RATIONALITY OF MODULI OF ELLIPTIC FIBRATIONS WITH FIXED MONODROMY

by

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ABSTRACT. — We prove rationality results for moduli spaces of elliptic K3 surfaces and elliptic rational surfaces with fixed monodromy groups.

Contents

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0. Introduction

Let X be an algebraic variety of dimension n over $\mathbb{C}$. One says that X is rational if its function field $\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(x_1, \ldots, x_n)$. The study of rationality properties of fields of invariants $\mathbb{C}(X)^{\mathrm{G}} = \mathbb{C}(X/\mathrm{G})$ is a classical theme in algebraic geometry. For a finite group $\mathrm{G} \subset \mathrm{PGL}_n$ acting on $X = \mathbb{P}^{n-1}$ the problem is referred to as Noether's problem (1916). It is still unsolved for $n = 4$. Another class of examples is provided by *moduli spaces*. Birationally, they are often representable as quotients of simple varieties, like projective spaces or Grassmannians, by actions of linear algebraic groups, like PGL_2. Rationality is known for each of the following moduli spaces:

- curves of genus ≤ 6 [**18**], [**32**], [**20**], [**21**], [**31**];
- hyperelliptic curves [**18**], [**7**];
- plane curves of degrees $4n + 1$ and $3n$ [**33**], [**19**];
- Enriques surfaces [**24**];
- polarized K3 surfaces of degree 18 [**32**];
- stable vector bundles (with various numerical characteristics) on curves, Del Pezzo surfaces, $\mathbb{P}^3$ [**22**], [**5**], [**11**],[**25**],[**29**];

and in many other cases. For excellent surveys we refer to [**12**] and [**33**]. We will study rationality properties of moduli spaces of smooth nonisotrivial Jacobian elliptic fibrations over curves

$$\pi \colon \mathcal{E} \to C$$

with fixed global monodromy group $\tilde{\Gamma} = \tilde{\Gamma}(\mathcal{E}) \subset \mathrm{SL}_2(\mathbb{Z})$. In [**8**] we developed techniques aimed at the classification of possible global monodromies $\tilde{\Gamma}$. The present paper gives a natural application of these techniques.

Let $\mathcal{B}$ be an irreducible algebraic family of Jacobian elliptic surfaces. Then the set of subgroups $\tilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\tilde{\Gamma}$ is the (global) monodromy group of some $\mathcal{E}$ in this family is finite. Moreover, for every such group $\tilde{\Gamma}$ the subset of fibrations with this monodromy

$$\mathcal{B}_{\tilde{\Gamma}} := \{b \in \mathcal{B} \mid \tilde{\Gamma}(\mathcal{E}_b) = \tilde{\Gamma}\}$$

is an algebraic (not necessarily closed) subvariety of $\mathcal{B}$.

Generalizing this observation, we introduce (maximal) *parameter spaces* $\mathcal{F}_{\tilde{\Gamma}}$ of elliptic fibrations with fixed global monodromy $\tilde{\Gamma}$ (considered up to fiberwise birational transformations acting trivially on the base of the elliptic fibration). These parameter spaces can be represented as quotients of quasi-projective varieties by algebraic groups. In particular, we can consider *irreducible connected components* of the parameter space $\mathcal{F}_{\tilde{\Gamma}}$, which we call *moduli spaces*. Even though these moduli spaces need not be algebraic varieties, we can still make sense of their birational type.

Theorem. *Let $\tilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z})$ be a proper subgroup of finite index. Then all moduli spaces of (Jacobian) elliptic rational or elliptic K3 surfaces with global monodromy $\tilde{\Gamma}$ are rational.*

Notice that the finite index condition in the theorem is not a restriction since it always holds for nonisotrivial Jacobian elliptic fibrations, considered in this paper.

Corollary. *For all $\tilde{\Gamma}$ with moduli $\mathcal{F}_{\tilde{\Gamma}}$ of dimension > 0 there exists a number field K such that there are infinitely many nonisomorphic elliptic K3 surfaces over K with global monodromy $\tilde{\Gamma}$.*

REMARK 0.1. — Our method shows that many other classes of moduli of elliptic surfaces over $\mathbb{P}^1$ with fixed monodromy are rational or unirational. However, we cannot expect a similar result for all moduli spaces of elliptic surfaces over higher genus curves, since the moduli space of higher genus curves itself is not uniruled (by a result of Harris and Mumford [15]).

We proceed to give a more detailed description of our approach. First of all, we can work not with the monodromy group $\tilde{\Gamma}$ itself but rather with its image

$$\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$$

under the natural projection $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$. Let

$$\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

be the upper half-plane and

$$\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}.$$

The natural j-map

$$j \;:\; C \to \mathbb{P}^1 = \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z})$$

decomposes as

$$j = j_\Gamma \circ j_{\mathcal{E}},$$

where

$$\begin{array}{rcccl} j_{\mathcal{E}}: & C & \to & M_\Gamma & = & \overline{\mathcal{H}}/\Gamma \\ j_\Gamma: & M_\Gamma & \to & \mathbb{P}^1 & = & \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z}). \end{array}$$

Here M_Γ is the *j-modular curve* corresponding to Γ; it is equipped with a special triangulation, obtained as the pullback of the standard triangulation of $\mathbb{S}^2 = \mathbb{P}^1(\mathbb{C})$ (by two triangles with vertices at $0, 1$ and ∞) under the map j_Γ (which ramifies only over $0, 1$ and ∞). We call the obtained triangulation of M_Γ a j_Γ-triangulation. Let T_Γ be the preimage in M_Γ of the closed interval $[0, 1] \subset \mathbb{P}^1$. The graph T_Γ is our main tool in the combinatorial analysis of Γ.

Denote by $\chi(\mathcal{E})$ the Euler characteristic of $\mathcal{E}$. It splits equivalence classes of Jacobian elliptic surfaces (modulo fiberwise birational transformations) into *algebraic families*. In particular, if $C = \mathbb{P}^1$ then the *algebraic variety* $\mathcal{F}_r$ parametrizing (equivalence classes of) Jacobian elliptic surfaces with given $\chi(\mathcal{E})$ is irreducible; here we put $r = \chi(\mathcal{E})/12$. Our goal is to analyze the birational type of (irreducible components)

$$\mathcal{F}_{r,\tilde{\Gamma}} \subset \mathcal{F}_r$$

parametrizing fibrations with fixed monodromy group $\tilde{\Gamma}$. It suffices to study parameter spaces $\mathcal{F}_{r,\Gamma}$ corresponding to $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$, since every irreducible component of $\mathcal{F}_{r,\tilde{\Gamma}}$ coincides with a component of $\mathcal{F}_{r,\Gamma}$.

From now on we assume that $C = M_\Gamma = \mathbb{P}^1$. Denote by $\mathcal{R}_{d,\Gamma}$ the space of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ (of degree d) with prescribed ramification (encoded in T_Γ). The spaces $\mathcal{F}_{r,\Gamma}$ are quotients, by $\mathrm{PGL}_2 \times H_\Gamma$, of fibrations over $\mathcal{R}_{d,\Gamma}$ with fibers (Zariski open subsets of) $\mathrm{Sym}^\ell(\mathbb{P}^1)$ (for appropriate d and ℓ). Here PGL_2 acts (on the left) by changing the parameter on the base $C = \mathbb{P}^1$ and H_Γ is the group of automorphisms of $M_\Gamma = \mathbb{P}^1$ stabilizing the embedded graph T_Γ (acting on the right). The nontriviality of H_Γ means that there is a $\Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$ containing Γ as a normal subgroup with $H_\Gamma = \Gamma'/\Gamma$. So in most cases in order to prove rationality of $\mathcal{F}_{r,\Gamma}$ it is sufficient to establish it for $\mathrm{PGL}_2\backslash\mathcal{R}_{d,\Gamma}$, which can be deduced from general rationality results for PGL_2-quotients (see [9], [18]). To cover *all* cases we need to set up a rather extensive combinatorial analysis.

Here is a roadmap of the paper. In Section 1 we discuss finite covers $M_\Gamma \to \mathbb{P}^1$ in the spirit of Grothendieck's "Dessins d'Enfants" program (see [28],[34] and the references therein) and introduce the invariants $\mathrm{GD}(\Gamma), \mathrm{RD}(\Gamma)$ and $\mathrm{ET}(\Gamma)$. In "ideal" cases $\mathrm{ET}(\Gamma)$ coincides with the number of triangles in the j_Γ-triangulation of M_Γ (the notation $\mathrm{ET}(\Gamma)$ stands for "*Effective Triangles*"). In Section 2 we recall basic facts about elliptic fibrations and introduce the invariant $\mathrm{ET}(\mathcal{E})$. For an "ideal" elliptic fibration one has $\mathrm{ET}(\Gamma) = \mathrm{ET}(\mathcal{E})$. In Section 3 we discuss *moduli* of elliptic fibrations with fixed monodromy. In Sections 7 and 8 we formulate and prove several rationality results for PGL_2 and related quotients. In Section 5 we classify families of rational elliptic surfaces and elliptic K3 surfaces with different monodromy groups. In Section 4, we study relations between the combinatorics of the graph Γ and the topology of $\mathcal{E}$. And finally, in Section 10 we list (certain) relevant subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ (represented by trivalent graphs T_Γ). There are too many monodromy groups of elliptic K3 surfaces to be drawn on paper, but we show how to obtain them from our list by simple operations.

1. Finite covers

Let Γ be a subgroup of finite index in $\mathrm{PSL}_2(\mathbb{Z})$. The latter is isomorphic to a free product $\mathbb{Z}/3 * \mathbb{Z}/2$. Consider the map

$$\overline{\mathcal{H}}/\Gamma = M_\Gamma \xrightarrow{j_\Gamma} \mathbb{P}^1 = \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z}),$$

ramified over the points $0, 1, \infty \in \mathbb{P}^1$. Denote their preimages in M_Γ by A, B and I, respectively. The possible ramification orders are 3 or 1 for A-points, 2 or 1 for B-points and arbitrary for I-points. The points $0, 1$ and ∞ subdivide the circle $\mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1$ into three segments and, together with the upper and lower hemisphere, define a decomposition of $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^2$ into three triangles. This induces a special triangulation of M_Γ with vertices in A, B and I-points which we call the *j_Γ-triangulation*. The preimage of the segment $[0,1] \subset \mathbb{P}^1$ defines a graph T_Γ which determines the j_Γ-triangulation uniquely. Interior vertices of T_Γ are marked by A_6 and ends are marked by either A_2 or B_2.

NOTATIONS 1.1. — The *graph datum* $\mathrm{GD}(\Gamma)$ of T_Γ is the formal sum

$$\mathrm{GD}(\Gamma) := [a_6 A_6 + a_2 A_2 + b_2 B_2],$$

where a_i ($i = 6, 2$) is the number of A_i vertices and b_2 is the number of B_2-vertices. Denote by $\tau^0 = \tau^0(\Gamma)$ the number of vertices of T_Γ (including the ends), by $\tau^1 = \tau^1(\Gamma)$ the number of edges and by $\tau^2 = \tau^2(\Gamma) = \pi_0(M_\Gamma \setminus T_\Gamma)$.

