main goal of this paper is to explain in detail how our approach works in the simplest case: $\mathbb{P}^3$ considered as the wonderful compactification of PGL_2 over $\mathbb{Q}$. The counting problem itself is trivial, but it allows us to focus on the method, which covers (verbatim) wonderful compactifications of rank one semi-simple groups of adjoint type and highlights the technical difficulties one faces for groups of higher rank. Compactifications of anisotropic forms of semi-simple groups of adjoint type are treated in [15].

2. Basic definitions and results

Notation . —

- $\mathcal{V} = \mathcal{V}_{\mathbb{Q}} := \{2, 3, 5, \ldots, p, \ldots, \infty\}$ - valuations of $\mathbb{Q}$;
- $G = PGL_2$, A the (diagonal) torus, N upper triangular unipotent matrices, $P = NA$ the Borel subgroup;
- $\mathfrak{a} \simeq \mathbb{Z}$ - Lie algebra of A, $\mathfrak{a}^*$ its character lattice;
- $K_p = G(\mathbb{Z}_p)$, $K_f = \prod_p K_p$, $K_\infty = SO(2)$ and $K = K_\infty \times K_f$;
- for $v \in \mathcal{V}$, let $G(\mathbb{Q}_v) = N_v A_v K_v$ and $K_v A_v^+ K_v$ be the Iwasawa, resp. Cartan decompositions;
-
$$\mathfrak{S} = \mathfrak{S}_{\mathsf{c}} := \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \Big| a \geq \mathsf{c} > 0 \right\} \subset G(\mathbb{R}) \hookrightarrow G(\mathbb{A})$$
a Siegel domain, $G(\mathbb{A}) = G(\mathbb{Q}) \cdot \mathfrak{S} \cdot \Omega$, for some compact $\Omega \subset G(\mathbb{A})$;
- $dg = \prod_v dg_v = dn\, da\, dk$ normalized Haar measure,
$$\int_{K_v} dk_v = 1, \quad \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} dn = 1;$$
- $\mathrm{vol}_p(\ell) := \mathrm{vol}(K_p a_p^\ell K_p)$, where $a_p^\ell = \begin{pmatrix} 1 & 0 \\ 0 & p^\ell \end{pmatrix}$;
- $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$;
- $\Delta \in \mathfrak{U}$ - the standard Casimir element (Laplacian);
- $\mathsf{T}_\delta = \{s \in \mathbb{C} \,|\, \Re(s) > 4 + \delta\}$;
- $\mathsf{L}^2 := \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, unless noted otherwise, $\|\cdot\|_2$ the L^2-norm.

Represent an element in $A(\mathbb{Q}_v)$ in the form
$$\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix},$$
(with $a_v \in \mathbb{Q}_v$) and define two local height functions:
$$\begin{array}{rrcl} H_v : & g_v = k_v a_v k_v' & \mapsto & |a_v|_v^{1/2} \\ \chi_{v,P} : & g_v = n_v a_v k_v & \mapsto & |a_v|_v \end{array} \tag{2.1}$$
Define the global heights by
$$H := \prod_v H_v \quad \text{and} \quad \chi_P := \prod_v \chi_{v,P} \tag{2.2}$$

Remark 2.1. — For $\gamma \in P(\mathbb{Q})$ we have $\chi_P(\gamma g) = \chi_P(g)$ (by the product formula) and χ_P^{-1} is the usual height on $\mathbb{P}^1(\mathbb{Q}) = P(\mathbb{Q})\backslash G(\mathbb{Q})$.

Remark 2.2. — The group G has a (canonical) compactification

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \hookrightarrow \mathbb{P}(\mathrm{End}(V)) = \mathbb{P}^3 = \{(a : b : c : d)\}$$

which is equivariant for the action of G on both sides. A standard height on $\mathbb{P}^3(\mathbb{Q})$ is

$$\sqrt{a^2 + b^2 + c^2 + d^2} \cdot \prod_p \max(|a|_p, |b|_p, |c|_p, |d|_p).$$

Its local factors are identical with (2.1).

The main object of interest is the *height zeta function*

$$\mathcal{Z}(s, g) := \sum_{\gamma \in G(\mathbb{Q})} H(\gamma g)^{-s}. \tag{2.3}$$

The convergence of the series in the domain $\Re(s) \gg 0$ (for fixed g) is a special case of a general fact: let X be any projective algebraic variety over a number field F, and H any height induced from a projective embedding of X. Then the height zeta function

$$\mathcal{Z}(H, s) = \sum_{x \in X(F)} H(x)^{-s}$$

converges absolutely and uniformly on compacts in the domain $\Re(s) \gg 0$.

Proposition 2.3. — *There exists a $\sigma > 0$ such that the series*

$$\sum_{\gamma \in G(\mathbb{Q})} H(\gamma g)^{-s}$$

converges absolutely and uniformly on compacts in the domain $\mathsf{T}_\sigma \times G(\mathbb{A})$ to a function $\mathcal{Z}(s, g)$ which

(1) *is continuous in g;*
(2) *is bounded on $G(\mathbb{Q}) \backslash G(\mathbb{A})$;*
(3) *has bounded Δ-derivatives.*

In particular, in this domain, $\mathcal{Z}(s, g)$ and all its Δ-derivatives are in L^2.

Proof. — By reduction theory, $G(\mathbb{A}) = G(\mathbb{Q}) \cdot \mathfrak{S} \cdot \Omega$, where $\Omega \subset G(\mathbb{A})$ is some compact and $\mathfrak{S}$ a Siegel domain. Now we use the following easy property of the height: there exist constants $\mathsf{c}, \mathsf{r} > 0$ such that

– $H(\gamma a \omega) \geq \mathsf{c} H(\gamma)^{\mathsf{r}}$ for all $a \in \mathfrak{S}, \omega \in \Omega$ and $\gamma \in G(\mathbb{Q})$.

In particular, there exists an $\mathsf{r} > 0$ such that for all $g \in G(\mathbb{A})$ one has

$$\mathcal{Z}(\Re(s), g) \leq \mathcal{Z}(\mathsf{r}\Re(s), e) < \infty \quad \text{for} \quad \Re(s) \gg 0.$$

Since Δ commutes with the K-action, it suffices to prove (3) on matrices in A_∞^+. Explicit formulas for the height and for Δ give the result. □

A solution of Problem 1.1 in our special case is given by

Theorem 2.4. — *There exists an $\epsilon > 0$ such that $\mathcal{Z}(s, e)$ admits a meromorphic continuation to $\mathsf{T}_{-\epsilon}$ with a unique simple pole at $s = 4$.*

In the analysis of $\mathcal{Z}(s, g)$ we use the Eisenstein series:

$$E(s, g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(s, \gamma g), \tag{2.4}$$

where $\chi(s, g) := \chi_P(g)^{s+1/2}$. The idea of the proof of Theorem 2.4 is to first establish an identity of continuous L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$

$$\mathcal{Z}(s, g) = \mathcal{Z}^{\text{res}}(s) + \mathcal{Z}^{\text{cusp}}(s, g) + \mathcal{Z}^{\text{eis}}(s, g), \quad \text{for} \quad \Re(s) \gg 0, \tag{2.5}$$

where

$$\mathcal{Z}^{\text{res}}(s) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mathcal{Z}(s, g)\, dg = \int_{G(\mathbb{A})} H(g)^{-s}\, dg \tag{2.6}$$

is the contribution from the trivial representation, $\mathcal{Z}^{\text{cusp}}(s, g)$ is the projection of $\mathcal{Z}(s, g)$ onto the cuspidal spectrum and

$$\mathcal{Z}^{\text{eis}}(s, g) := \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mathcal{Z}(s, g) \overline{E(it, g)} dg \right) E(it, g) dt \tag{2.7}$$

is the projection onto the continuous spectrum. This is accomplished in Propositions 5.1 and 7.1. Then we use (2.5) to meromorphically continue $\mathcal{Z}(s, e)$ (see Propositions 3.4, 7.6 and 5.1). Finally, a Tauberian theorem implies

Corollary 2.5. — *There is a constant $\mathsf{c} > 0$, such that*

$$\#\{\gamma \in G(\mathbb{Q}) \,|\, H(\gamma) \leq B\} = \mathsf{c}B^4(1 + o(1)), \quad \textit{as } B \to \infty.$$

3. Heights and height integrals

Let $\varphi_p = \varphi_p(\chi_p)$ be a bi-K_p-invariant function on $G(\mathbb{Q}_p)$ such that

$$\varphi_p(a_p^\ell) = \frac{p^{-\frac{\ell}{2}}}{1+p^{-1}}\left(\frac{1-\bar{\chi}_p(p)^2/p}{1-\bar{\chi}_p(p)^2}\chi_p(p^\ell) + \frac{1-\chi_p(p)^2/p}{1-\chi_p(p)^2}\bar{\chi}_p(p^\ell)\right), \tag{3.1}$$

for $\ell \geq 1$, where χ_p is a nontrivial unramified quasi-character of $\mathbb{Q}_p^*$,

$$\chi_p(p) = \bar{\chi}_p(p)^{-1} = p^{\alpha_p}, \text{ with parameter } \alpha_p \in \mathbb{C}^*. \tag{3.2}$$

We write also $\varphi_p = \varphi_p(s,\cdot)$, if $\alpha_p = s$ for all p. Define

$$I_v(s) := \int_{G(\mathbb{Q}_v)} H_v(g_v)^{-s}\, dg_v, \quad I_f(s) := \prod_p I_p(s) \tag{3.3}$$

and, for $\varphi_p = \varphi_p(\chi_p)$ and $\varphi = \prod_p \varphi_p$,

$$I_p(s, \varphi_p(\chi_p)) := \int_{G(\mathbb{Q}_v)} \varphi_p(g_p) H_p(g_p)^{-s}\, dg_p, \quad I_f(s,\varphi) := \prod_p I_p(s,\varphi).$$

LEMMA 3.1. — *The functions $I_p(s)$ are holomorphic in T_{-2}. Moreover, $I_f(s)$ is holomorphic in T_0 and admits a meromorphic continuation to T_{-2} with an isolated simple pole at $s = 4$.*

Proof. — We have

$$\mathrm{vol}_p(\ell) = \begin{cases} p^\ell(1+p^{-1}) & \text{if } \ell > 0, \\ 1 & \text{if } \ell = 0 \end{cases} \tag{3.4}$$

so that $I_p(s)$ is given by

$$1 + (1+p^{-1})\sum_{\ell\geq 1} p^{-\frac{\ell s}{2}} p^\ell = (1-p^{-(\frac{s}{2}-1)})^{-1}(1+p^{-\frac{s}{2}}) = \zeta_p(\frac{s}{2}-1)(1+p^{-\frac{s}{2}}),$$

where ζ_p is the local factor of the Riemann zeta function ζ (the sum converges absolutely and uniformly on compacts in T_{-2}). The Euler product $\prod_p(1+p^{-\frac{s}{2}})$ converges (uniformly on compacts in T_{-2}) to a holomorphic function. It suffices to recall the analytic properties of ζ. □

LEMMA 3.2. — *Assume that for all p, $|\Re(\alpha_p)| < \mathsf{r}$. Then $I_p(s,\varphi_p)$ is holomorphic for $\Re(s) > 2\mathsf{r}+1$. Moreover, for all $\epsilon > 0$ there exists a constant $\mathsf{c} = \mathsf{c}(\epsilon)$ such that $I_f(s,\varphi)$ is holomorphic and*

$$|I_f(s,\varphi)| \le \mathsf{c}, \quad \textit{for} \quad \Re(s) > 2\mathsf{r}+3+\epsilon.$$

Proof. — Combining (3.1) with (3.4) we have

$$I_p(s,\varphi_p) = 1 + \frac{1-\bar{\chi}_p^2(p)/p}{1-\bar{\chi}_p^2(p)} \sum_{\ell>0} p^{-\ell\frac{s-1}{2}} \chi_p^\ell(p) + \frac{1-\chi_p^2(p)/p}{1-\chi_p^2(p)} \sum_{\ell>0} p^{-\ell\frac{s-1}{2}} \bar{\chi}_p^\ell(p).$$

For $\Re(s) > 2\mathsf{r}+1$ the series converges absolutely (and uniformly) to

$$I_p(s,\varphi_p) = \frac{(1-p^{-\frac{s-1}{2}}\chi_p(p))^{-1}(1-p^{-\frac{s-1}{2}}\bar{\chi}_p(p))^{-1}}{\zeta_p(s)}. \tag{3.5}$$

The corresponding Euler product is holomorphic and bounded by some constant $\mathsf{c}(\epsilon)$, provided $\Re(s) > 2\mathsf{r}+3+\epsilon$. □

LEMMA 3.3. — *For all $\epsilon > 0$ and $\mathsf{n} \in \mathbb{N}$ there is a $\mathsf{c} = \mathsf{c}(\epsilon,\mathsf{n})$ such that*

$$I_\infty(s) \quad \textit{and} \quad I_{\mathsf{n},\infty}(s) := \int_{G(\mathbb{R})} \Delta^{\mathsf{n}} \cdot H_\infty(g_\infty)^{-s}\, dg_\infty$$

are holomorphic for all $s \in \mathsf{T}_{-2+\epsilon}$ with absolute value bounded by c.

Proof. — Write

$$g = k \begin{pmatrix} 1 & \\ & a \end{pmatrix} k', \quad \text{with} \quad k,k' \in SO(2),\ 0 < a \le 1. \tag{3.6}$$

By a standard integration formula (cf. p. 142 of [11])

$$I_\infty(\sigma) \ll \int_0^1 a^{\frac{\sigma}{2}}(a^{-1}-a)\, da^* < \int_0^1 a^{\frac{\sigma}{2}-1}\, da^* < \infty$$

(for $\sigma > 2$). A similar explicit computation proves the second claim. □

PROPOSITION 3.4. — *The function $\mathcal{Z}^{\mathrm{res}}(s)$ is holomorphic in T_0 and admits a meromorphic continuation to T_{-2} with an isolated pole at $s = 4$.*

Proof. — By definition,

$$\mathcal{Z}^{\mathrm{res}}(s) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma\in G(\mathbb{Q})} H(\gamma g)^{-s} dg = \int_{G(\mathbb{A})} H(g)^{-s}\, dg = I_f(s)\cdot I_\infty(s).$$

It suffices to apply Lemma 3.1 and 3.3. □

4. Eisenstein series

NOTATION . —

– $\int_{\mathbb{R}} := \int_{-\infty}^{+\infty}$;

– $d\mu_{\mathsf{n}}(t) := (1+t^2)^{-\mathsf{n}}dt$ - a measure on $\mathbb{R}$;

– $c(s) := \prod_v c_v(s)$, with $c_p(s) = \frac{\zeta_p(2s)}{\zeta_p(2s+1)}$ and $c_\infty(s) = \pi\frac{\Gamma(s)}{\Gamma(s+1/2)}$;

– $W(t) := 1 - \sum_\eta \frac{2\Re(\eta)}{\Re(\eta)^2+(t-\Im(\eta))^2}$, sum over all poles, with multiplicity, of $c(s)$ with $\Re(\eta) < 0$;

– $E(s,g)$ - Eisenstein series (2.4), $E_s := E(s,e)$ and

$$E_P(s,g) := \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(s,ng)dn;$$

– truncations:

$$\wedge_T E(s,g) := \sum\nolimits_T E_P(s,\gamma g) \quad \text{and} \quad \wedge^T E(s,g) := E(s,g) - \wedge_T E(s,g),$$

the sum over $\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})$, with $\chi_P(\gamma g) \geq T > 0$.

We recall some basic facts from the theory of Eisenstein series. We follow closely the exposition in [**8, 9**].

THEOREM 4.1. —

(1) *the poles of $E(s,g)$ coincide with the poles of $c(s)$;*
(2) *away from the poles, $E(s,g) = E(1-s,g)$;*
(3) *away from the poles,*

$$\int_K E(s,kg)\,dk = E(s,e)\varphi(s,g),$$

where $\varphi(s,\cdot) = \prod_v \varphi_v(s,\cdot)$ is the spherical function of the corresponding principal series representation;

(4) $\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} E(it,g)dg = 0$.

Proof. — Property (4) follows easily from the fact that the p-adic principal series representations corresponding to a purely imaginary parameter do not have the trivial representation as a quotient. □

The following key facts about Eisenstein series will be crucial for the analysis of the height zeta function. In the special case at hand, they could be deduced from standard facts about the Riemann zeta function. However, we give proofs applicable in more general situations.

THEOREM 4.2. — *For* $\mathsf{n} \gg 0$, *one has*

(1) $\int_{\mathbb{R}} \| \wedge^T E(it,\cdot)\|_2^2 d\mu_{\mathsf{n}}(t) < \infty$;

(2) $\int_{\mathbb{R}} \|E(it,\cdot)\|_{2,\Omega}^2 d\mu_{\mathsf{n}}(t) < \infty$, *where* $\Omega \subset G(\mathbb{A})$ *is a compact subset;*

(3) *for all* $g \in G(\mathbb{A})$

$$\int_{\mathbb{R}} | \wedge^T E(it,g)| d\mu_{\mathsf{n}}(t) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |E(it,g)|^2 d\mu_{\mathsf{n}}(t) < \infty;$$

(4) *the function* $g \mapsto \int_{\mathbb{R}} |E(it,g)| d\mu_{\mathsf{n}}(t)$ *is continuous.*

We will need the following

LEMMA 4.3. — *We have*

$$\| \wedge^T E(it,\cdot)\|_2^2 = 2T - \frac{c'(it)}{c(it)} + \Psi(it,T), \tag{4.1}$$

where $\Psi(it,T)$ *is a bounded function.*

Proof. — Following [9], p. 102, we get, away from the poles

$$\| \wedge^T E(s,\cdot)\|_2^2 = (s+\bar{s})^{-1}\left(e^{T(s+\bar{s})} - |c(s)|^2 e^{-T(s+\bar{s})}\right) + \Psi(s-\bar{s},T), \tag{4.2}$$

where

$$\Psi(s,T) := \frac{1}{s-\bar{s}}\left(\overline{c(s)}e^{(s-\bar{s})T} - c(s)e^{-(s-\bar{s})T}\right).$$

The singularities from $(s+\bar{s})^{-1}$ and $(s-\bar{s})^{-1}$ are removable ([8], p. 231). For $s = \sigma + it$, and $\sigma \to 0$ the right side of (4.2) equals

$$\lim_{\sigma\to 0} \frac{1}{2\sigma}\left\{e^{2T\sigma} - |c(\sigma+it)|^2 e^{-2T\sigma}\right\} + \Psi(it,T) = 2T - \frac{c'(it)}{c(it)} + \Psi(it,T).$$

Since $|c(it)| = 1$, it suffices to show that Ψ is bounded near $t = 0$. Write

$$\Psi(it, T) = \overline{c(it)}\frac{e^{2itT} - 1}{2it} - c(it)\frac{e^{-2itT} - 1}{2it} + \frac{c(it) - \overline{c(it)}}{2it}.$$

Since $c(0) \in \mathbb{R}$ and $c(it)$ is differentiable in t, $\lim_{t\to 0} \Psi(it, T)$ exists. □

Corollary 4.4. — *The function $s \mapsto c(s)$ is bounded in any region $0 \le \Re(s) \le \epsilon$, $\Im(s) \ge 1$.*

Proof. — Follows from the positivity of $\| \wedge^T E(s, \cdot)\|^2$ and (4.2). □

Lemma 4.5. — *For $\mathsf{n} \gg 0$,*

$$\int_{\mathbb{R}} \left| \frac{c'(it)}{c(it)} \right| d\mu_{\mathsf{n}}(t) < \infty.$$

Proof. — The only pole of $c(s)$ in the right half-plane is at $s = 1/2$. By Theorem 6.9 in [12], there is a $\mathsf{q} > 0$, such that

$$\frac{c'(it)}{c(it)} = \log(\mathsf{q}) + \frac{1}{t^2 + \frac{1}{4}} + (1 - W(t)).$$