REMARK 1.2. — For given a_2, b_2 there is a unique group with

$$\mathrm{GD}(\Gamma) = [A_6 + a_2 A_2 + b_2 B_2].$$

Forgetting the markings of T_Γ we obtain a connected unmarked *topological* graph T_Γ^u with (possibly some) ends and all interior vertices of valency 3 — a *trivalent graph*.

LEMMA 1.3. — *Let X be a compact orientable Riemann surface of genus $g(X)$ and $T^u \subset X$ an embedding of a connected trivalent graph such that*

- *the set $X \setminus T^u$ is a disjoint union of topological cells;*
- *all interior vertices of T^u are trivalent;*
- *the ends of T^u are arbitrarily marked by two colors A_2 and B_2.*

Then there exist a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ and a unique complex structure on X such that $X = M_\Gamma$ and $T^u = T_\Gamma^u$.

Proof. — Assume that we have an embedded graph $T^u \subset X$ satisfying the conditions above. Mark by A all trivalent vertices and enlarge the graph T^u by putting a B-vertex in the middle of any edge bounded by two A-vertices. Put one I-vertex into every connected component of $X \setminus T^u$ and connect all I-vertices with A and B-vertices at the boundary of the corresponding domain. By assumption, every connected component of $X \setminus T^u$ is contractible. Consider the boundary of the individual cell. Every A-vertex of the boundary is connected by edges to B-vertices only. Similarly, the B-vertices are connected by edges only to A-vertices. Hence every triangle of the induced triangulation has vertices colored by three colors: A, B and I. This gives a j_Γ-triangulation of X. Following Alexander [1], we observe that a j_Γ-triangulation defines a map

$$h\,:\,X \to \mathbb{P}^1$$

which is cyclically ramified over A, B and I (see [8]). The trivalence of T^u implies that h has only 3 or 1-ramifications over $0 \in \mathbb{P}^1$ and only 2 or 1-ramifications over $1 \in \mathbb{P}^1$. Since $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/3 * \mathbb{Z}/2$ there is exactly one subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ (of finite index) which corresponds to the covering $X \to \mathbb{P}^1$. Any graph T_Γ^u constructed via a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ satisfies the conditions above. Indeed, we have already described the j_Γ-triangulation on M_Γ. Triangles adjacent to a given I-vertex constitute a contractible cell and the division of M_Γ into neighborhoods of I-vertices is a cellular decomposition of M_Γ. Hence after removing I-vertices with open edges from them we obtain the preimage of $[0,1]$. If we forget the B-vertices which lie between two A-vertices we obtain the graph T_Γ^u. Thus $T_\Gamma^u \subset X = M_\Gamma$ is the boundary of this cellular decomposition and T_Γ is simply T_Γ^u with an A, B-marking of the ends. □

REMARK 1.4. — Graphs which are isotopic in X (modulo diffeomorphisms of X of degree 1) define conjugated subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

REMARK 1.5. — Even if we omit the condition of compactness of X we still get a bijection between conjugacy classes of subgroups of finite index of $\mathrm{PSL}_2(\mathbb{Z})$ and embedded trivalent graphs with marked ends.

REMARK 1.6. — The topology of X restricts the topology of T^u_Γ. The graph T^u_Γ must contain some 1-skeleton of X. In particular, the map $\pi_1(T^u_\Gamma) \to \pi_1(X)$ is surjective. Hence T^u_Γ can be a tree only if $X = \mathbb{S}^2$.

For $X = \mathbb{P}^1$ the connectedness of T^u guarantees that all the components of $X \backslash T^u$ are contractible. Hence we can classify graphs in $X = \mathbb{P}^1$ by drawing them on the plane. In general, connectedness of T^u is necessary but not sufficient.

DEFINITION 1.7. — *Define*

$$\begin{aligned} \mathrm{ET}(\Gamma) &:= 6\tau^0 = 6(a_6 + a_2 + b_2) \\ \Delta(\Gamma) &:= 6a_6 + 2a_2 \end{aligned}$$

Thus both $\mathrm{ET}(\Gamma)$ and $\Delta(\Gamma)$ depend only on the marking of the ends but not on the embedding of the graph. Observe that $\Delta(\Gamma)$ is the *number of triangles* in the corresponding j_Γ-triangulation of M_Γ and that

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma] = \Delta(\Gamma)/2.$$

REMARK 1.8. — If M_Γ arises from an elliptic fibration as in the Introduction then $\Delta(\Gamma)/2$ equals the number of *Dehn twists* in Γ around the multiplicative singular fibers.

NOTATIONS 1.9. — Let $f : C \to \mathbb{P}^1$ be a cover of degree d and $p \in \mathbb{P}^1$ a ramification point of f. The *local ramification datum* is an $\mathbb{N}$-valued vector $v = (v_k)$, ($\sum v_k = d$), where v_k is the order of ramification of f at a point $c_k \in f^{-1}(p)$. A *reduced local ramification datum* is a vector $\overline{v}$ obtained from v by omitting all entries $v_k = 1$. The vector v is defined up to permutation of the entries.

For $f = j_\mathcal{E} : C \to M_\Gamma = \mathbb{P}^1$ we have distinguished ramification points, namely those over A- and B-vertices of the graph $T_\Gamma \subset M_\Gamma$. The (global) $j_\mathcal{E}$-ramification datum is the vector

$$\mathrm{RD}(j_\mathcal{E}) := [v_{1,A}, \ldots, v_{n,A}, v_{n+1,B}, \ldots, v_{n',B}, \overline{v}_{n'+1}, \ldots, \overline{v}_{n''}],$$

where the $v_{i,A}$ are local ramification data over A-vertices for $i = 1, \ldots, n$, (resp. $v_{i,B}$ for B-vertices, $i = n+1, \ldots, n'$) and $\overline{v}_i$ are *reduced* local ramification data for unspecified other points in M_Γ for $i > n'$ (distinct from A- and B-vertices of M_Γ).

For $f = j_\Gamma : M_\Gamma \to \mathbb{P}^1$ the distinguished (and the only) ramification points are $0, 1, \infty$. We write

$$\mathrm{RD}(j_\Gamma) := [v_0, v_1, v_\infty]$$

for the global j_Γ-ramification datum.

EXAMPLE 1.10. — Assume that $\mathrm{GD}(\Gamma) = [nA_6 + A_2 + B_2]$ is the graph datum of $T_\Gamma \subset M_\Gamma$ and let $j_\mathcal{E} : \mathbb{P}^1 \to M_\Gamma = \mathbb{P}^1$ be a finite cover. Then the $j_\mathcal{E}$-ramification datum

$$\mathrm{RD}(j_\mathcal{E}) = [(2,3)_A, (2,2,1)_B, (2), (2)]$$

means that $\deg(j_\mathcal{E}) = 5$, that $j_\mathcal{E}$ has ramification points of order 2 and 3 over one point $A_2 \in T_\Gamma$ and $(2,2,1)$ over one B_2-point and ramifications of order 2 over two other unspecified points in M_Γ.

2. Elliptic fibrations

In this section we briefly recall some basic facts of Kodaira's theory [23] of elliptic fibrations. For more details we refer to [3], [14] and [35]. Let

$$\pi\colon \mathcal{E} \to C$$

be a smooth nonisotrivial relatively minimal Jacobian elliptic fibration over a smooth projective curve C. This means that:

- $\mathcal{E}$ is a smooth compact complex projective surface and π is a proper holomorphic map;
- the generic fiber of π is a smooth curve of genus 1;
- the fibers of $\mathcal{E}$ do not contain exceptional curves of the first kind, i.e., rational curves F such that $(F^2) = -1$ (*relative minimality*);
- there exists a (global) zero section $s : C \to \mathcal{E}$ (*Jacobian elliptic fibration*);
- the j-function which assigns to each smooth fiber $\pi^{-1}(p) = \mathcal{E}_p \subset \mathcal{E}$ its j-invariant is a nonconstant rational function on C (*nonisotriviality*).

It is well known that $s^2 < 0$. We define

$$\mathrm{ET}(\mathcal{E}) := -24s^2.$$

LEMMA 2.1. — *We have*

$$\mathrm{ET}(\mathcal{E})/2 = -12s^2 = \chi(\mathcal{E}) = c_2(\mathcal{E}).$$

Proof. — Well known, but we decided to include an argument. Since $\mathcal{E}$ is smooth and relatively minimal its canonical bundle $K_{\mathcal{E}}$ is induced from a one-dimensional bundle K on the base C. The sheaf $\pi^*K(C)$ is a subsheaf of $K_{\mathcal{E}}$. Since there are singular fibers we have the following equality

$$h^0(\mathcal{E},\Omega^1) = h^1(\mathcal{E},\mathcal{O}) = g$$

where g is the genus of C. By Riemann-Roch we obtain

$$\chi(\mathcal{O}) = 1 - g + h^0(\mathcal{E}, K_{\mathcal{E}}) = \chi(\mathcal{E})/12.$$

We also know that $s^2 + sK_{\mathcal{E}} - 2g + 2 = 0$ (genus formula). Therefore,

$$1 - g + h^0(\mathcal{E}, K_{\mathcal{E}}) = \deg(K) - 2g + 2 = \chi(\mathcal{E})/12$$

since $\deg(K) > 2g - 2$ and hence $h^1(C, K) = 0$. Further,

$$sK_{\mathcal{E}} = \deg(K).$$

Thus $s^2 + sK_{\mathcal{E}} - 2g + 2 = 0$ transforms to $s^2 + \chi(\mathcal{E})/12 = 0$. □

Let $C^{\mathrm{sing}} = \{p_1, \ldots, p_k\} \subset C$ be the set of points on the base corresponding to singular fibers. The topological Euler characteristic $\chi(\mathcal{E}) = c_2(\mathcal{E})$ is equal to the sum of Euler characteristics of the singular fibers $\mathcal{E}_{p_i} = \pi^{-1}(p_i)$ (since every generic fiber has Euler characteristic equal to 0). Therefore,

$$\mathrm{ET}(\mathcal{E}) = \sum_{p_i \in C^{\mathrm{sing}}} \mathrm{ET}(\mathcal{E}_{p_i}),$$

where the summation runs over all singular fibers of $\mathcal{E}$ and $\mathrm{ET}(\mathcal{E}_{p_i})$ is the contribution from the corresponding singular fiber. Since the fibration is Jacobian every singular fiber has a unique representative from Kodaira's list and it is defined by the local monodromy. The possible types of singular fibers and their ET-contributions are:

	ET		ET
I_0		I_0^*	12
I_n	$2n$	I_n^*	$2n+12$
II	4	IV^*	16
III	6	III^*	18
IV	8	II^*	20

Here I_0 is a smooth fiber, I_n is a multiplicative fiber with n-irreducible components. The types II, III and IV correspond to the case of potentially good reduction. More precisely, the neighborhood of such a fiber is a (desingularization of a) quotient of a local fibration with smooth fibers by an automorphism of finite order. The corresponding order is 4 for the case III and 3 in the cases II, IV. The fibers of type I_0^*, (resp. $\mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$) are obtained from fibers I_0 (resp. $\mathrm{I}_n, \mathrm{IV}, \mathrm{III}, \mathrm{II}$) (after changing the local automorphism by the involution $x \mapsto -x$ in the local group structure of the fibration). We shall call them $*$-*fibers* in the sequel.