By Theorem 7.1 of [12], there is a polynomial Q with positive coefficients such that

$$\left| \int_{-T}^{+T} \frac{c'(it)}{c(it)}\, dt \right| \le Q(T),$$

for all $T \in \mathbb{R}_{>0}$. By definition, for all t, $W(t) > 1$, so that

$$\left| \int_{-T}^{+T} \frac{c'(it)}{c(it)}\, dt \right| \ge \int_{-T}^{+T} (W(t) - 1)dt - \int_{-T}^{+T} \left(\log(\mathsf{q}) + \frac{1}{t^2 + \frac{1}{4}} \right) dt,$$

and

$$\int_{-T}^{+T} W(t)dt \le 2T + Q(T) + \int_{-T}^{+T} \left(\log(\mathsf{q}) + \frac{1}{t^2 + \frac{1}{4}} \right) dt.$$

In particular, there is a polynomial R with positive coefficients such that for all $T \in \mathbb{R}_{>0}$ one has

$$\int_{-T}^{+T} W(t)\, dt \le R(T).$$

Now we use the following easy statement: if a function $f : \mathbb{R} \to \mathbb{R}_{>0}$ and a polynomial R with positive coefficients are such that

$$\int_{-T}^{+T} f(t)\, dt \leq Q(T) \text{ then } \int_{\mathbb{R}} f(t)\, d\mu_{\mathsf{n}}(t) < \infty, \quad \text{for } \mathsf{n} \gg 0.$$

If follows that, for $\mathsf{n} \gg 0$,

$$\int_{\mathbb{R}} W(t) d\mu_{\mathsf{n}}(t) < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left| \frac{c'(it)}{c(it)} \right| d\mu_{\mathsf{n}}(t) < \infty.$$

□

Combining Lemma 4.3 with Lemma 4.5 we see that for all polynomials R there exists an $\mathsf{n}' > 0$ such that for all $\mathsf{n} > \mathsf{n}'$ one has

$$\mathcal{J}^T := \int_{\mathbb{R}} |R(it)| \cdot \| \wedge^T E(it, \cdot)\|_2^2 d\mu_{\mathsf{n}}(t) < \infty. \tag{4.3}$$

This proves (1) of Theorem 4.2.

Next, fix a compact $\Omega \subset G(\mathbb{A})$, $g \in \Omega$, and let $\| \cdot \|_{\infty,\Omega}$, resp. $\| \cdot \|_{2,\Omega}$ be the $\mathsf{L}^\infty(\Omega)$, resp. $\mathsf{L}^2(\Omega)$, norms. We bound $\|E(it, \cdot)\|_{2,\Omega}$ by

$$\| \wedge_T E(it, \cdot)\|_{2,\Omega} + \| \wedge^T E(it, \cdot)\|_{2,\Omega} \leq \mathsf{c}\| \wedge_T E(it, \cdot)\|_{\infty,\Omega} + \| \wedge^T E(it, \cdot)\|_2,$$

for some $\mathsf{c} > 0$. We proceed to estimate

$$\mathcal{J}_T := \int_{\mathbb{R}} |R(it)| \cdot \| \wedge_T E(it, \cdot)\|_{\infty,\Omega}^2\, d\mu_{\mathsf{n}}(t). \tag{4.4}$$

Observe that

$$\wedge_T E(it, g) = \sum_{\gamma \in \mathcal{S}} E_P(it, \gamma g) \mathcal{X}_T(\chi_P(\gamma g)),$$

where $\mathcal{S}$ is a finite subset of $P(\mathbb{Q})\backslash G(\mathbb{Q})$, depending only on $\mathfrak{S}$ and T, and $\mathcal{X}_T$ is the characteristic function of $[T, \infty)$. Thus

$$| \wedge_T E(it, g)| \leq \sum_{\gamma \in \mathcal{S}} |\chi(it, \gamma g)| + |\chi(-it, g)| \leq 2 \sum_{\gamma \in \mathcal{S}} \|\chi(0, \gamma \cdot)\|_{\infty,\Omega},$$

and the sum on the right side is bounded. It follows that the integral $\mathcal{J}_T$ converges, for $\mathsf{n} \gg 0$. Combined with (4.3) this proves (2) of Theorem 4.2.

An argument based on Sobolev's lemma (see the appendix or the proof of Lemma 3.4.7 [14]), shows that there exists a polynomial R such that

$$|E(it, g)| \leq |R(it)| \cdot \|E(it, \cdot)\|_{2,\Omega}.$$

Hence the integral in (3), Theorem 4.2, is bounded by $\mathcal{J}^T + \mathcal{J}_T$, from (4.3) and (4.4)). This proves (3) and (4).

5. Eisenstein integrals

For $g \in G(\mathbb{A})$, and $s, w \in \mathbb{C}$, we put (formally)

$$\hat{\mathcal{Z}}(s,w) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mathcal{Z}(s,g)E(-w,g)\,dg. \tag{5.1}$$

The main technical result of this section is the following

Proposition 5.1. — *For $s \in \mathsf{T}_{-1}$, the integral*

$$\mathcal{Z}^{\mathrm{eis}}(s,g) := \frac{1}{2\pi i}\int_{\mathbb{R}} \hat{\mathcal{Z}}(s,it)E(it,g)\,dt \tag{5.2}$$

(1) *is absolutely and uniformly convergent to a holomorphic in s and continuous in g function;*
(2) $\mathcal{Z}^{\mathrm{eis}}(s,g) \in \mathsf{L}^2$.

Proof. — Since $\int_K E(s,gk)dk = E(s,e)\varphi(s,g)$, (formally) $\hat{\mathcal{Z}}(s,w)$ equals

$$\int_{G(\mathbb{A})} H(g)^{-s}E(-w,g)\,dg = \left(\int_{G(\mathbb{A})} H(g)^{-s}\varphi(-w,g)\,dg\right)\cdot E(-w,e).$$

The local integrals

$$I_v(s,w) := \int_{G(\mathbb{Q}_v)} H_v(g)^{-s}\varphi_v(-w,g)dg_v$$

have been computed for $v = p$ in (3.5) (the character corresponding to the spherical function $\varphi_v(w,\cdot)$ is $|\cdot|_p^w$, see (3.1) and (3.2)). We get

$$I_f(s,w) = \prod_p I_p(s,w) = \frac{\zeta(\frac{s-1}{2}-w)\zeta(\frac{s-1}{2}+w)}{\zeta(s)}. \tag{5.3}$$

In particular, for $\epsilon \in (0,1)$ and all $w = \sigma + it$ with $0 \le \sigma \le \epsilon$, the map

$$s \mapsto I(s,w) = I_f(s,w)\cdot I_\infty(s,w)$$

is holomorphic for $s \in \mathsf{T}_{-1+\epsilon}$. Furthermore, for s and w as above,

$$|I(s,\sigma+it)| \ll_{\mathsf{n},\epsilon} (1+t^2)^{-\mathsf{n}}. \tag{5.4}$$

Indeed, the function $\varphi_\infty(it,\cdot)$ is bounded. Moreover, by Lemma 4.1, it is an eigenfunction for Δ. We now apply repeatedly integration by parts (with respect to Δ) combined with Lemma 3.3 to $I_\infty(s,w)$ and use the standard bounds for ζ.

We have

$$\int_{\mathbb{R}} |\hat{\mathcal{Z}}(s,it)|\cdot|E(it,g)|\,dt \le \int_{\mathbb{R}} |I(s,it)|\cdot|E(it,e)|\cdot|E(it,g)|\,dt$$

Let $\mathsf{K}\subset\mathsf{T}_{-1}$ and $\Omega\subset G(\mathbb{A})$ be compact sets. By the estimates above, for every $\mathsf{n}\in\mathbb{N}$, there is a constant $\mathsf{c}=\mathsf{c}(\mathsf{n},\mathsf{K})$ such that

$$\int_{\mathbb{R}} \sup_{s\in\mathsf{K}} |I(s,it)|\cdot|E(it,e)|^2\,dt \le \mathsf{c}\int_{\mathbb{R}} |E(it,e)|^2\,d\mu_{\mathsf{n}}(t). \tag{5.5}$$

To show the absolute and uniform convergence of the integral (5.2) for $(s,g)\in\mathsf{K}\times\Omega$ we need to check that, for $\mathsf{n}\gg 0$,

$$\int_{\mathbb{R}} |E(it,e)|\cdot\|E(it,\cdot)\|_{\infty,\Omega}\,d\mu_{\mathsf{n}}(t)<\infty,$$

or, by Cauchy-Schwartz, that

$$\int_{\mathbb{R}} |E(it,e)|^2\,d\mu_{\mathsf{n}}(t)<0 \quad\text{and}\quad \int_{\mathbb{R}} \|E(it,\cdot)\|_{\infty,\Omega}\,d\mu_{\mathsf{n}}(t)<\infty,$$

which follows from Theorem 4.2.

To prove that $\mathcal{Z}^{\mathrm{eis}}(s,\cdot)\in\mathsf{L}^2$, for $s\in\mathsf{T}_{-1}$, write $E_s:=E(s,e)$,

$$2\pi i\cdot\mathcal{Z}^{\mathrm{eis}}(s,g)=\mathcal{J}^T(s,g)+\mathcal{J}_T(s,g),$$

$$\mathcal{J}^T(s,g):=\int_{\mathbb{R}} I(s,it)E_{-it}\wedge^T E(it,g)\,dt,$$

similarly $\mathcal{J}_T(s,g)$, with $\wedge^T$ replaced by $\wedge_T$. We bound $\|\mathcal{J}^T(s,\cdot)\|_2^2$ as

$$\begin{aligned}
&\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |\int_{\mathbb{R}} I(s,it)E_{-it}\wedge^T E(it,g)dt|\cdot|\int_{\mathbb{R}} \overline{I(s,i\tau)}E_{i\tau}\overline{\wedge^T(i\tau,g)}d\tau|dg\\
&\leq \int_{\mathbb{R}^2} |I(s,it)I(s,i\tau)E_{it}E_{i\tau}| \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |\wedge^T E(it,g)\wedge^T E(i\tau,g)|dgdtd\tau\\
&\leq \int_{\mathbb{R}^2} |I(s,it)I(s,i\tau)E_{it}E_{i\tau}|\cdot\|\wedge^T E(it,\cdot)\|_2\cdot\|\wedge^T E(i\tau,\cdot)\|_2 dtd\tau\\
&= \left(\int_{\mathbb{R}} |I(s,it)E_{it}|\cdot\|\wedge^T E(it,\cdot)\|_2 dt\right)^2
\end{aligned}$$

by repeatedly applying Fubini's theorem, and then Cauchy-Schwartz to the inner integral. Thus it suffices to bound the integrals

$$\int_{\mathbb{R}} |E(it,e)|d\mu_{\mathsf{n}}(t) \text{ and } \int_{\mathbb{R}} \|\wedge^T E(it,\cdot)\|_2^2\, d\mu_{\mathsf{n}}(t),$$

which has been accomplished in Theorem 4.2.