REMARK 2.2. — The invariant $\mathrm{ET}(\mathcal{E}_p)$ has a monodromy interpretation. Namely, every element of a local monodromy at $p \in C^{\mathrm{sing}}$ has a minimal representation as a product of elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. The length of this representation equals $\mathrm{ET}(\mathcal{E}_p)/2$. This explains the equality $\mathrm{ET}(\mathcal{E}_p^*) = \mathrm{ET}(\mathcal{E}_p) + 12$ — the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is a product of 6 elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (elementary Dehn twists).

3. Moduli spaces

Every Jacobian elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ admits a *Weierstrass model* $\bar{\mathcal{E}}$. Its geometric realization is given as follows: there exists a pair of sections

$$\begin{array}{rcl} g_2 & \in & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)), \\ g_3 & \in & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r)) \end{array}$$

such that $\mathcal{E}$ is given by

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \tag{3.1}$$

inside $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r))$, subject to conditions

- the discriminant $\Delta = g_2^3 - 27g_3^2$ is not identically 0;
- for every point $p \in \mathbb{P}^1$ we have

$$\min(3\nu_p(g_2), 2\nu_p(g_3)) < 12, \tag{3.2}$$

where ν_p is the valuation corresponding to $p \in \mathbb{P}^1$

(see [14] or [13], Section 7).

Two pairs (g_2, g_3) and (g_2', g_3') define isomorphic Jacobian elliptic surfaces $(\mathcal{E}, s)$ and $(\mathcal{E}', s')$ iff there exits an $h \in \mathrm{GL}_2(\mathbb{C})$ transforming (g_2, g_3) into (g_2', g_3') under the natural action of GL_2 on (the GL_2-linearized) $\mathcal{O}_{\mathbb{P}^1}(r)$. We define $\mathcal{F}_r$ as the set of isomorphism classes of pairs (g_2, g_3) subject to the conditions above.

The parameter space $\mathcal{F}_r$ has a natural structure of a (categorical) quotient of some open subvariety U_r of the sum of two linear GL_2-representations

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r))$$

by the action of GL_2. Equivalently, $\mathcal{F}_r$ is a (categorical) quotient of the open subvariety $U'_r = U_r/\mathbb{G}_m$ of the weighted projective space

$$\mathbb{P}_{4r,6r}(4r+1, 6r+1)$$

by the action of PGL_2.

LEMMA 3.1. — *The variety U'_r is a disjoint union of locally closed subvarieties $U'_{r,\tilde{\Gamma}}$, each preserved under the action of PGL_2, such that for every $u \in U'_{r,\tilde{\Gamma}}$ one has $\tilde{\Gamma}(\mathcal{E}_u) = \tilde{\Gamma}$.*

Proof. — The monodromy group of an elliptic fibration does not change under deformations preserving the topological type of singular fibers (it is encoded in the topology of the smooth part). Thus it can change only on algebraic subvarieties where the topological type of singular fibers changes. The monodromy group of $\mathcal{E}_u$ is upper-semicontinuous under changes of u - it can only drop on a closed subset of the parameter space. For $u \in U'_r$ the corresponding j-map has a decomposition $j_u = j_\Gamma \circ j_{\mathcal{E}}$. By (3.1),(3.2), j_u is nonconstant of degree $\leq 12r$. Thus

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma] = \deg(j_\Gamma) \leq 12r$$

and the set of possible Γ is finite. Similarly, there are finitely many $\tilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z})$ which can appear as monodromy groups of elliptic fibrations parametrized by U'_r. Since Γ defines j_Γ there are only finitely many possible j_Γ which can appear for $u \in U'_r$. For every fixed j_Γ, a decomposition of $j_u = j_\Gamma \circ s, u \in U_r$, where s is a rational map $s : \mathbb{P}^1 \to \mathbb{P}^1, \deg(s) \geq 1$, determines a $\mathbb{G}_m$-homogeneous set of algebraic equations on the pair (g_2, g_3) defining the point $u \in U_r$. Thus there is a closed algebraic subvariety $U_{r,\Gamma}$ where $j_u = j_\Gamma \circ s$. The monodromy of the fibration $\mathcal{E}_u$ for $u \in U_{r,\Gamma}$ surjects onto Γ unless u is in $U_{r,\Gamma'}$, where $\Gamma' \subset \Gamma$. There are finitely many such Γ' (with the above degree bound) and hence finitely many algebraic varieties $U_{r,\Gamma'}$ such that for every $u \in U_{r,\Gamma} - \bigcup U_{r,\Gamma'}$ the monodromy group of $\mathcal{E}_u$ surjects onto $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$.

Let M be an irreducible component of $U'_{r,\Gamma} - \bigcup U'_{r,\Gamma'}$. The monodromy group of $\mathcal{E}_u, u \in M$ is either constant on M or drops when the topology of singular fibers changes. This can occur only on a finite number of closed algebraic irreducible subvarieties M_i in M. Since the monodromy group surjects onto Γ it can only drop from the group $\tilde{\Gamma}$ at a generic point if the map $\tilde{\Gamma} \to \Gamma$ has a kernel of order 2. If the above map is an isomorphism then the monodormy group is constant on M. If the monodromy group on M does change on M_i then $\tilde{\Gamma}_i$ (for a generic $u \in M_i$) is isomoprphic to Γ and hence does not change on M_i. The varieties M and M_i are all preserved under the PGL_2-action. Thus there is one monodromy group $\tilde{\Gamma}$ for a generic point of M. Renaming $M - \bigcap M_i$ as $U'_{r,\tilde{\Gamma}}$ and M_i as $U'_{r,\tilde{\Gamma}_i}$ we obtain the algebraic stratification of the lemma. □

The unstable points of the PGL_2-action on the weighted projective space correspond to sections g_2, g_3 with high order of vanishing at some point p. Namely $\nu_p(g_2) > 2r, \nu_p(g_3) > 3r$. However, the inequality (3.2) implies that $6r < 12$. Thus, for $r \geq 2$, $\mathcal{F}_r$ is a PGL_2-quotient of some open subvariety of the semistable locus

$$\mathbb{P}^{\mathrm{ss}}_{4r,6r}(4r+1, 6r+1) \subset \mathbb{P}_{4r,6r}(4r+1, 6r+1).$$

It follows that $\mathcal{F}_r$ is a quasi-projective algebraic variety. This variety is clearly unirational and in fact rational by [18].

Moreover, for $r \geq 2$ we can define a set of *subvarieties* $\mathcal{F}_{r,\tilde{\Gamma}} \subset \mathcal{F}_r$ such that for every $b \in \mathcal{F}_{r,\tilde{\Gamma}}$ the corresponding Jacobian elliptic surface $(\mathcal{E}_b, s)$ has global monodromy group $\tilde{\Gamma}$.

REMARK 3.2. — Notice that the maps $j_{\mathcal{E}}$ for elliptic fibrations corresponding to different points of the same irreducible component of $\mathcal{F}_{r,\tilde{\Gamma}}$ can have different $\mathrm{RD}(j_{\mathcal{E}})$, even over the A_2 or B_2-ends of $T_\Gamma \subset M_\Gamma$. Thus, for a given irreducible component, we have the notion of a *generic* ramification datum $\mathrm{RD}(j_{\mathcal{E}})$ and its *degenerations*.

The case $r = 1$, corresponding to rational elliptic surfaces, is more subtle - the subvariety U'_1 contains unstable points. The quasi-projective locus of semistable points $U^{\mathrm{ss}\prime}_r$ is a disjoint union of locally closed PGL_2-semistable subsets $U^{\mathrm{ss}}_{r,\tilde{\Gamma}}$; taking quotients we obtain varieties $\mathcal{F}_{1,\tilde{\Gamma}}$ parametrizing rational elliptic fibrations with global monodromy $\tilde{\Gamma}$.

Let $W'_1 = U'_1 - U^{\mathrm{ss}\prime}_1$ be the complement. It consists of pairs (g_2, g_3) with

$$g_2 = l^3 f_2, \quad g_3 = l^4 f_3,$$

where l is a linear form (vanishing at a point p and) coprime to f_2, f_3 and $\deg(f_2) = 1, \deg(f_3) = 2$. For $w \in W'_1$ we have $\deg(j) \leq 4$. The case of $\tilde{\Gamma} \neq \mathrm{SL}_2(\mathbb{Z})$ corresponds to $\deg(j_\Gamma) \geq 2$. Thus we have to consider two cases:

- $\deg(j_\Gamma) = \deg(j_{\mathcal{E}}) = 2$;
- $\deg(j_\Gamma) \leq 4, \deg(j_{\mathcal{E}}) = 1$.

The first case does not occur since $j^{-1}(0)$ has ramification of type $(3,1)$ (by the assumption that f_2 is coprime to l and that $3\nu_p(g_2) < 12$). Thus the j-map cannot be decomposed even locally into a product of two maps. The second case leads to

LEMMA 3.3. — *If $w \in W_1'$ and $\tilde{\Gamma}(\mathcal{E}_w) \neq \mathrm{SL}_2(\mathbb{Z})$ then $\deg(j_{\mathcal{E}}) = 1$ and one has one of the following graph and ramification data:*

$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_\Gamma)$
$[A_6 + A_2]$	$[(3,1)_0, (2,2)_1, (3,1)_\infty]$
$[A_6 + A_2 + 2B_2]$	$[(3,1)_0, (2,1,1)_1, (4)_\infty]$
$[A_6 + 3B_2]$	$[(3)_0, (1,1,1)_1, (3)_\infty]$
$[A_6 + B_2]$	$[(3)_0, (2,1)_1, (2,1)_\infty]$

Proof. — The formula $j = lf_2^3/(lf_2^3 - f_3^2)$ shows that j_Γ has a point with local ramification datum $(3,1)$ or (3), corresponding to

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(\mathcal{E})] = 4 \text{ or } 3.$$

Since only two more branch points are allowed and one of them is 1 (with local ramifications 1 or 2), the Euler characteristic computation gives the ramification data listed in the statement plus one more:

$$[(3,1)_0, (2,2)_1, (2,2)_\infty].$$

However, this datum is impossible for topological reasons (the only possible graph datum is $[A_6 + A_2]$ and there is a unique embedded graph T_Γ with this datum).