We turn to $\mathcal{J}_T(s,\cdot)$. It suffices to consider the integral over $\mathfrak{S}_T$ (since $\mathfrak{S}^T$ is compact). For $T>1$ we have, for all $g\in G(\mathbb{A})$,

$$\wedge_T E(it,g) = E_P(it,g) = \chi(it,g) + c(it)\chi(-it,g).$$

Substituting this into the definition of $\mathcal{J}_T$ (and rewriting the integral $\int_{\mathbb{R}}$) we see that we need to bound the $\mathsf{L}^2(\mathfrak{S}_T)$-norms of

$$\int_{\Re(w)=0} I(s,w)E_{-w}\chi(w,\cdot)dw \quad \text{and} \quad \int_{\Re(w)=0} I(s,w)E_{-w}c(w)\chi(-w,\cdot)\,dw$$

By Corollary 4.4, $c(s)$ is bounded in the region $\Re(s)\in[0,\epsilon]$, $\Im(s)\geq 1$ and we can shift the contour of integration of the first integral slightly to the left and that of the second to the right. This gives

$$\chi(-\varepsilon,\cdot)\int_{\Re(w)=-\varepsilon} I(s,w)E_{-w}dw \quad \text{and } \chi(-\varepsilon,\cdot)\int_{\Re(w)=\varepsilon} I(s,w)E_{-w}dw.$$

Now it suffices to observe that, for $\varepsilon>0$,

$$\|\chi(-\varepsilon,\cdot)\|_{2,\mathfrak{S}_T} = \int_T^\infty a^{2(-\varepsilon+\frac{1}{2})}a^{-1}da^* < \infty,$$

and to use the estimate (5.4). This completes the proof of Proposition 5.1

□

6. P-series

Definition 6.1. — Let $\phi : G(\mathbb{A}) \to \mathbb{C}$ satisfy

– $\phi(ngk) = \phi(g)$, for $n \in N(\mathbb{A}), k \in K$ and $g \in G(\mathbb{A})$;
– $\phi(ag) = \phi(g)$, for $a \in A(\mathbb{Q})$, $g \in G(\mathbb{A})$;
– $\phi : A(\mathbb{Q})\backslash A(\mathbb{A}) \to \mathbb{C}$ is smooth of compact support.

For $g \in G(\mathbb{A})$, set

$$\theta_\phi(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) \quad \text{and} \quad \hat{\theta}_\phi(s) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta_\phi(g) E(s,g) dg.$$

Since ϕ is left $P(\mathbb{Q})$-invariant and compactly supported modulo $P(\mathbb{Q})$ the sum defining θ_ϕ is finite so that $\theta_\phi \in \mathsf{C}_c^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and $\hat{\theta}_\phi$ is meromorphic, analytic in $\Re(s) \geq 0$, except possibly at $s = 1/2$.

In the following we identify $\mathbb{A}^*$ with $A(\mathbb{A})$:

$$a := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

write da^*, resp. da, for the corresponding Haar measure and regard ϕ as being (simultaneously) in $\mathsf{C}_c^\infty(\mathbb{Q}^* \cdot \hat{\mathbb{Z}}^* \backslash \mathbb{A}^*) = \mathsf{C}_c^\infty(\mathbb{R}_{>0}^*)$.

For $\Re(s) \gg 0$ we may use Fubini to derive

$$\begin{aligned}
\hat{\theta}_\phi(s) &= \int_{P(\mathbb{Q})\backslash G(\mathbb{A})} \theta_\phi(g)\chi(s,g)dg \\
&= \int_{N(\mathbb{A})A(\mathbb{A})\backslash G(\mathbb{A})} \theta_{\phi,P}(g)\chi(s,g)dg \\
&= \int_{A(\mathbb{Q})\backslash A(\mathbb{A})} \theta_{\phi,P}(a)\chi(s,a)|\delta(a)|_{\mathbb{A}}^{-1} da \\
&= \int_{\mathbb{Q}^*\backslash \mathbb{A}^*} \theta_{\phi,P}(a)|a|_{\mathbb{A}}^{s-1/2} da^*,
\end{aligned}$$

where $\theta_{\phi,P}$ is the constant term of θ_ϕ. By Bruhat's lemma,

$$\theta_\phi(g) = \phi(g) + \sum_{\gamma \in N(\mathbb{Q})} \phi(\mathsf{w}\gamma g) \quad \text{and} \quad \theta_{\phi,P}(g) = \phi(g) + \int_{N(\mathbb{A})} \phi(\mathsf{w}ng)dn,$$

(with w the nontrivial element of the Weyl group), so that

$$\hat{\theta}_\phi(s) = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \phi(a)|a|_{\mathbb{A}}^{s-1/2} da^* + \int_{\mathbb{Q}^*\backslash\mathbb{A}^*}\int_{N(\mathbb{A})} \phi(\mathsf{w}na)dn \cdot |a|_{\mathbb{A}}^{s-1/2}da^*.$$

To justify the above equation note that the double integral on the right is absolutely convergent. Indeed, for $g \in G(\mathbb{A})$ and ϕ as above, set

$$f_\phi(s,g) = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \phi(ag)|a|_{\mathbb{A}}^{-1/2-s} da^*.$$

Then

$$f_\phi(s,nag) = |a|_{\mathbb{A}}^{1/2+s} f_\phi(s,g)$$

and the double integral may be written as

$$\int_{N(\mathbb{A})} f_\phi(s,\mathsf{w}n)dn = c(s)\cdot f_\phi(-s,e). \tag{6.1}$$

The Euler product defining $c(s)$ and hence the integral (6.1) converge for $\Re(s) \gg 0$. We have, for general ϕ, and, at first for $\Re(s) \gg 0$ and then, by analytic continuation, for all s,

$$\hat{\theta}_\phi(s) = f_\phi(-s,e) + c(s)f_\phi(s,e). \tag{6.2}$$

Notice that $\hat{\theta}_\phi(it)$ is rapidly decreasing ($f_\phi(it,e)$ is essentially the Fourier transform of a function in $\mathsf{C}_c^\infty(\mathbb{R})$) and define, for $g \in G(\mathbb{A})$,

$$\theta_\phi^c(g) := \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\theta}_\phi(it)E(it,g)dt. \tag{6.3}$$

Proposition 6.2. — *If $\phi \in \mathsf{C}_c^\infty(\mathbb{R}_{>0}^*)$ then θ_ϕ^c is continuous and in L^2.*

Proof. — From (6.3) and (6.6) we have $|\theta_\phi^c(g)| \le \mathcal{J}^T(g) + \mathcal{J}_T(g)$, with

$$\mathcal{J}^T(g) := \int_{\mathbb{R}} |\wedge^T E(it,g)|d\mu_{\mathsf{n}}(t) \quad \text{and} \quad \mathcal{J}_T(g) := \int_{\mathbb{R}} |\wedge_T E(it,g)|d\mu_{\mathsf{n}}(t).$$

Both integrals are finite by Theorem 4.2, (3). To prove continuity, let Ω be a pre-compact neighborhood of g and recall that (by the same theorem)

$$\int_{\mathbb{R}} \sup_{g\in\Omega} |E(it,g)|d\mu_{\mathsf{n}}(t) < \infty \quad \text{for} \quad \mathsf{n} \gg 0.$$

□

THEOREM 6.3. — *For all* $\phi \in \mathsf{C}_c^\infty(\mathbb{Q}^* \cdot \hat{\mathbb{Z}}^* \backslash \mathbb{A}^*)$ *one has*

$$\theta^c_\phi = \theta_\phi - \langle \theta_\phi, \mathbf{1} \rangle \mathbf{1}.$$

Proof. — We first prove the identity for constant terms:

$$\theta^c_{\phi,P} = \theta_{\phi,P} - \langle \theta_\phi, \mathbf{1} \rangle \mathbf{1}.$$

By Theorem 4.2, $g \mapsto \int_{\mathbb{R}} |E(-it, g)| d\mu_{\mathsf{n}}(t)$ is continuous so that

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{\mathbb{R}} |\hat{\theta}_\phi(it) E(-it, ng)| dt dn < \infty.$$

It follows that, for $g \in G(\mathbb{A})$,

$$\theta^c_{\phi,P}(g) = \int_{\mathbb{R}} \hat{\theta}_\phi(it) E_P(-it, g) dt = \int_{\mathbb{R}} \hat{\theta}_\phi(it) \left(\chi(it, g) + c(-it)\chi(-it, g) \right) dt.$$

Since

$$c(-s)\hat{\theta}_\phi(s) = f_\phi(s, e) + c(-s) f_\phi(-s, e) = \hat{\theta}_\phi(-s)$$

we obtain that

$$\theta^c_{\phi,P}(g) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{\theta}_\phi(it)\chi(it, g) dt. \tag{6.4}$$

We now compare $\theta^c_{\phi,P}$ and $\theta_{\phi,P}$. We have already seen that

$$\hat{\theta}_\phi(s) = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \theta_{\phi,P}(a)|a|_{\mathbb{A}}^{s-1/2} da^*, \quad \text{for} \quad \Re(s) \gg 0. \tag{6.5}$$