If $\deg(j) = 3$ then one has a cyclic point of order 3, leading to the data above. □

COROLLARY 3.4. — *Every irreducible component $W'_{1,\tilde{\Gamma}} \subset W_1'$ such that $\tilde{\Gamma}(\mathcal{E}_w) \neq \mathrm{SL}_2(\mathbb{Z})$ for $w \in W'_{1,\tilde{\Gamma}}$ is rational.*

Consider an irreducible component $\mathcal{F}_{r,\tilde{\Gamma}}$ and the corresponding decomposition $j = j_\Gamma \circ j_{\mathcal{E}}$. Here

$$j_{\mathcal{E}} = (j_{\mathcal{E},2}, j_{\mathcal{E},3}) \,:\, \mathbb{P}^1 \to M_\Gamma = \mathbb{P}^1$$

is a pair of homogeneous polynomials in 2 variables. Let

$$\mathcal{G} = \{(g_2, g_3)\} \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r))$$

be the subset corresponding to smooth elliptic fibrations. Put

$$\mathcal{J}_\Gamma := \{j \mid \exists j_{\mathcal{E}} \,:\, \mathbb{P}^1 \to M_\Gamma \ s.t.\ j = j_\Gamma \circ j_{\mathcal{E}}\}.$$

LEMMA 3.5. — *If $j \in \mathcal{J}_\Gamma \cap \mathcal{J}_{\Gamma'}$ with $\Gamma \neq \Gamma'$ then there exist an $h \in \mathrm{PSL}_2(\mathbb{Z})$, a group $\Gamma'' \subset \Gamma \cap h\Gamma' h^{-1}$ and a map $j''_{\mathcal{E}} \,:\, \mathbb{P}^1 \to M_{\Gamma''}$ such that $j = j_{\mathcal{E}''} \circ j_{\Gamma''}$.*

Proof. — The monodromy group and its image in $\mathrm{PSL}_2(\mathbb{Z})$ are uniquely determined by the smooth part of the elliptic fibration. Therefore, in any smooth family of elliptic surfaces

$$\Gamma(\text{generic fiber}) \supseteq \Gamma(\text{special fiber}).$$

Since Γ is defined modulo conjugation by elements in $\mathrm{SL}_2(\mathbb{Z})$ the claim follows. ∎

COROLLARY 3.6. — *We have a decomposition $\mathcal{G} = \bigsqcup \mathcal{G}_\Gamma$ into a finite (disjoint) union of algebraic GL_2-stable subvarieties such that for all $g = (g_2, g_3) \in \mathcal{G}_\Gamma$ the monodromy group $\tilde{\Gamma}(\mathcal{E}_g) \subset \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of a central $\mathbb{Z}/2$-extension of Γ.*

REMARK 3.7. — For a given $g \in \mathcal{G}_\Gamma$ the map $j_{\mathcal{E}}$ is not unique. Let $j_{\mathcal{E}}$ and $j'_{\mathcal{E}}$ be two such maps. Then $j_{\mathcal{E}} = h_\Gamma \circ j'_{\mathcal{E}}$, where $h_\Gamma \in \mathrm{Aut}(T_\Gamma)$ is an automorphism of M_Γ, preserving T_Γ.

LEMMA 3.8. — *We have a decomposition*

$$\mathcal{G}_\Gamma = \bigsqcup_k \mathcal{G}_{\tilde{\Gamma},k}$$

into a finite union of algebraic irreducible GL_2*-stable subvarieties such that $\tilde{\Gamma}(\mathcal{E}_g) = \tilde{\Gamma}$ for all $g \in \mathcal{G}_{\tilde{\Gamma},k}$.*

Proof. — Assume that some $g \in \mathcal{G}_\Gamma$ belongs to $\mathcal{G}_{\tilde{\Gamma},1} \cap \mathcal{G}_{\tilde{\Gamma},2}$, where $\mathcal{G}_{\tilde{\Gamma},1}, \mathcal{G}_{\tilde{\Gamma},2}$ are different (nonconjugated) lifts of Γ into $\mathrm{SL}_2(\mathbb{Z})$. Lemma 3.5 implies that there exists a *proper* subgroup $\Gamma'' \subset \Gamma$ such that g belongs to $\mathcal{G}_{\Gamma''}$, contradiction. ∎

Let $\mathcal{G}_{\tilde{\Gamma}} = \mathcal{G}_{\tilde{\Gamma},k}$ be an irreducible component of $\mathcal{G}_\Gamma$ as in Lemma 3.8 and $g \in \mathcal{G}_{\tilde{\Gamma}}$ its generic point. It determines a set of $*$-fibers on the base $\mathbb{P}^1$. We denote their number by ℓ. Choose (one of) the $j_{\mathcal{E}_g}$, with ramification datum $\mathrm{RD} = \mathrm{RD}(j_{\mathcal{E}_g})$. We get a map

$$\phi_{\mathcal{U}} \ : \ \mathcal{U}_g \to \mathcal{U}_{j_g} \times (\mathbb{P}^1)^\ell,$$

where $\mathcal{U}_g \subset \mathcal{G}_{\tilde{\Gamma}}$ is a neighborhood of g and $\mathcal{U}_{j_g} \subset \mathcal{R}(\mathrm{RD})$ is a neighborhood of the map $j_g = j_{\mathcal{E}_g}$ in the space

$$\mathcal{R}(\mathrm{RD}) := \{ j \ : \ \mathbb{P}^1 \to \mathbb{P}^1 \mid \mathrm{RD}(j) = \mathrm{RD} \}$$

of rational maps with ramification datum RD.

LEMMA 3.9. — *The map $\phi_{\mathcal{U}}$ is a local (complex analytic) surjection.*

Proof. — It suffices to show that the variation of j in the space of maps with $\mathrm{RD}(j) = \mathrm{RD}(j_{\mathcal{E}_g})$ lifts to a variation of the elliptic fibration $\mathcal{E}_g$. Observe that M_Γ minus the preimages of $\{0, 1, \infty\}$ carries a natural "projective" local system PL_Γ with fiber $\mathbb{Z} \oplus \mathbb{Z}$ modulo ± 1. This local system lifts onto a similar local system $j^* PL_\Gamma$ on $\mathbb{P}^1$. Since the fundamental group of $\mathbb{P}^1$ minus $j^{-1}(\{0, 1, \infty\})$ is a free group the projective local system $j^* PL_\Gamma$ can be lifted to a linear local system with fiber $\mathbb{Z} \oplus \mathbb{Z}$.

By a theorem of Kodaira, a nonisotrivial elliptic fibration over B is uniquely determined by the nonconstant map $j : B \to \mathbb{P}^1 = \overline{\mathcal{H}}/\Gamma$ and the lifting of the induced projective monodromy on $\mathbb{P}^1 \setminus j^{-1}(\{0, 1, \infty\})$ with $(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z}/2)$ as a fiber to the linear monodromy with $\mathbb{Z} \oplus \mathbb{Z}$ as a fiber. Therefore it suffices to lift the variation of $j_{\mathcal{E}}$ to a variation of the corresponding local system with fiber

$$H_1(\mathcal{E}_t) = \mathbb{Z} \oplus \mathbb{Z}, \;\; t \in \mathbb{P}^1 - j_{\mathcal{E}}^{-1}(\{0, 1, \infty\}).$$

Any such lifting corresponds to an elliptic fibration with Jacobian map $j_{\mathcal{E}}$. However, its topological type depends on the lifting.

A linear lifting is defined by the choice of local monodromy elements in $\mathrm{SL}_2(\mathbb{Z})$ at every point in the preimage of the corresponding projective monodromy $g \in \mathrm{PSL}_2(\mathbb{Z})$. Namely, for every local projective monodromy $g \in \mathrm{PSL}_2(\mathbb{Z})$ we have a choice $g_m, -g_m$ where g_m corresponds to the minimal lifting - the lifting with a minimal Betti number for the corresponding singular fiber (see the next section for a more detailed discussion or [23]). The topological type $r(\mathcal{E}_g)$ is a function of $\mathrm{RD}(\mathcal{E})$ and the number of nonminimal liftings l . The existence of a lifting depends only on the product of local liftings (1 or -1), which can be calculated using $\mathrm{RD}(j)$ and l. In particular, since it is 1 for $\mathcal{E}_g$ the same holds for a variation j of $j_{\mathcal{E}_g}$ (with constant l).

According to Kodaira, a simultaneous lifting of the variation of $j_{\mathcal{E}}$ and the variation of the linear system for $\mathcal{E}_g$ as above is equivalent to the existence of a variation of the elliptic surface $\mathcal{E}_g$. This completes the proof of the lemma. □

COROLLARY 3.10. — *Let $\mathcal{F}'_{r,\tilde{\Gamma}} \subset \mathcal{F}_{r,\tilde{\Gamma}}$ be an (irreducible) component with generic ramification datum* RD. *Then $\mathcal{F}'_{r,\tilde{\Gamma}}$ surjects (rationally) onto the quotient of the variety of rational maps $\mathcal{R}(\mathrm{RD})$ by H_Γ.*

Proof. — Since both $\mathcal{F}'_{r,\tilde{\Gamma}}$ and $\mathcal{R}(\mathrm{RD})$ are algebraic varieties the local complex analytic surjection from Lemma 3.9 extends to an algebraic correspondence. Moreover, two decompositions of the map j as $j = j_\Gamma \circ j_{\mathcal{E}}$ differ by an element in H_Γ. This gives a map to the quotient space, which is a (global) rational surjection. □

PROPOSITION 3.11. — *Every irreducible component* $\mathcal{F}_{r,\tilde{\Gamma}}$ *contains an open part* $\mathcal{F}'_{r,\tilde{\Gamma}}$ *with the following properties:*

- $\mathcal{F}'_{r,\tilde{\Gamma}}$ *is a quotient of an algebraic variety* $U'_{r,\tilde{\Gamma},\ell}$ *by the (left) action of* PGL_2 *and (right) action of a subgroup* H_Γ *of* $\mathrm{Aut}(T_\Gamma)$;
- $U'_{r,\tilde{\Gamma},\ell}$ *admits a fibration with fiber (an open subset of)* $\mathrm{Sym}^\ell(\mathbb{P}^1)$ *and base the variety* $\mathcal{R}_{r,\Gamma}$ *of maps* $f : \mathbb{P}^1 \to M_\Gamma$ *with fixed local ramification data over* A_2 *and* B_2*-points of* $T_\Gamma \subset M_\Gamma$;
- *the action of* PGL_2 *on* $U'_{r,\tilde{\Gamma},\ell}$ *is induced from the standard* PGL_2*-action on* $\mathbb{P}^1$;
- *the group* $\mathrm{Aut}(T_\Gamma)$ *is a subgroup of* PGL_2 *(acting on* M_Γ*).*

Proof. — Elliptic surfaces parametrized by a smooth irreducible variety have the same $\mathrm{ET}(\mathcal{E})$, which depends on the number ℓ of $*$-fibers in $\mathcal{E}$, on the degree of $j_\mathcal{E}$ and on the ramification properties over the ends of T_Γ. Once ℓ is fixed, for any given $j_\mathcal{E}$, the $*$-mark can be placed over arbitrary ℓ-points of $\mathbb{P}^1$. Their position defines a unique surface $\mathcal{E}$. This implies that $U'_{r,\tilde{\Gamma},\ell}$ is fibered with fibers (birationally) isomorphic to $\mathrm{Sym}^\ell(\mathbb{P}^1) = \mathbb{P}^\ell$. The ramification properties of $j_\mathcal{E}$ remain the same on the open part of $U'_{r,\tilde{\Gamma},\ell}$ (since the number of $*$-fibers remains the same). Thus the base of the above fibration is the space of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1 = M_\Gamma$ with fixed ramification locus. Any such map defines an elliptic surface $\mathcal{E}$ with given Γ (see [8]). The PGL_2-action on $U'_{r,\tilde{\Gamma},\ell}$ identifies points corresponding to isomorphic surfaces $\mathcal{E}$. Additional nontrivial isomorphisms correspond to exterior automorphisms of Γ, coming from the action on M_Γ, i.e., automorphisms of the graph T_Γ. □

REMARK 3.12. — If the $\mathrm{PGL}_2 \times \mathrm{Aut}(T_\Gamma)$-action on $U'_{r,\tilde{\Gamma},\ell}$ is almost free then the rationality of $\mathrm{PGL}_2\backslash U'_{r,\tilde{\Gamma},0}/\mathrm{Aut}(T_\Gamma)$ implies the rationality the corresponding quotients for all ℓ. In the other cases the degree of $j_\mathcal{E}$ is small and they are handled separately (see Section 9).