For $\Re(s) \geq 1$, the function

$$a \mapsto \theta_{\phi,P}(a)|a|_{\mathbb{A}}^{-1/2+s}$$

is left-invariant under $\hat{\mathbb{Z}}^*$ and in $\mathsf{L}^1(\mathbb{Q}^* \cdot \hat{\mathbb{Z}}^* \backslash \mathbb{A}^*) = \mathsf{L}_1(\mathbb{R}^*_{>0})$. On the other hand, $c(s)$ is bounded in the region $0 \leq \Re(s) \leq \sigma_0$ (for any $\sigma_0 > 0$), $|\Im(s)| \geq 1$ (by Corollary 4.4). It follows from (6.2) that in this domain

$$\hat{\theta}_\phi(\sigma + it) = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \theta_{\phi,P}(a)|a|_{\mathbb{A}}^{-1/2+\sigma}|a|_{\mathbb{A}}^{it} da^*$$

is rapidly decreasing in t and we may apply Fourier inversion so that

$$\theta_{\phi,P}(a) = |a|_{\mathbb{A}}^{-\sigma+1/2} \int_{\mathbb{R}} \hat{\theta}_\phi(\sigma + it)|a|_{\mathbb{A}}^{-it} dt. \tag{6.6}$$

But we also have

$$\theta^c_{\phi,P}(g) = \frac{1}{2\pi i}\int_{\Re(s)=0} \hat{\theta}_\phi(s)|a|_{\mathbb{A}}^{1/2-s}ds. \tag{6.7}$$

We shift the contour to $\Re(s) = \sigma$, taking $\sigma > 1/2$. The shift is justified by using (6.6) and the fact that $f_\phi(\sigma + it, e)$ is rapidly decreasing in t, uniformly in σ, for $|\sigma| \leq \sigma_0$. It follows that

$$\theta^c_{\phi,P}(a) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma} \hat{\theta}_\phi(s)|a|_{\mathbb{A}}^{1/2-s}ds - \mathrm{res}_{s=1/2}\hat{\theta}_\phi(s)|a|_{\mathbb{A}}^{1/2-s}.$$

Now it suffices to compute the residue of $c(s)$ at $s = 1/2$ and to see that

$$f_\phi(1/2, e) = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \phi(a)|a|_{\mathbb{A}}^{-1}da^* = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta_\phi(g)dg,$$

(for appropriately normalized measures).

Let us now set

$$\theta^{\mathrm{cusp}}_\phi := \theta_\phi - \langle\theta_\phi, \mathbf{1}\rangle \cdot \mathbf{1} - \theta^c_\phi.$$

We have proved that $\theta^{\mathrm{cusp}}_\phi$ is continuous, in L_2 and has a vanishing constant term. Thus $\theta^{\mathrm{cusp}}_\phi$ is a cusp form, i.e., orthogonal to all the P-series θ_ϕ. To complete the proof of Theorem 6.3 it suffices to prove that θ^c_ϕ is orthogonal to all cusp forms. Since $\theta^c_\phi \in \mathsf{L}^2$, it will suffice to prove that

$$\langle\theta^c_\phi, \psi * \alpha\rangle = 0$$

for all cusp forms ψ and all $\alpha \in \mathsf{C}^\infty_c(G(\mathbb{A}))$. Replacing ψ by $\psi * \alpha$ we may assume that ψ is rapidly decreasing (and continuous). Recall that

$$\theta^c_\phi(g) = \int_{\mathbb{R}} \hat{\theta}_\phi(t)E(-it, g)dt.$$

Since $\langle E(-it, \cdot), \psi\rangle = 0$, for all $t \in \mathbb{R}$, it suffices to prove that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\int_{\mathbb{R}} |E(it, g)\psi(g)|d\mu_{\mathsf{n}}(t)dg < \infty \quad \text{for } \mathsf{n} \gg 0.$$

As before, we consider the integral

$$\mathcal{J}^T := \int_{\mathfrak{S}}\int_{\mathbb{R}} | \wedge^T E(it, g)| \cdot |\psi(g)|d\mu_{\mathsf{n}}(t)dg$$

and a similar integral $\mathcal{J}_T$ with $\wedge^T$ replaced by $\wedge_T$. Using Theorem 4.2,

$$\mathcal{J}^T \ll \|\psi\|_2 \cdot \int_{\mathbb{R}} \| \wedge^T E(it, \cdot)\|_2 d\mu_{\mathfrak{n}}(t) < \infty.$$

To treat the integral $\mathcal{J}_T$ we decompose $\mathfrak{S} = \mathfrak{S}^T \cup \mathfrak{S}_T$. Since $\mathfrak{S}_T$ is compact, ψ continuous and $| \wedge_T E(it, g)|$ bounded in this domain, the double integral is absolutely convergent. It remains to estimate

$$\int_{\mathfrak{S}^T} \int_{\mathbb{R}} | \wedge_T E(it, g)| \cdot |\psi(g)| d\mu_{\mathfrak{n}}(t) dg.$$

To complete the proof of Theorem 6.3 observe that for $T > 1$,

$$| \wedge_T E(it, g)| = |E_P(it, g)| \le 2\chi(0, g)$$

and, since ψ is rapidly decreasing,

$$\int_{\mathfrak{S}^T} |\psi(g)| \cdot \chi(0, g) dg < \infty.$$

□

Proposition 6.4. — *For $\Re(s) \gg 0$, the function $\mathcal{Z}^{\mathrm{eis}}(s, \cdot)$ is orthogonal to all cusp forms and to the constant function.*

Proof. — Recall that

$$\mathcal{Z}^{\mathrm{eis}}(s, g) = \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{\mathcal{Z}}(s, it) E(-it, g) dt,$$

where

$$\hat{\mathcal{Z}}(s, it) = I(s, it) E(it, e) \quad \text{with} \quad I(s, w) = \int_{G(\mathbb{A})} H(g)^{-s} \varphi_w(g) dg.$$

As above, we show first that the double integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{\mathbb{R}} \hat{\mathcal{Z}}(s, it) E(-it, g) \psi(g) dt dg,$$

is absolutely convergent, for ψ continuous and bounded, and that it converges to zero. The last integral is majorized by

$$\int_{\mathbb{R}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |E(it, e)| \cdot |E(-it, g)| \cdot |\psi(g)| d\mu_{\mathfrak{n}}(t) dg.$$

We set

$$\mathcal{J}^T := \int_{\mathbb{R}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} |E(-it,e)| \cdot | \wedge^T E(it,g)| \cdot |\psi(g)| d\mu_{\mathsf{n}}(t) dg.$$

and a similar integral $\mathcal{J}_T$ (with $\wedge^T$ replaced by $\wedge_T$). By Cauchy-Schwartz, to treat $\mathcal{J}^T$ it suffices to consider

$$\int_{\mathbb{R}} |E(-it,e)| \cdot \| \wedge^T E(it,\cdot)\|_2 d\mu_{\mathsf{n}}(t),$$

and again by Cauchy-Schwartz, the two integrals

$$\int_{\mathbb{R}} |E(-it,e)|^2 d\mu_{\mathsf{n}}(t) \quad \text{and} \quad \int_{\mathbb{R}} \| \wedge^T E(it,\cdot)\|_2^2 d\mu_{\mathsf{n}}(t),$$

which are finite by Theorem 4.2. To bound $\mathcal{J}_T$ we decompose $\mathfrak{S} = \mathfrak{S}_T \cup \mathfrak{S}^T$. The contribution from (the compact) $\mathfrak{S}_T$ is estimated using the boundedness of $\wedge_T E(it,\cdot) \cdot \psi(\cdot)$. To treat $\mathfrak{S}^T$ recall that in this domain (for $T > 1$), $\wedge_T E(it,\cdot) = E_P(it,\cdot)$ and

$$|E_P(it,g)\psi(g)| \le \mathsf{c} \cdot \chi(0,g).$$

Using the fact that the right side is integrable on $\mathfrak{S}$ and, once again, Theorem 4.2, we see that $\mathcal{J}^T$ is also finite. Since

$$\int_{\mathbb{R}} |E(-it,e)| d\mu_{\mathsf{n}}(t) \tag{6.8}$$

is finite for $\mathsf{n} \gg 0$, the integral J^T is also finite.

As we have remarked, $\langle E(-it,\cdot), \psi(\cdot)\rangle = 0$, for $t \in \mathbb{R}$, at least if ψ is rapidly decreasing. It remains to recall Proposition 4.1 (4):

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} E(it,g) dg = 0.$$

This completes the proof. □

We will need the following

Lemma 6.5. — *For all $\phi \in \mathsf{C}_c^\infty(\mathbb{R}^*_{>0})$, we have*

$$\langle \mathcal{Z}^{\mathrm{eis}}(s,\cdot), \theta_\phi\rangle = \langle \mathcal{Z}(s,\cdot), \theta_\phi^c\rangle.$$

Proof. — First observe that

$$\langle \mathcal{Z}^{\mathrm{eis}}, \theta_\phi \rangle = \langle \hat{\mathcal{Z}}, \hat{\theta}_\phi \rangle_{\mathsf{L}^2(\mathbb{R})} = \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{\mathcal{Z}}(s, it)\hat{\theta}_\phi(t) dt. \tag{6.9}$$

For this it suffices to prove that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{\mathbb{R}} |\hat{\mathcal{Z}}(s,it)E(it,g)\theta_\phi(g)| dt dg < \infty$$

We have $\hat{\mathcal{Z}}(s,it) = I(s,it)E(it,e)$, where, by (5.4), $I(s,it)$ is rapidly decreasing in t, and

$$\int_{G(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{R}} |E(-it,e)E(it,g)\theta_\phi(g)| d\mu_{\mathsf{n}}(t) dg < \infty,$$

for $\mathsf{n} \gg 0$, by Theorem 4.2. To show that

$$\langle \hat{\mathcal{Z}}, \hat{\theta}_\phi \rangle_{\mathsf{L}^2(\mathbb{R})} = \langle \mathcal{Z}, \theta_\phi^c \rangle$$

it suffices to check that, for $\Re(s) \gg 0$,

$$\int_{G(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{R}} |\mathcal{Z}(s,g)E(it,g)\hat{\theta}_\phi(it)| dt dg < \infty.$$