Most of the graphs T_Γ have trivial automorphisms. In particular, any nontrivial automorphism acts on the ends of the graph. In general, automorphisms of the pair (M_Γ, T_Γ) correspond to elements of Γ'/Γ where $\Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$ is a maximal subgroup with the property that Γ is a normal subgroup of Γ'.

LEMMA 3.13. — *The group* $\mathrm{Aut}(T_\Gamma)$ *acts freely on the set of ends and end-loops.*

Proof. — Consider $j_\Gamma : M_\Gamma \to \mathbb{P}^1$. Then $h \in \mathrm{Aut}(T_\Gamma)$ is any element in $\mathrm{PGL}_2(\mathbb{C})$ such that $j_\Gamma(hz) = j_\Gamma(z)$ for all $z \in T_\Gamma \subset M_\Gamma$. If h stabilizes an end or an end-loop of T_Γ then it stabilizes the unique adjacent vertex and its other end. Any element of $\mathrm{PGL}_2(\mathbb{C})$ preserving a closed interval is the identity. □

COROLLARY 3.14. — *For $r \leq 2$, the only possible groups $\mathrm{Aut}(T_\Gamma)$ are cyclic, dihedral or subgroups of $\mathfrak{S}_4$. More precisely, for graphs with one end $\mathrm{Aut}(T_\Gamma) = 1$ and graphs with two ends $\mathrm{Aut}(T_\Gamma)$ is a subgroup of $\mathbb{Z}/2$.*

LEMMA 3.15. — *Let*

$$\mathcal{R} := \{f \,:\, \mathbb{P}^1 \to \mathbb{P}^1\}$$

be the space of rational maps with ramifications over exactly 0 and ∞. Then $\mathcal{R}$ is a $\mathbb{G}_m$-fibration over the product of symmetric spaces $\mathrm{Sym}^{m_i}(\mathbb{P}^1)$.

Proof. — Indeed any two cycles c_1 and c_2 of fixed degree are equivalent on $\mathbb{P}^1$. Therefore, there is a rational function f on $\mathbb{P}^1$ with $c_1 = f^{-1}(0)$ and $c_2 = f^{-1}(\infty)$. If c_1, c_2 do not intersect then $\deg(f) = \deg(c_1) = \deg(c_2)$. The function f is defined modulo multiplication by a constant. The space of cycles $c_1 = \sum_i n_i p_i$ is a product of symmetric powers $\mathrm{Sym}^m(\mathbb{P}^1)$ where m is the number of equal n_i. □

4. Combinatorics

In this section we investigate relations between $\mathrm{ET}(\mathcal{E})$ and $\mathrm{ET}(\Gamma)$. We keep the notations of the previous sections.

LEMMA 4.1. — *Let $j \,:\, \mathcal{E} \to C$ be an elliptic fibration. Then*

$$\mathrm{ET}(\mathcal{E}) = \deg(j_{\mathcal{E}})\Delta(\Gamma) + 8\alpha_2 + 4\alpha_1 + 6\beta_1 + 12\ell. \tag{4.1}$$

Here α_1 and α_2 equal the number of points over A_2-ends of T_Γ with ramification multiplicity $1 \pmod 3$ and $2 \pmod 3$, respectively, β_1 is the number of odd ramification points over the B_2-ends and ℓ is the number of $*$-fibers of $\mathcal{E}$.

Proof. — The summand $\deg(j_{\mathcal{E}})\Delta(\Gamma)$ corresponds to multiplicative fibers of $\mathcal{E}$. The next summands are the contributions of those singular fibers of $\mathcal{E}$ which are in the preimage of A_2 or B_2-ends of T_Γ. If the ramification order at a point p over a B_2-end is even then the corresponding fiber with minimal ET is smooth and hence does not contribute to $\mathrm{ET}(\mathcal{E})$. If it is odd then the fiber with minimal ET is of type III and we have to add $6\beta_1$. Similarly, for the preimages of A_2-ends and $*$-twists. □

COROLLARY 4.2. — *In particular,*

$$\mathrm{ET}(\mathcal{E}) \leq \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma) + 12\ell,$$

with equality if

$$\begin{aligned} 2\alpha_2 + \alpha_1 &= a_2 \cdot \deg(j_{\mathcal{E}}) \\ \beta_1 &= b_2 \cdot \deg(j_{\mathcal{E}}). \end{aligned}$$

DEFINITION 4.3. — *We call* T_Γ saturated *if all vertices of* T_Γ^u *are trivalent and a* tree *if it is contractible.*

REMARK 4.4. — For saturated graphs $\Delta(\Gamma) = 12\mathrm{rk}\,\pi_1(T_\Gamma)$, where

$$\mathrm{rk}\,\pi_1(T_\Gamma) = \mathrm{rk}\,H_1(T_\Gamma)$$

is the number of independent closed loops of $T_\Gamma \subset M_\Gamma$.

The following simple procedures produce new graphs:

– If T_1 and T_2 are (unmarked) trivalent graphs we can join T_1 and T_2 along two edges. For the resulting graph T' we have

$$\mathrm{ET}(T') = \mathrm{ET}(T_1) + \mathrm{ET}(T_2) + 12.$$

If T_i are marked and the marking of the ends of T' is induced from the marking of the corresponding ends of T_1 and T_2 then

$$\Delta(T') = \Delta(T_1) + \Delta(T_2) + 12.$$

– We can glue an end p of T_1 to an edge of T_2. In this case

$$\mathrm{ET}(T') = \mathrm{ET}(T_1) + \mathrm{ET}(T_2).$$

The change of Δ depends on the marking of the end:

$$\Delta(T') = \begin{cases} \Delta(T_1) + \Delta(T_2) + 6 & \text{if } p = B_2 \\ \Delta(T_1) + \Delta(T_2) + 4 & \text{if } p = A_2. \end{cases}$$

REMARK 4.5. — Any connected graph T can be uniquely decomposed into a union of a saturated graph and a union of trees.

LEMMA 4.6. — $\mathrm{ET}(\Gamma)$ *is divisible by* 12.

Proof. — Every vertex of T_Γ has either one or three incoming edges. Therefore, the number of edges

$$\tau^1 = \frac{1}{2}(\tau_1^0 + \tau_3^0),$$

(τ_i^0 is the number of vertices with i-edges). Thus $\tau^0 = \tau_1^0 + \tau_3^0$ is even and since $\mathrm{ET}(\Gamma) = 6\tau^0$ we are done. □

EXAMPLE 4.7. — If T_Γ is a tree with $k+2$ vertices then

$$\begin{array}{ll} \mathrm{ET}(\Gamma) & = 12k+12 \\ \Delta(\Gamma) & \begin{cases} = 6k & \text{if all ends are } B_2, \\ > 6k & \text{otherwise.} \end{cases} \end{array}$$

LEMMA 4.8. — *For all Γ one has*

$$\Delta(\Gamma) \geq \mathrm{ET}(\Gamma)/2 + 6(\mathrm{rk}\, H_1(T_\Gamma) - 1).$$

Proof. — A direct computation shows that for saturated graphs one has an equality. Suppose that T_Γ is a concatenation of a saturated graph T_{sat} and a tree T_{tree}. The number of ends drops by one and the number of A_6 vertices increases by 1. Thus the tree will add $12k+12$ to $\mathrm{ET}(\Gamma)$ but $\Delta(\Gamma)$ will change by $6k+6$. Finally, the ratio $\Delta(\Gamma)/\mathrm{ET}(\Gamma)$ only increases if we change B_2- to A_2-markings for some ends. Indeed, $\Delta(\Gamma)$ increases without changing $\mathrm{ET}(\Gamma)$. □

COROLLARY 4.9. — *If $\Delta(\Gamma) = \mathrm{ET}(\Gamma)/2$ then T_Γ is a concatenation of a loop L and some trees. Moreover, all the ends of T_Γ are of type B_2.*

PROPOSITION 4.10. — *Let $\mathcal{E} \to C$ be an elliptic fibration with $\mathrm{ET}(\mathcal{E}) < \mathrm{ET}(\Gamma)$. Then:*

- *$M_\Gamma = \mathbb{P}^1$ and T_Γ is a tree without A_2-ends and with $\mathrm{ET}(\Gamma) > 24$;*
- *$\deg(j_{\mathcal{E}}) = 2$ and it is ramified in all (B_2) ends of T_Γ (and, possibly, some other points);*
- *$\mathcal{E}$ has 1 or 2 singular fibers of type I_n.*

Proof. — From 4.1 and 4.8 we conclude that $\mathrm{rk}\, H_1(T_\Gamma) = 0$ which implies that T_Γ is a tree and $M_\Gamma = \mathbb{P}^1$. By Lemma 2.1 and our assumption, $\mathrm{ET}(\Gamma) > 24$, which implies that $\deg(j_{\mathcal{E}}) \leq 2$. If $\deg(j_{\mathcal{E}}) = 1$, we apply Corollary 4.2 and get a contradiction to the assumption. For $\deg(j_{\mathcal{E}}) = 2$ combine Definition 1.7 and (4.1):

$$\mathrm{ET}(\mathcal{E}) = \mathrm{ET}(\Gamma) + 4a_2 + 4\alpha_1 + 8\alpha_2 + 6\beta_1 - 12.$$

Since α_1, resp. β_1 is twice the number of unramified A_2, resp. B_2-ends, and α_2 is the number of ramified A_2-ends we see that if at least one of them is not zero, then $\mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma)$. The claim follows. □

COROLLARY 4.11. — *For every elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ one has*

$$\mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma).$$

Further, if $\deg(j_{\mathcal{E}}) = 2$ and $j_{\mathcal{E}}$ is ramified over only one B_2-point then

$$\mathrm{ET}(\mathcal{E}) \geq 2\,\mathrm{ET}(\Gamma) - 12.$$

Proof. — If $\deg(j_{\mathcal{E}}) = 2$ and $\mathbb{C} = \mathbb{P}^1$ then $j_{\mathcal{E}}$ ramifies in two points. If neither of these points is B_2 then, by Lemma 4.1, $\mathrm{ET}(\mathcal{E}) \geq 2\,\mathrm{ET}(\Gamma)$. If both of these points are B_2-points then the covering $j_{\mathcal{E}}$ corresponds to a subgroup Γ' of index 2 in Γ and $C = M_{\Gamma'}$, contradiction. Otherwise, the claimed inequality follows from Lemma 4.1. □

5. Elliptic K3 surfaces with $\deg(j_{\mathcal{E}}) > 1$

In this section we assume that $C = \mathbb{P}^1$, that $j_{\mathcal{E}} > 1$ and that Γ is a proper subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. We consider

$$\begin{array}{rcl} \text{general families}: \ \mathrm{ET}(\mathcal{E}) - 12\ell & = & \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma), \\ \text{special families}: \ \mathrm{ET}(\mathcal{E}) - 12\ell & < & \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma). \end{array}$$

In Section 3 we showed that the main building block in the construction of moduli space of elliptic surfaces with fixed Γ is the space of rational maps $j_{\mathcal{E}} : C \to M_\Gamma$ of fixed degree and ramification restrictions over certain points. For a *general* family there are no such restrictions and the corresponding moduli spaces are rational by classical results of invariant theory for actions of PGL_2 and its algebraic subgroups (see Section 7). For *special* families the corresponding space of rational maps is more complicated.

LEMMA 5.1. — *There are no special families of elliptic* K3 *surfaces with*

$$\mathrm{ET}(\Gamma) = 48, 36.$$

Proof. —

- If $\mathrm{ET}(\Gamma) = 48$ then $\Delta(\Gamma) \geq 18$ and $\deg(j_{\mathcal{E}}) \leq 2$. However, $\deg(j_{\mathcal{E}}) = 2$ contradicts Corollary 4.11 ($\mathrm{ET}(\mathcal{E}) \geq 96 - 24 > 48$).
- If $\mathrm{ET}(\Gamma) = 36$ and $\Delta(\Gamma) > 16$ then $\deg(j_{\mathcal{E}}) = 2$, contradicting to 4.11. We are left with $\Delta(\Gamma) = 16, 14, 12$ for $\deg(j_{\mathcal{E}}) = 3$ and $\Delta(\Gamma) = 12$ for $\deg(j_{\mathcal{E}}) = 4$.
- If $\deg(j_{\mathcal{E}}) = 4$ then T_Γ is a tree with $\mathrm{GD}(\Gamma) = [2A_6 + 4B_2]$. By Lemma 4.1, all ramifications over the B_2-ends are even, which contradicts $C = \mathbb{P}^1$ (compute $\chi(C)$).
- If $\deg(j_{\mathcal{E}}) = 3$ then T_Γ is a tree (by 4.8) and

$$\mathrm{GD}(\Gamma) = [2A_6 + a_2A_2 + (4 - a_2)B_2]$$

  with $a_2 \leq 2$. We have

$$48 \geq \mathrm{ET}(\mathcal{E}) \geq 3(12 + 2a_2) + 4\alpha_1 + 8\alpha_2 + 6\beta_1,$$

where $\beta_1 \geq 2$ (since $\deg(j_{\mathcal{E}})$ is odd there is odd ramification over some B_2-end). Therefore, $a_2 = 0$ and consequently, $\beta_1 \geq 4$, contradiction.

□

LEMMA 5.2. — *If T_Γ is not a tree and $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{2,\tilde{\Gamma}}$) then*

$\mathrm{ET}(\Gamma)$	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
24	4	$[2A_6 + 2B_2]$	$[(2,2)_B, (2,2)_B, (2), (2)]$
24	3	$[2A_6 + 2B_2]$	$[(2,1)_B, (2,1)_B, (2), (2)]$
24	3	$[2A_6 + A_2 + B_2]$	$[(3)_A, (2,1)_B, (2)]$
12	6	$[A_6 + A_2]$	$[(3,3)_A, (3,3)_A, (2), (2)]$
12	5	$[A_6 + A_2]$	$[(3,1,1)_A]$
12	$5 \leq d \leq 8$	$[A_6 + B_2]$	$[\beta = (\beta_i)_B, (2)^{d'}_B]$,

where

$$\beta_i \in \mathbb{N}, \ \sum \beta_i = d, \ \#\mathrm{odd}\ \beta_i \leq 8 - d$$

and

$$d' = 2d - \#\mathrm{nonzero}\ \beta_i.$$

Proof. — Follows from Lemma 4.1. First observe that $\Delta(\Gamma) \leq 16$, which implies that $a_6 = 2$ and $a_2 \leq 2$. If $a_2 = 2$ then $\Delta(\Gamma) = 16$ and

$$\alpha_1 = \alpha_2 = \beta_1 = 0.$$

Hence both A_2-ends have a 3-cyclic ramification and the cover corresponds to a subgroup $\Gamma' \subset \Gamma$ of index 3. This excludes $\mathrm{GD}(\Gamma) = [2A_6 + 2A_2]$. If $\deg(j_{\mathcal{E}}) = 4$ then $\Delta(\Gamma) = 12$ which implies that all preimages of B_2-ends have even ramification. The description of all other ramification data follows similarly from Lemma 4.1. Notice that the (omitted) possibilities

$\mathrm{ET}(\Gamma)$	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
12	6	$[A_6 + A_2]$	$[(6)_A, (3,3)_A]$
12	5	$[A_6 + A_2]$	$[(3,2)_A]$

are degenerations of the listed cases (see Remark 3.2). □

LEMMA 5.3. — *If T_Γ is a tree and $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{2,\tilde{\Gamma}}$) then*

	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
j_1	4	$[A_6+A_2+2B_2]$	$[(1,1,1,1)_A,(2,2)_B,(2,2)_B,(2),(2)]$
j_2	4	$[A_6+A_2+2B_2]$	$[(3,1)_A,(2,2)_B,(2,2)_B]+*$
j_3	4	$[A_6+A_2+2B_2]$	$[(3,1)_A,(2,2)_B,(2,1,1)_B,(2)]$
j_4	3	$[A_6+2A_2+B_2]$	$[(3)_A,(1,1,1)_A,(2,1)_B]$
j_5	3	$[A_6+A_2+2B_2]$	$[(1,1,1)_A,(2,1)_B,(2,1)_B,(2),(2)]$
j_6	3	$[A_6+A_2+2B_2]$	$[(3)_A,(1,1,1)_B,(2,1)_B,(2)]$

or $\mathrm{GD}(\Gamma)=[A_6+3B_2]$ *and*

	$\deg(j_{\mathcal{E}})$	$\mathrm{RD}(j_{\mathcal{E}})$
j_7	8	$[(2,2,2,2)_B,(2,2,2,2)_B,(2,2,2,2)_B,(2),(2)]$
j_8	6	$[(2,2,2)_B,(2,2,2)_B,(2,2,1,1)_B,(2),(2)]$
j_9	6	$[(2,2,2)_B,(2,2,2)_B,(2,2,2)_B,(2)]+*$
j_{10}	5	$[(2,2,1)_B,(2,2,1)_B,(2,2,1)_B,(2),(2)]$
j_{11}	4	$[(2,1,1)_B,(2,1,1)_B,(2,2)_B,(2),(2)]$
j_{12}	4	$[(2,1,1)_B,(2,2)_B,(2,2)_B,(2)]+*$
j_{13}	3	$[(1,1,1)_B,(2,1)_B,(2,1)_B]$
j_{14}	3	$[(2,1)_B,(2,1)_B,(2,1)_B,(2)]+*$

or $\mathrm{ET}(\Gamma)=12$ *and* $\mathrm{GD}(\Gamma)=[2A_2]$ *with* $\deg(j_{\mathcal{E}})=4-10,12$.

(In the above tables, $+*$ means that there exists a moduli space of elliptic surfaces with the same $\mathrm{RD}(j_{\mathcal{E}})$ *and with an additional $*$-fiber over an unspecified point.)*

Proof. — Assume that $\mathrm{ET}(\Gamma)=24$ and T_Γ is a tree with

$$\mathrm{GD}(\Gamma)\neq[A_6+3B_2].$$

First observe that $\deg(j_{\mathcal{E}})\le 6$, since $\Delta(\Gamma)\ge 8$. If $\deg(j_{\mathcal{E}})\ge 5$ then, by 4.1, $\mathrm{GD}(\Gamma)=[A_6+A_2+2B_2]$. If $\deg(j_{\mathcal{E}})=6$ then $j_{\mathcal{E}}$ has to be completely ramified over all ends and no other ramifications are allowed by Euler characteristic computation. Therefore, it is a group-covering and can't be $j_{\mathcal{E}}$. If $\deg(j_{\mathcal{E}})=5$ then there are two odd ramifications over B_2-ends, and by 4.1, $\mathrm{ET}(\Gamma)>48$.

We are left with

$$\begin{aligned}\mathrm{GD}(\Gamma)&=[A_6+3A_2],\\&=[A_6+2A_2+B_2],\\&=[A_6+A_2+2B_2]\end{aligned}$$

and $3\le\deg(j_{\mathcal{E}})\le 4$. If there are at least two A_2-ends without 3-cyclic ramification points over them then $\mathrm{ET}(\mathcal{E})>48$ (see 4.1). The first case is impossible: $\deg(j_{\mathcal{E}})=$

4 does not occur (the degree is not divisible by 3), if $\deg(j_{\mathcal{E}}) = 3$ and there is at most one 3-cyclic ramification over an A_2-end then, by 4.1, $\mathrm{ET}(\mathcal{E}) > 48$, contradiction. Consider the second case and $\deg(j_{\mathcal{E}}) = 4$. Then $\Delta(\Gamma) = 10$ and $4\alpha_1+8\alpha_2+6\beta_1 \leq 8$. Since $\alpha_1 \geq 2$ we have $\alpha_2 = \beta_1 = 0$ and $\alpha_1 = 2$. The only possible

$$\mathrm{RD}(j_{\mathcal{E}}) = [(3,1)_A, (3,1)_A, (2,2)_B],$$

which corresponds to a group covering, contradiction.

Similarly, if $\mathrm{GD}(\Gamma) = [A_6 + A_2 + 2B_2]$ and $\deg(j_{\mathcal{E}}) = 4$ then $\Delta(\Gamma) = 8$ and $4\alpha_1 + 8\alpha_2 + 6\beta_1 \leq 16$. We have $\alpha_1 \geq 1$ and $8\alpha_2 + 4\alpha_1 = 16$ or 4. In the first case, both B_2-ends are completely ramified, and we get j_1. The second case splits into subcases: $\beta_1 = 0$ or 2, leading to j_2, resp. j_3. If $\deg(j_{\mathcal{E}}) = 3$, then if $\mathrm{GD}(\Gamma) = [A_6 + 2A_2 + B_2]$ then exactly one of the A_2-ends has cyclic ramification. It follows that $\beta_1 = 1$, which leads to j_4. If $\mathrm{GD}(\Gamma) = [A_6 + A_2 + 2B_2]$ there are two subcases: there is cyclic ramification over the A_2-end or not. In the first subcase, possible $\mathrm{RD}(j_{\mathcal{E}})$ include $[(2,1)_B, (2,1)_B]$, which is excluded as it gives a group covering. The other case leads to j_6. In the second subcase, we get j_5.

Consider the case $T_\Gamma = A_6 + 3B_2$. Here $\Delta(\Gamma) = 6$ and

$$\mathrm{ET}(\mathcal{E}) \geq 6\deg(j_{\mathcal{E}}) + 6n,$$

where n is a number of points with odd ramification over B_2-vertices. It follows that

$$48 \geq 6\deg(j_{\mathcal{E}}) + 6\beta_1$$

and $\beta_1 \geq 3$ if $\deg(j_{\mathcal{E}})$ is odd and the number of odd ramifications over *each* B_2-end is congruent to $\deg(j_{\mathcal{E}})$ modulo 2.

If $\deg(j_{\mathcal{E}}) = 8$ then all preimages of B_2-vertices are $2n$-ramified. If $\deg(j_\Gamma)$ is odd then $\mathrm{ET}(\mathcal{E}) \geq 6\deg(j_\Gamma) + 18$, which excludes $\deg(j_\Gamma) = 7$. Now assume $\deg(j_\Gamma) = 6$. The number of possible odd ramifications over any B_2-end is even by 4.1 and it cannot exceed 2. There are two possibilities listed above. Assume that $\deg(j_{\mathcal{E}}) = 5$. The minimal possible ramifications are $(2,2,1)$ over all B_2-ends. Since $10 - 6 = 4$ we can add two more points.