Since $\mathcal{Z}(s,g)$ is bounded in g, for $\Re(s) \gg 0$, and $\hat{\theta}_\phi(it)$ is rapidly decreasing, it suffices to recall that

$$\int_{\mathbb{R}} |E(it,\cdot)| d\mu_{\mathsf{n}}(t) \in \mathsf{L}^2, \quad \text{for} \quad \mathsf{n} \gg 0.$$

□

7. The cuspidal spectrum

Write $G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{R}) \cdot K_f$ and let $\Gamma = \mathrm{pr}_\infty(G(\mathbb{Z}))$. The map

$$\begin{array}{rcl} j : \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})) & \to & \mathsf{L}^2(\Gamma\backslash G(\mathbb{R})) \\ \phi & \mapsto & \phi|_{G(\mathbb{R})} \end{array}$$

is an isometry. We consider the right regular representation of $G(\mathbb{R})$ on

$$\mathsf{H} := \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} = \mathsf{L}^2(\Gamma\backslash G(\mathbb{R}))$$

and denote by $\mathsf{H}^\infty \subset \mathsf{H}$ the subspace of smooth vectors. If $M = \Gamma\backslash G(\mathbb{R})$ and ω is the gauge form on M whose accociated measure is a fixed right-invariant measure on $\Gamma\backslash G(\mathbb{R})$, then $\mathsf{H}^\infty = \mathsf{H}^\infty(M)$ as defined in the

Appendix. Write $\mathsf{L}_0^2 := \mathsf{L}_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ for the space of cusp forms on $G(\mathbb{A})$ and $\mathsf{H}_0 = \mathsf{L}_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f}$ for the closed $G(\mathbb{R})$-stable subspace of H. Let

$$\mathcal{P} : \mathsf{H} \to \mathsf{H}_0$$

be the orthogonal projection, it maps H^∞ to H_0^∞, the subspace of smooth vectors in H_0. Moreover, $\mathsf{H}_0^\infty \subset \mathsf{H}^\infty$. We have decompositions

$$\mathsf{L}_0^2 = \oplus_\pi \mathsf{H}_\pi \subset \mathsf{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})), \quad \text{and} \quad \mathsf{H}_0 = \oplus \mathsf{H}_\pi^{K_f}, \tag{7.1}$$

as a $G(\mathbb{A})$-modules, into a countable direct sum of closed irreducible subspaces (each occuring with finite multiplicity). Let $\mathcal{A}_0$ be the set of all π occuring in H_0. Note that each $\mathsf{H}_\pi^{K_f}$ contains an essentially unique K_∞-fixed vector ϕ_π normalized so that $\|\phi_\pi\|_2 = 1$. Let $\mathsf{H}_0(M) \subset \mathsf{L}^2(M,\omega)$ be the Hilbert subspace spanned by the ϕ_π's. We remark that the ϕ_π are necessarily in H^∞ and are eigenfunctions of Δ, with eigenvalue, say, λ_π. By Proposition 8.8, for $\phi \in \mathsf{H}_0 \cap \mathsf{H}^\infty$, the Fourier series

$$\sum_{\pi \in \mathcal{A}_0} \langle \phi, \phi_\pi \rangle \phi_\pi \tag{7.2}$$

converges to ϕ, uniformly of compact sets.

We apply the above arguments to the height zeta function as follows. For $\Re(s) \gg 0$, put $\mathcal{Z}^{\mathrm{res}} = \langle \mathcal{Z}, \mathbf{1} \rangle \mathbf{1}$ and define $\mathcal{Z}^{\mathrm{cusp}}$ so that

$$\mathcal{Z}(s,\cdot) = \mathcal{Z}^{\mathrm{res}}(s) + \mathcal{Z}^{\mathrm{cusp}}(s,\cdot) + \mathcal{Z}^{\mathrm{eis}}(s,\cdot).$$

Proposition 7.1. — *For $\Re(s) \gg 0$,*

$$\mathcal{Z}^{\mathrm{cusp}}(s,\cdot) = \mathcal{P}(\mathcal{Z}(s,\cdot)) \in \mathsf{L}_0^2.$$

Proof. — By Proposition 5.1,

$$\mathcal{Z}^{\mathrm{eis}}(s,\cdot) \in \mathsf{L}^2$$

for $\Re(s) \gg 0$ and is continuous. Same holds for all its Δ-derivatives. It follows that $\mathcal{Z}^{\mathrm{cups}}(s,\cdot) \in \mathsf{L}^2$ and is also continuous.

For $\phi \in \mathsf{C}_c^\infty(\mathbb{R}_{>0}^*)$ we have

$$\langle \mathcal{Z}^{\mathrm{cusp}}(s,\cdot), \theta_\phi \rangle = \langle \mathcal{Z}(s,\cdot), \theta_\phi \rangle - \langle \mathcal{Z}^{\mathrm{eis}}(s,\cdot), \theta_\phi \rangle - \langle \mathcal{Z}(s,\cdot), \mathbf{1} \rangle \mathbf{1}.$$

By Lemma 6.5, $\langle \mathcal{Z}^{\mathrm{eis}}(s,\cdot), \theta_\phi \rangle = \langle \mathcal{Z}(s,\cdot), \theta_\phi^c \rangle$ and, by Theorem 6.3,

$$\theta_\phi^c = \theta_\phi - \langle \theta_\phi, \mathbf{1} \rangle \mathbf{1}.$$

We get at once

$$\langle \mathcal{Z}^{\mathrm{cusp}}(s,\cdot), \theta_\phi\rangle = 0$$

and $\mathcal{Z}^{\mathrm{cusp}} \in \mathsf{L}^2_0$. (Here we used the right K-invariance of $\mathcal{Z}^{\mathrm{cusp}}(s,\cdot)$.)

To prove that $\mathcal{Z}^{\mathrm{cusp}} = \mathcal{P}(\mathcal{Z})$, we need to show that

$$\langle \mathcal{Z}^{\mathrm{cusp}}(s,\cdot), \psi\rangle = \langle \mathcal{Z}(s,\cdot), \psi\rangle,$$

for all $\psi \in \mathsf{L}^2_0$. This is an immediate consequence of Proposition 6.4. □

We proceed to investigate the meromorphic properties of the Fourier expansion of $\mathcal{Z}^{\mathrm{cusp}}$ as in (7.2). For $v = p$, let φ_p be the corresponding local spherical function with parameter $\alpha_p \in \mathbb{C}^*$. They are given by (3.1) and (3.2). The crucial facts in our further analysis are

Theorem 7.2. — [10] *For $\pi = \otimes_\nu \pi_\nu \in \mathcal{A}_0$ let φ_p be the normalized spherical function corresponding to π_p and α_p its parameter. Then*

$$|\Re(\alpha_p)| \leq 1/6.$$

Remark 7.3. — Any nontrivial uniform bound towards the Ramanujan conjecture suffices for our purposes.

Theorem 7.4. — *For all $\mathsf{r} > 0$ there is a $\mathsf{c} > 0$ such that*

$$\mathcal{Z}_\Delta(s) := \sum_{\pi\in\mathcal{A}_0,\lambda_\pi\neq 0} \frac{\|\phi\|_\infty^{\mathsf{r}}}{|\lambda_\pi|^{\mathsf{n}}} < \infty, \quad \textit{for all} \quad \mathsf{n} > \mathsf{c}. \tag{7.3}$$

Proof. — Estimate the $\|\cdot\|_\infty$-norm of eigenfunctions in terms of the corresponding eigenvalues as in the Appendix and use the following fact: there exist constants $\mathsf{c}, \mathsf{r} > 0$ such that the number of linearly independent Δ-eigenfunctions with eigenvalue less than B is bounded by $\mathsf{c}(1 + B^{\mathsf{r}})$, for all $B \geq 0$ (see [12], for example). □

Lemma 7.5. — *For all $\epsilon > 0$ and $\mathsf{n} \in \mathbb{N}$ there is a constant $\mathsf{c} = \mathsf{c}(\epsilon, \mathsf{n})$ such that for all $s \in \mathsf{T}_{-2/3+\epsilon}$ and all π the function*

$$\langle \mathcal{Z}(s,\cdot), \phi_\pi\rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mathcal{Z}(s,g)\overline{\phi_\pi(g)}dg$$

is holomorphic, with absolute value bounded by $\mathsf{c}\|\phi\|_\infty|\lambda_\pi|^{-\mathsf{n}}$ (for $\lambda_\pi \neq 0$).

Proof. — For ϕ_π (with $\lambda_\pi \neq 0$) and s such that $\mathcal{Z}(s,\cdot) \in \mathsf{L}^2$

$$\langle \mathcal{Z}(s,\cdot), \phi_\pi \rangle = \int_{G(\mathbb{A})} H(g)^{-s}\overline{\phi_\pi(g)}\, dg = \lambda_\pi^{-\mathsf{n}} \int_{G(\mathbb{A})} H(g)^{-s}\overline{\Delta^{\mathsf{n}}\phi_\pi(g)}\, dg$$
$$= \lambda_\pi^{-\mathsf{n}} \int_{G(\mathbb{A})} \Delta^{\mathsf{n}} H(g)^{-s}\overline{\phi_\pi(g)}\, dg.$$

Since the (right) actions of Δ and K commute, $\Delta^{\mathsf{n}} H(g)^{-s}$ is invariant under K (which has volume 1), so that the above expression equals
(7.4)
$$\lambda_\pi^{-\mathsf{n}} \int_K \int_{G(\mathbb{A})} \Delta^{\mathsf{n}} H(kg)^{-s}\overline{\phi_\pi(g)} dg dk = \lambda_\pi^{-\mathsf{n}} \int_{G(\mathbb{A})} \Delta^{\mathsf{n}} H(g)^{-s} \int_K \overline{\phi_\pi(kg)} dk dg.$$

As is well-known,

$$\int_K \phi_\pi(kg) = \varphi_\pi(g) \cdot \phi_\pi(e),$$

where φ_π is the spherical function attached to π, i.e.,

$$\varphi_\pi(g) = \langle \pi(g)\phi_\pi, \phi_\pi \rangle, \quad g \in G(\mathbb{A}).$$

Indeed, the functional

$$\phi \mapsto \int_K \phi(k\cdot) dk$$

is a bounded, left K-invariant functional on H_π^∞ (and H_π). Thus it is proportional to the functional

$$\phi \mapsto \langle \phi, \phi_\pi \rangle.$$

Taking $\phi = \pi(g)\phi_\pi$, we find that the proportionality constant is $\phi_\pi(e)$.