In $\deg(j_{\mathcal{E}}) = 4$ we could have further RD:

$$\begin{aligned}\mathrm{RD}(j_{\mathcal{E}}) &= [(2,2)_B, (2,1,1)_B, (2,2)_B, (2)],\\ &= [(2,2)_B, (2,1,1)_B, (2,1,1)_B, (2), (2)],\\ &= [(2,2)_B, (2,1,1)_B, (2,1,1)_B, (3)]\end{aligned}$$

but they are obtained as degenerations of j_{12} and j_{13}.

The only $\mathrm{GD}(\Gamma)$ which allow $\deg(j_{\mathcal{E}}) \geq 12$ are $[A_2+B_2]$ and $[2A_2]$. The first case corresponds to $\mathrm{PSL}_2(\mathbb{Z})$ (which we don't consider). The second case corresponds to subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ of index 2. For a generic $\mathcal{E}$ in each moduli space the ramification datum $\mathrm{RD}(j_{\mathcal{E}})$ is one of the following:

$$\mathrm{RD}(j_{\mathcal{E}}) = [(3,\dots,3_{n_1},1,\dots,1)_A,(3,\dots,3_{n_1},1,\dots,1)_A,(2)^d]+*,$$

where n_1, n_2, d are nonnegative integers such that

$$\deg(j_{\mathcal{E}}) - (n_1+n_2) \leq 4,$$

$$3n_1, 3n_2 \leq \deg(j_{\mathcal{E}}) \text{ and, } d \leq 2(\deg(j_{\mathcal{E}} - (n_1+n_2+1))).$$

(In particular, $d \leq 4$). □

6. Rational elliptic surfaces with $\deg(j_{\mathcal{E}}) > 1$

LEMMA 6.1. — *There are no special families of rational elliptic surfaces with* $\mathrm{ET}(\Gamma) = 24$.

Proof. — If $\deg(j_{\mathcal{E}}) = 2$ then $j_{\mathcal{E}}$ cannot be ramified over more than one B_2-end (otherwise it is a group covering). Therefore, we can apply Corollary 4.11 and get $\mathrm{ET}(\mathcal{E}) > 2\cdot 24 - 12 > 24$, contradiction (to 2.1). Thus $\deg(j_{\mathcal{E}}) = 3$ or 4 and $a_6 = 1$. Moreover, $\Delta(\Gamma) \leq 8$. This leaves the cases:

$$\begin{aligned} \mathrm{GD}(\Gamma) &= [A_6+3B_2], \\ &= [A_6+A_2+2B_2]. \end{aligned}$$

In the first case $\deg(j_{\mathcal{E}}) = 3$ is impossible, and $\deg(j_{\mathcal{E}}) = 4$ leads to

$$\mathrm{RD}(j_{\mathcal{E}}) = [(2,2)_B,(2,2)_B,(2,2)_B,(2,2)_B]$$

which corresponds to a group covering. In the second case $\deg(j_{\mathcal{E}}) \neq 4$ (since $\Delta(\Gamma) = 8$) and $\deg(j_{\mathcal{E}}) = 3$ implies that $\beta_1 \geq 2$ and $\mathrm{ET}(\mathcal{E}) \geq 36$, contradiction. □

LEMMA 6.2. — *If $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{1,\tilde{\Gamma}}$) then*

	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
j_{15}	6	$[2A_2]$	$[(3,3)_A,(3,3)_A,(2),(2)]$
j_{16}	4	$[2A_2]$	$[(3,1)_A,(3,1)_A,(2),(2)]$
j_{17}	3	$[2A_2]$	$[(3)_A,(1,1,1)_A,(2),(2)]$
j_{18}	4	$[A_6+B_2]$	$[(2,2)_B,(2),(2),(2),(2)]$
j_{19}	3	$[A_6+B_2]$	$[(2,1)_B,(2),(2)]$
j_{20}	3	$[A_6+A_2]$	$[(3)_A,(2),(2)]$

Proof. — If $a_6\geq 1$ then $\deg(j_{\mathcal{E}})=4$ or 3. In the first case $a_2=0$ and $\mathrm{GD}(\Gamma)=[A_6+B_2]$ and we have complete ramification over the B_2-end. This gives j_{18}. In the second case the ramification over B_2 is $(2,1)_B$ and we get j_{19}. If $\mathrm{GD}(\Gamma)=[A_6{+}A_2]$ then $\deg(j_{\mathcal{E}})=3$ and $\alpha_1=\alpha_2=0$, leading j_{20}.

It remains to consider $\mathrm{GD}(\Gamma)=[2A_2]$. We apply the same formulas as in the proof of Lemma 5.3, with the inequality

$$\deg(j_{\mathcal{E}})-n_1-n_2\leq 2.$$

We have $\deg(j_{\mathcal{E}})\leq 6$ and $\alpha_1=\alpha_2=0$. Notice that $\deg(j_{\mathcal{E}})=5$ is impossible. □

7. General rationality results

NOTATIONS 7.1. — We will denote by $\mathfrak{S}_n$ the symmetric group on n letters, by $\mathfrak{A}_n$ the alternating group, by $\mathfrak{D}_n$ the dihedral group and by $\mathfrak{C}_n=\mathbb{Z}/n$ the cyclic group. In particular, $\mathfrak{S}_2=\mathfrak{C}_2=\mathbb{Z}/2$ and $\mathfrak{D}_2=\mathbb{Z}/2\times\mathbb{Z}/2$ (sometimes we prefer the notation $\mathfrak{S}_2$ over $\mathfrak{C}_2$ to stress that the action is by permutation). We write $\mathrm{Gr}(k,n)$ for the Grassmannian of k-planes in a vector space of dimension n and V_d for the space of binary forms of degree d. We will denote by $\mathrm{GL}_2,\mathrm{PGL}_2,\mathbb{G}_m$ etc. the corresponding *complex* algebraic groups. For a group G, we denote by Z_g the centralizer of $g\in\mathrm{G}$ and by Z_{G} its center. We denote by $\mathrm{M}_2=V_1\oplus V_1$ the space of 2×2-matrices. We write $\mathcal{V}\stackrel{V}{\longrightarrow}X$ or simply $\stackrel{V}{\longrightarrow}X$ for a locally trivial (in Zariski topology) fibration $\mathcal{V}$ over X with generic fiber V. We will often write G-map (etc.), instead of G-equivariant map.

We say that two algebraic varieties X and X' are birational, and write $X\sim X'$, if $\mathbb{C}(X)=\mathbb{C}(X')$. A variety X of dimension n is *rational* if $X\sim\mathbb{A}^n$, k*-stably rational* if $X\times\mathbb{A}^k\sim\mathbb{A}^{n+k}$ and *stably rational* if there exists such a $k\in\mathbb{N}$. We say that X is *unirational* if X is dominated by $\mathbb{A}^n$. The first basic result, a theorem of Castelnuovo from 1894, is:

THEOREM 7.2. — *A unirational surface is rational.*

Already in dimension three, one has strict inclusions

$$\text{rational} \subsetneq \text{stably rational} \subsetneq \text{unirational}$$

(see the counterexamples in [16], [2],[10],[6]). There is a very extensive literature on rationality for various classes of varieties. We will use the following facts:

LEMMA 7.3. — *Let $S \to B$ be a ruled surface with base B and $\pi : \mathcal{C} \to S$ a conic bundle over S. Assume that the restriction of π to a generic $\mathbb{P}^1 \subset S$ is a conic bundle with at most three singular fibers. Then $\mathcal{C} \sim \mathbb{A}^2 \times B$.*

LEMMA 7.4. — *Let $\pi : \mathcal{C} \to S$ be a conic bundle over an irreducible variety S and $Y \subset \mathcal{C}$ a subvariety such that the restriction of π to Y is a surjective finite map of odd degree. Then $\mathcal{C}$ has a section and $\mathcal{C} \sim S \times \mathbb{A}^1$.*

Let G be an algebraic group. A (good) *rational action* of G is a homomorphism

$$\rho_{\mathrm{rat}} : \mathrm{G} \to \mathrm{Bir}(X)$$

such that there exists a birational model X' of X with the property that ρ_{rat} extends to a (regular) morphism $\mathrm{G} \times X' \to X'$. We consider only rational actions. We write $X \sim_{\mathrm{G}} Y$ for a G-birational ($=$ G-equivariant birational) isomorphism between X and Y. We will denote by $\mathrm{G}\backslash X$ a model for the field of invariants $\mathbb{C}(X)^{\mathrm{G}}$.

Let $E \to X$ be a vector bundle. A *linear* action of G on E is a rational action which preserves the subspace of fiberwise linear functions on E. In particular, there is a linear G-action on regular and rational sections of E.

We are interested in rationality properties of quotient spaces for the actions of PGL_2, its subgroups and products of PGL_2 with finite groups. The finite subgroups of PGL_2 are

$$\mathfrak{C}_n, \mathfrak{D}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5.$$

We denote by $\tilde{\mathfrak{C}}_n, \tilde{\mathfrak{D}}_n$ etc. their lifts to GL_2 (as central $\mathfrak{C}_2$-extensions). We denote by

$$\mathrm{B}, \mathrm{T} = \mathbb{C}^*, \mathrm{N}_\mathrm{T}$$

the upper-triangular group, the standard maximal torus and the normalizer of this torus in PGL_2 and by

$$\tilde{\mathrm{B}}, \tilde{\mathrm{T}}, \mathrm{N}_{\tilde{\mathrm{T}}}$$

the corresponding subgroups in GL_2 (or SL_2).

Let V be an n-dimensional vector space, $\tilde{\mathrm{G}} \subset \mathrm{GL}(V)$ a subgroup and G its projection to $\mathrm{PGL}(V)$, acting naturally on $\mathbb{P}(V)$. Determining the rationality of quotients $\mathrm{G}\backslash\mathbb{P}(V)$ (at least for finite groups) is known as Noether's problem.

COROLLARY 7.5 (of Theorem 7.2). — *For all $n \leq 3$ the space $\mathrm{G}\backslash\mathbb{P}(V)$ is rational.*

THEOREM 7.6. — [29],[36] *A quotient of $\mathbb{P}(V)$ by a (projective) action of a connected solvable group, a torus or a finite abelian subgroup of a torus is rational.*

A fundamental rationality result is the following theorem of Katsylo:

THEOREM 7.7. — [17] *For any representation V of GL_2 or PGL_2 the quotient $\mathrm{PGL}_2\backslash\mathbb{P}(V)$ is rational.*

In general, the quotients need not be rational (see Saltman's counterexamples in [30]). We now describe some partial results for $n = 4$, which we will use later on.

DEFINITION 7.8. — *A finite group $\tilde{\mathrm{G}} \subset \mathrm{GL}_n = \mathrm{GL}(V)$ is called* imprimitive *if there exists a decomposition $V = \oplus_\alpha V^\alpha$ such that for all α and $\tilde{g} \in \tilde{\mathrm{G}}$ there is an α' with $\tilde{g}V^\alpha = V^{\alpha'}$. Otherwise, G is called* primitive.