Thus the integral in (7.4) is computed as

$$I_f(s,\overline{\varphi}) \cdot \int_{G(\mathbb{R})} \Delta^{\mathsf{n}} H_\infty(g_\infty)^{-s}\overline{\varphi_\infty(g_\infty)}\, dg_\infty \cdot \overline{\phi_\pi(e)} \tag{7.5}$$

Combining Lemma 3.2 with Theorem 7.2 (giving $\mathsf{r} = 1/6$), we find that $I_f(s,\overline{\varphi})$ is holomorphic for $s \in \mathsf{T}_{-2/3}$ and *uniformly* bounded by a constant $\mathsf{c}(\epsilon)$ for $s \in \mathsf{T}_{-2/3+\epsilon}$ (and $\epsilon > 0$). Since ϕ is bounded, Lemma 3.3 shows that

$$\int_{G(\mathbb{R})} \Delta^{\mathsf{n}} H_\infty(g_\infty)^{-s}\overline{\varphi_\infty(g_\infty)}\, dg_\infty$$

is absolutely convergent, in T_{-2}, to a holomorphic function which, in $\mathsf{T}_{-2+\epsilon}$, is bounded by $\mathsf{c}\|\phi\|_\infty$ for some constant $\mathsf{c} = \mathsf{c}(\epsilon, \mathsf{n})$. □

PROPOSITION 7.6. — *For all $\epsilon > 0$ the function*

$$\mathcal{Z}^{\mathrm{cusp}}(s,g) := \sum_{\pi\in\mathcal{A}_0} \langle \mathcal{Z}(s,\cdot), \phi_\pi\rangle \phi_\pi(g) \tag{7.6}$$

is holomorphic in s and continuous in $g \in G(\mathbb{A})$ for all $s \in \mathsf{T}_{-2/3+\epsilon}$.

Proof. — Combine Theorem 7.4 and Lemma 7.5. The uniform convergence on compacts $\Omega \subset G(\mathbb{A})$ follows from the estimate

$$\sum_{\pi} \sup_{g\in\Omega} |\langle \mathcal{Z}(s,\cdot), \phi_\pi\rangle \phi_\pi(g)| \ll_{\mathsf{n},\epsilon,\Omega} \sum_{\pi\in\mathcal{A}_0} \frac{\|\phi_\pi\|_\infty^2}{|\lambda_\pi|^{\mathsf{n}}}$$

and the convergence of the spectral zeta function (7.3) for $\mathsf{n} \gg 0$. □

8. Appendix

NOTATION . —

- $dx = dx_1 \cdots dx_n$ - Lebesgue measure on $\mathbb{R}^n$, $|x|^2 = x_1^2 + \cdots + x_n^2$;
- $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ - a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$ the corresponding differential operator;
- $\mathbb{B}, \mathbb{B}' \subset \mathbb{R}^n$ - open balls such that the closure $\bar{\mathbb{B}}' \subset \mathbb{B}$;
- $\mathsf{H}^\infty(\mathbb{B}) := \{u \in \mathsf{C}^\infty(\mathbb{B}) \mid \partial^\alpha u \in \mathsf{L}^2(\mathbb{B}), \quad \forall \alpha\}$;
- Δ - a fixed second order elliptic operator on $\mathbb{B}$;
- $u \mapsto \hat{u}$ the usual Fourier transform.

For $u \in \mathsf{C}_c^\infty(\mathbb{B}) \subset \mathsf{C}_c^\infty(\mathbb{R}^n)$, put

$$\|u\|_{\mathbb{B}}^2 := \|u\|_{(2,0),\mathbb{B}}^2 = \int_{\mathbb{B}} |u(x)|^2 dx = \int_{\mathbb{R}^n} |u(x)|^2 dx$$

and, more generally,

$$\|u\|_{(2,\mathsf{r}),\mathbb{B}} := \sum_{|\alpha|\le \mathsf{r}} \|\partial^\alpha u\|_{\mathbb{B}}, \quad \|u\|_{(2,-1)}^2 := \int_{\mathbb{R}^n} (1+|x|^2)^{-1} |\hat{u}(x)|^2 dx.$$

Since ∂^α preserves $\mathsf{H}^\infty(\mathbb{B})$, we can extend the norm $\|\cdot\|_{(2,\mathsf{r}),\mathbb{B}}$ to $\mathsf{H}^\infty(\mathbb{B})$.

PROPOSITION 8.1. — *For every* $\mathsf{r} \geq 2$*, there is a* $\mathsf{c} = \mathsf{c}_{\mathsf{r},\mathbb{B}} > 0$ *such that*

$$\|u\|_{(2,\mathsf{r}),\mathbb{B}'} \leq \mathsf{c}\left(\|u\|_{\mathbb{B}} + \|\Delta u\|_{(2,\mathsf{r}-2),\mathbb{B}}\right), \quad \textit{for all} \quad u \in \mathsf{H}^{\infty}(\mathbb{B}). \tag{8.1}$$

Proof. — Induction on r. Fix $\psi \in \mathsf{C}_c^{\infty}(\mathbb{B})$ with $0 \leq \psi \leq 1$ and $\psi = 1$ on $\bar{\mathbb{B}}'$. Let $v = u \cdot \psi \in \mathsf{C}^{\infty}(\mathbb{B})$. By Corollary 6.27, p. 267 in [6],

$$\|v\|_{(2,2),\mathbb{B}} \leq \mathsf{c}\,(\|\Delta v\|_{\mathbb{B}} + \|v\|_{\mathbb{B}})\,.$$

Next we have

$$\Delta v = \Delta(\psi \cdot u) = \psi \cdot \Delta u + \sum_{j=1}^{n} \psi_j \frac{\partial u}{\partial x_j} + \psi_0 \cdot u, \tag{8.2}$$

with ψ_j $(0 \leq j \leq n)$ fixed in $\mathsf{C}_c^{\infty}(\mathbb{B})$. To prove the assertion for $\mathsf{r} = 2$ it suffices to show that with ϕ fixed in $\mathsf{C}^{\infty}(\mathbb{B})$,

$$\|\phi \cdot \frac{\partial u}{\partial x_j}\|_{\mathbb{B}} \leq \mathsf{c}_1\,(\|u\|_{\mathbb{B}} + \|\Delta u\|_{\mathbb{B}})\,. \tag{8.3}$$

For this we write

$$\phi \cdot \frac{\partial u}{\partial x_i} = \frac{\partial (u\phi)}{\partial x_i} - u\frac{\partial \phi}{\partial x_i}$$

and note that $w := \phi \cdot u \in \mathsf{C}_c^{\infty}(\mathbb{B})$. Apply (6.25), p. 262 of [6] to obtain

$$\|w\|_{(2,1),\mathbb{B}} \leq \mathsf{c}\left(\|\Delta w\|_{(2,-1)} + \|w\|_{\mathbb{B}}\right).$$

The following inequality will imply (8.3):

$$\|\Delta w\|_{(2,-1)} \leq \mathsf{c}_1\,(\|\Delta u\|_{\mathbb{B}} + \|u\|_{\mathbb{B}})\,. \tag{8.4}$$

Again, for ϕ as above, we have

$$\Delta w = \Delta(\phi \cdot u) = \phi \cdot \Delta u + \sum_{j=1}^{n} \phi_j \frac{\partial u}{\partial x_j} + \phi_0 \cdot u,$$

with ϕ_j's fixed in $\mathsf{C}_c^{\infty}(\mathbb{B})$. To prove (8.4) it suffices to bound $\|\phi \cdot \frac{\partial u}{\partial x_j}\|_{(2,-1)}$ (for $\phi \in \mathsf{C}_c^{\infty}(\mathbb{B})$), or equivalently, $\|\frac{\partial w}{\partial x_j}\|_{(2,-1)}$. We have in fact

$$\begin{aligned}\|\frac{\partial w}{\partial x_j}\|_{(2,-1)}^2 &= \int_{\mathbb{R}^n} (1+|x|^2)^{-1} x_j^2 |\hat{w}(x)|^2 dx \\ &\leq \int_{\mathbb{R}^n} |\hat{w}(x)|^2 dx \leq \int_{\mathbb{R}^n} |w(x)|^2 dx \leq \|u\|_{\mathbb{B}}^2\end{aligned}$$

This completes the proof of (8.1) for $\mathsf{r} = 2$.

Suppose that $\mathsf{r} \geq 2$ and assume that the claim holds for all s with $2 \leq \mathsf{s} \leq \mathsf{r}$. Choose a ball $\mathbb{B}_1 \subset \mathbb{R}^n$ with

$$\mathbb{B}' \subset \bar{\mathbb{B}}' \subset \mathbb{B}_1 \subset \bar{\mathbb{B}}_1 \subset \mathbb{B}.$$

For $\psi \in \mathsf{C}^\infty(\mathbb{B}_1)$ and v as before, we have

$$\|u\|_{(2,\mathsf{r}),\mathbb{B}'} \leq \|v\|_{(2,\mathsf{r}),\mathbb{B}_1} \leq \mathsf{c}(\|\Delta v\|_{(2,\mathsf{r}-2),\mathbb{B}_1} + \|v\|_{\mathbb{B}_1}), \tag{8.5}$$

again, by Corollary 6.27 in [6]. We need only bound $\|\Delta v\|_{(2,\mathsf{r}-2),\mathbb{B}_1}$. For this we have first from (8.2)

$$\partial^\alpha(\Delta v) = \partial^\alpha(\psi \cdot \Delta u) + \partial^\alpha(\psi_0 \cdot u) + \sum_{j=1}^n \partial^\alpha(\psi_j \frac{\partial u}{\partial x_j}), \tag{8.6}$$

where now $\psi_j \in \mathsf{C}_c^\infty(\mathbb{B}_1)$ (for $0 \leq j \leq n$). It suffices to bound the L^2-norms of the terms on the right in (8.6) (for $|\alpha| \leq \mathsf{r} - 2$) in terms of $\|u\|_{\mathbb{B}}$ and $\|\Delta u\|_{(2,\mathsf{r}-2),\mathbb{B}}$. By Leibniz' rule,

$$\|\partial^\alpha(\psi_0 \cdot u)\|_{\mathbb{B}_1} \leq \mathsf{c}_2\|u\|_{(2,r-2),\mathbb{B}_1},$$

and we may use induction on r, provided $\mathsf{r} \geq 4$. Suppose then that $\mathsf{r} = 3$ and set $w = \psi_0 \cdot u \in \mathsf{C}_c^\infty(\mathbb{B}_1)$. Since here $|\alpha| \leq 1$, we have trivially,

$$\|\partial^\alpha w\|_{\mathbb{B}_1} \leq \|w\|_{(2,1),\mathbb{B}_1}.$$

Applying [6] once more, this time to $\mathbb{B}_1$,

$$\begin{aligned} \|w\|_{(2,1),\mathbb{B}_1} &\leq \mathsf{c}_1(\|\Delta w\|_{(2,-1)} + \|w\|_{\mathbb{B}_1}) \\ &\leq \mathsf{c}_2(\|\Delta u\|_{\mathbb{B}_1} + \|u\|_{\mathbb{B}_1}) \\ &\leq \mathsf{c}_2(\|\Delta u\|_{\mathbb{B}} + \|u\|_{\mathbb{B}}), \end{aligned}$$

(the second inequality follows from (8.4)). Next we have, with $|\alpha| \leq \mathsf{r}-2$,

$$\|\partial^\alpha(\psi\Delta u)\|_{\mathbb{B}} \leq \mathsf{c}_3\|\Delta u\|_{(2,\mathsf{r}-2),\mathbb{B}},$$

and finally

$$\|\partial^\alpha(\psi_j \frac{\partial u}{\partial x_j})\|_{\mathbb{B}_1} \leq \mathsf{c}_4\|u\|_{(2,\mathsf{r}-1),\mathbb{B}}$$

and we may apply the induction hypothesis to arrive at (8.1). □

Corollary 8.2. — *Suppose that* $\mathsf{r} > 0$ *is even. Then there is a constant* $\mathsf{c} = \mathsf{c}_\mathsf{r} > 0$ *such that for all* $u \in \mathsf{H}^\infty(\mathbb{B})$

$$\|u\|_{(2,\mathsf{r}),\mathbb{B}'} \leq \mathsf{c}\left(\|u\|_\mathbb{B} + \|\Delta u\|_\mathbb{B} + \cdots + \|\Delta^{\mathsf{r}/2} u\|_\mathbb{B}\right).$$

Proof. — We use induction on r. The case $\mathsf{r} = 2$ follows from Proposition 8.1. Suppose that $\mathsf{r} \geq 4$ and choose a ball $\mathbb{B}_1 \subset \mathbb{R}^n$ so that

$$\mathbb{B}' \subset \bar{\mathbb{B}}' \subset \mathbb{B}_1 \subset \bar{\mathbb{B}}_1 \subset \mathbb{B}.$$

By Proposition 8.1,

$$\|u\|_{(2,\mathsf{r}),\mathbb{B}'} \leq \mathsf{c}'_\mathsf{r}\left(\|u\|_{\mathbb{B}_1} + \|\Delta u\|_{(2,\mathsf{r}-2),\mathbb{B}_1}\right).$$

Further, we have by induction

$$\|\Delta u\|_{(2,\mathsf{r}-2),\mathbb{B}_1} \leq \mathsf{c}_{\mathsf{r}/2-1}\left(\sum_{j=0}^{\mathsf{r}/2-1} \|\Delta^{j+1} u\|_\mathbb{B}\right).$$

The claim follows combining these two inequalities. □

The following form of Sobolev's lemma will be useful for our purposes.

Proposition 8.3. — *Let* $\mathsf{r} > n/2$ *be an integer,* $\mathbb{B} \subset \mathbb{R}^n$ *an open ball and* $u \in \mathsf{H}^\infty(\mathbb{B})$*. Then* $u \in \mathsf{L}^\infty(\mathbb{B})$ *and there exists a* $\mathsf{c} = \mathsf{c}_{\mathsf{r},\mathbb{B}}$ *such that*

$$\sup_{x\in\mathbb{B}} |u(x)| \leq \mathsf{c}\|u\|_{(2,\mathsf{r}),\mathbb{B}}.$$

Corollary 8.4. — *Assume in addition that* r *is even. Then there exists a* $\mathsf{c} = \mathsf{c}_{\mathsf{r},\mathbb{B}}$*, such that*

$$\sup_{x\in\mathbb{B}'} |u(x)| \leq \mathsf{c}(\sum_{j=0}^{\mathsf{r}/2} \|\Delta^j u\|_\mathbb{B})$$

Let M be an n-dimensional manifold, ω a gauge form on M (a nowhere vanishing exterior n-form) and $d\mu$ the associated volume form. We define

$$\|f\|^2_\mathbb{B} := \int_\mathbb{B} |f(x)|^2 d\mu(x), \quad \text{and similarly} \quad \|f\|^2_M,$$

and the corresponding space $\mathsf{L}^2(M) = \mathsf{L}^2(M,\omega)$. Here $\mathbb{B}$ is a coordinate ball in M. We also fix a ball $\mathbb{B}'$ such that $\mathbb{B}' \subset \bar{\mathbb{B}}' \subset \mathbb{B}$. Let $\mathcal{G}$ be the Lie algebra of vector fields on M (for each $x \in M$, $\mathcal{G}_x = \mathcal{T}_x$, the tangent space to M at x). Let $\mathfrak{U} = \mathfrak{U}(\mathcal{G})$ be its universal enveloping algebra, regarded as the algebra of differential operators on M. Define $\mathsf{H}^\infty(M)$ to be the space of all C^∞-functions $f : M \to \mathbb{C}$ such that $\partial f \in \mathsf{L}^2(M)$ for all $\partial \in \mathfrak{U}$. We fix a second order elliptic operator $\Delta \in \mathfrak{U}$.

Proposition 8.5. — *Let $\mathsf{r} > n/2$ be an even integer. Then there exists a $\mathsf{c} = \mathsf{c}_{\mathbb{B}} > 0$ such that for all $f \in \mathsf{H}^\infty(M)$ one has*

$$\sup_{x\in\mathbb{B}'} |f(x)| \leq \mathsf{c} \left(\sum_{j=0}^{\mathsf{r}/2} \|\Delta^j f\|_M \right).$$

Proof. — Easy consequence of Corollary 8.4. □

Corollary 8.6. — *Let K be a compact and $\mathsf{r} > n/2$ an even integer. Then there exists a constant $\mathsf{c} = \mathsf{c}_K > 0$ such that for all $f \in \mathsf{H}^\infty(M)$ one has*

$$\sup_{x\in K} |f(x)| \leq \mathsf{c} \left(\sum_{j=0}^{\mathsf{r}/2} \|\Delta^j f\|_M \right).$$

Corollary 8.7. — *For any even integer $\mathsf{r} > n/2$ there exists a constant $\mathsf{c} = \mathsf{c}_{r,\mathbb{B}} > 0$ such that for any Δ-eigenfunction $f = f_\lambda \in \mathsf{C}^\infty(M)$ with eigenvalue $\lambda \neq 0$ and any $\mathbb{B}' \subset \mathbb{B}$ as above one has*

$$\sup_{x\in\mathbb{B}'} |f(u)| \leq \mathsf{c}|\lambda|^{\mathsf{r}/2}\|f\|_{\mathbb{B}}$$

Let M, ω, Δ be as above. Assume that

$$\langle \Delta f, f' \rangle = \langle f, \Delta f' \rangle, \quad \text{for all} \quad f, f' \in \mathsf{H}^\infty(M),$$

and let $\{f_j\}_{j\geq 1}$ be an orthonormal sequence of Δ-eigenfuctions in $\mathsf{H}^\infty(M)$. Let $\mathsf{H}_0(M) \subset \mathsf{L}^2(M)$ be the Hilbert subspace spanned by the f_j's and

$$\mathsf{H}_{0,\infty} := \{f \in \mathsf{H}^\infty(M) \,|\, \Delta^j f \in \mathsf{H}_0(M) \quad \text{for all} \quad j\}.$$

Proposition 8.8. — *Suppose that $f \in \mathsf{H}_{0,\infty}(M)$. Then the series*

$$\sum_{j\geq 1} \langle f, f_j \rangle f_j$$

converges uniformly on compacts, and in particular, pointwise, to f.

Proof. — Write $\Delta f_j = \lambda_j f_j$ (and note that $\lambda_j \in \mathbb{R}$). For $f \in \mathsf{H}_{0,\infty}(M)$ and $\mathsf{n} \in \mathbb{N}$ we set

$$f^{[\mathsf{n}]} := \sum_{j=1}^{\mathsf{n}} \langle f, f_j \rangle f_j.$$

We have

$$\Delta^{\mathsf{k}} f^{[\mathsf{n}]} = \sum_{j=1}^{\mathsf{n}} \langle \Delta^{\mathsf{k}} f, f_j \rangle f_j = \sum_{j=1}^{\mathsf{n}} \langle f, \Delta^{\mathsf{k}} f_j \rangle f_j = \sum_{j=1}^{\mathsf{n}} \langle f, f_j \rangle \lambda_j^{\mathsf{k}} f_j = \Delta^{\mathsf{k}} f^{[\mathsf{n}]}.$$

Consequently, for given $\epsilon > 0$, we have

$$\|\Delta^{\mathsf{k}}(f - f^{[\mathsf{n}]})\|_M \leq \epsilon$$

for $1 \leq \mathsf{k} \leq \mathsf{r}/2$ (with r as above), provided $\mathsf{n} \gg 0$. Since $f - f^{[\mathsf{n}]}$ belongs to $\mathsf{H}^{\infty}(M)$, the proof follows immediately from Corollary 8.6. □

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