REMARK 7.9. — There are 29 types of primitive subgroups of GL_4. For some of them, like

$$\mathfrak{A}_6, \mathfrak{A}_7, \mathrm{PSL}_2(\mathbf{F}_7), \mathfrak{S}_6,$$

rationality of the quotient is still unknown.

THEOREM 7.10. — [26] *For every primitive solvable subgroup* $\mathrm{G} \subset \mathrm{PGL}_4$ *the quotient* $\mathrm{G}\backslash\mathbb{P}^3$ *is rational.*

REMARK 7.11. — In [26] it is shown that

$$\mathrm{G}\backslash\mathbb{P}^3 \sim_{\mathrm{G}} \mathrm{G}'\backslash X_3,$$

where X_3 is the Segre cubic threefold and G' is a quotient of G. The problem is then reduced to the (easy) case of imprimitive actions.

We will also need to consider quotients by *nonlinear* actions.

LEMMA 7.12. — *The quotient of* GL_2 *(or* PGL_2*) by the involution* $i\,:\,x \mapsto x^{-1}$ *is rational.*

Proof. — The involution decomposes as a product $i = i_1 \circ i_2$, where

$$i_1\,:\,x := \begin{pmatrix} a & b \\ c & -a+d \end{pmatrix} \mapsto \begin{pmatrix} -a+d & -b \\ -c & a \end{pmatrix}$$

and

$$i_2\,:\,y \mapsto y \cdot \det(y)^{-1}.$$

are two commuting involutions. Another set of independent generators of $\mathbb{C}(a,b,c,d)$ is given by $\{a,b,c,\det(x)\}$ (write $d = (\det(x) + bc + a^2)/a$). Now the involutions take the form

$$i_1\,:\,(a,b,c) \to (-a,-b,-c)$$

and

$$i_2\,:\,\det(x) \to \det(x)^{-1}$$

and we can write down independent generators of the field of invariants. If

$$D := \frac{\det(x)+1}{\det(x)-1}$$

then

$$\begin{array}{cccc} i_2: & D & \mapsto & -D \\ i_1: & (a,b,c,D) & \mapsto & (-a,-b,-c,-D). \end{array}$$

This finishes the proof. □

A *(rational) slice* for the action of G is a subvariety $S \subset X$ such that the general G-orbit intersects S in exactly one point. (The slice S need not be a rational variety. To avoid confusion, we will always refer to S as a slice.) A subvariety $Y \subset X$ is called a (G, H)-*slice* (where $\mathrm{H} \subset \mathrm{G}$ is a subgroup) if $\mathrm{G}\cdot Y \sim X$ and $gy \in Y$ implies that $g \in \mathrm{H}$. Clearly, $\mathrm{G}\backslash X \sim \mathrm{H}\backslash Y$. Moreover, if $f : X \to X'$ is a G-equivariant morphism and Y' is a (G, H)-slice in X' then $f^{-1}(Y')$ is a (G, H)-slice in X.

NOTATIONS 7.13. — For (a reductive group) G acting (rationally) on X we denote by

$$\mathrm{St}_{gen} = \mathrm{St}_{gen}(\mathrm{G}, X)$$

the generic stabilizer (defined up to conjugacy). The action is called an af-action (*almost free*) if St_{gen} is trivial.

We use a more precise version of Theorem 7.7:

THEOREM 7.14. — [17] *Let $\rho : \mathrm{PGL}_2 \to \mathrm{PGL}(V)$ be a representation and $\tilde{\rho}$ a lifting of ρ to a representation of $\mathrm{GL}_2 \to \mathrm{GL}(V)$. Let*

$$\mathrm{G}'' := \mathrm{St}_{gen}(\mathrm{GL}_2, V) \text{ and } \mathrm{G} := \mathrm{GL}_2/\mathrm{G}''.$$

If the central $\mathfrak{C}_2 \not\subset \mathrm{G}''$ then

$$\mathbb{P}(V) \sim_{\mathrm{G}} \mathrm{G} \times S,$$

where S is a rational variety (with trivial G-action).
If $\mathfrak{C}_2 \subset \mathrm{G}''$ then

- *either the PGL_2-action on $\mathbb{P}(V)$ has* no *slice and $\mathrm{G}\backslash\mathbb{P}(V)$ is rational*
- *or*

$$\mathbb{P}(V) \sim_{\mathrm{G}} \mathrm{G} \times S,$$

  where the slice S is a rational variety (with trivial G-action).

We now explain some general techniques in the study of rationality of quotient varieties.

LEMMA 7.15. — *Let $E \to X$ be a vector bundle of rank $r = \mathrm{rk}\,(E)$. Let G be an (affine) reductive group acting on E such that the generic orbit of G in E projects isomorphically onto a generic orbit of G in X. Then*

$$E \sim_{\mathrm{G}} X \times \mathbb{A}^r$$

with trivial G*-action on the affine space* $\mathbb{A}^r$.

Proof. — Denote by O the G-orbit through a generic point in E. Shrinking (equivariantly) X, if necessary, we may assume that the map

$$\pi\,:\,H^0(X,E)\to H^0(O,E|_O)$$

is surjective. With our assumptions, there exists a basis $s_1,...,s_r$ such that for each j, the G-orbit of s_j projects isomorphically onto its image in X and generates a trivial 1-dimensional G-equivariant sub-bundle of the restriction $E|_O$ of E to the orbit O. It follows that $E|_O=\oplus_{j=1}^r\mathrm{G}\cdot s_j$. In particular, $H^0(O,E)$ contains the trivial G-module M generated by $s_1,...,s_r$. Moreover, M generates $H^0(O,E|_O)$ over every point of O. Since π is a map of G-modules and G is reductive $H^0(X,E)$ contains a submodule M' such that $\pi(M')=M$ (as G-modules). The elements of M' generate E over a generic point of X. A basis $s_1',...,s_r'$ of M' gives the desired splitting of the action. □

COROLLARY 7.16. — *Let* G *be a reductive group and*

$$E''\to E'\to X$$

a G*-equivariant sequence of vector bundles such that the generic* G*-orbit of* E' *projects isomorphically onto its image. Choose a generic* G*-equivariant section* s' *of* $E'\to X$ *and denote by* $E''_{s'}$ *the restriction of* E'' *to this section. Then*

$$E''\sim_{\mathrm{G}}E''_{s'}\times\mathbb{A}^{r'}$$

(where $r'=\mathrm{rk}\,E'$*), with trivial* G*-action on* $\mathbb{A}^{r'}$.

PROPOSITION 7.17. — *Let* X *be a variety with an action* $\rho\,:\,\mathrm{G}\to X$ *of a linear algebraic group* G. *Let* $E\to X$ *be a vector bundle and* $\tilde\rho\,:\,\tilde{\mathrm{G}}\to E$ *a* $\tilde{\mathrm{G}}$*-action lifting* ρ. *Consider a generic orbit* $\mathrm{G}\cdot x\subset X$ *and the linear action of* $\tilde{\mathrm{G}}$ *on the space of sections* $H^0(X,E)$.

Assume that $\tilde{\mathrm{G}}$ *is reductive and* V *is a linear representation of* $\tilde{\mathrm{G}}$ *which is contained in* $H^0(X,E)$. *Then there exists an affine open* $X'\subset X$ *such that the vector bundle* $E\to X'$ *admits a* $\tilde{\mathrm{G}}$*-map onto a* $\tilde{\mathrm{G}}$*-representation* V^*.

If the action of G on X is almost free we may think of X as being (birational to) a principal fibration over the quotient $\mathrm{G}\backslash X$ with fiber G. If G is *affine* we may assume that X and $\mathrm{G}\backslash X$ are also affine. Let us also recall a standard general construction of G-maps: if the ring $\mathbb{C}[X]$ is a direct sum of G-modules then any G-submodule $V\subset\mathbb{C}[X]$ defines a G-map $X\to\mathrm{Spec}(V)$. We also have a vector bundle version of the above construction: let $E \to X$ be a G-vector bundle and O a G-orbit through a generic point. Assume that $H^0(O,E)$ (the restriction of the space of sections to O) contains V as a submodule. We obtain a G-map

$$v\,:\,H^0(O,E)\to V^*$$

(the dual module, considered as a vector bundle over a point).

LEMMA 7.18. — *There exists a* G*-stable Zariski open* $U \subset X$ *and a rational* G*-map of* $H^0(U,E)\to V^*$ *extending* v*.*

Proof. — A generic orbit O has a G-equivariant neighborhood U, with U/G affine, such that

$$H^0(U,E) \twoheadrightarrow H^0(O,E).$$

The module $H^0(U,E)$ is a direct sum of finite dimensional irreducible G-modules. We can now take any submodule $V \subset H^0(U,E)$ which surjects isomorphically onto a submodule in $H^0(O,E)$. □

LEMMA 7.19. — *If* X *has an* af*-action of* PGL_2 *then*

$$X\times\mathbb{P}(V_{2d})\sim_{\mathrm{PGL}_2} X\times\mathbb{P}(V_{2d}),$$

with diagonal PGL_2*-action on the left and trivial* PGL_2*-action on* $\mathbb{P}(V_{2d})$ *on the right.*

Proof. — We know that $\mathbb{C}[\mathrm{PGL}_2]$, as a PGL_2-module, is sum of all even modules V_{2d}. This gives a PGL_2-map $s\,:\,X\to\mathbb{P}(V_{2d})$. The quotient

$$\mathrm{PGL}_2\backslash X\times\mathbb{P}(V_{2d})$$

is a projective bundle over the quotient $\mathrm{PGL}_2\backslash X$, with a section obtained from s. Therefore, it is birational to the product $(\mathrm{PGL}_2\backslash X)\times\mathbb{P}(V_{2d})$, which gives the claimed PGL_2-isomorphism. □

COROLLARY 7.20. — *Let* X *be a variety with an* af*-action of* PGL_2*. Then* X *is a* $(\mathrm{PGL}_2,\mathrm{N_T})$*-slice in*

$$X\times\mathbb{P}(V_2)$$

(with diagonal PGL_2*-action).*

LEMMA 7.21. — *Assume that* X *has an* af*-action* ρ *of* PGL_2. *Let* $\mathcal{V} \xrightarrow{V} X$ *be a vector bundle over* X *with an action* $\tilde{\rho}$ *of* GL_2 *lifting* ρ. *Assume that* X *contains a* PGL_2*-orbit* $Y \sim \mathrm{PGL}_2$ *such that the* GL_2*-module* $H^0(Y, \mathcal{V}_Y)$ *contains* V_d, *for some odd* d. *Then*

$$\mathbb{P}(\mathcal{V}) \sim_{\mathrm{PGL}_2} \mathrm{PGL}_2 \times S,$$

(with trivial PGL_2*-action on* S*). Otherwise,* $\mathcal{V}$ *is induced from a* GL_2*-vector bundle on* $\mathrm{PGL}_2 \backslash X$.

Proof. — Let Y be an orbit such that $H^0(Y, \mathcal{V}_Y)$ contains V_d, for some odd d. Shrinking X, if necessary, gives a surjective map of GL_2-modules

$$H^0(X, \mathcal{V}) \longrightarrow\!\!\!\!\!\rightarrow H^0(Y, \mathcal{V}_Y).$$

Since $H^0(Y, \mathcal{V}_Y)$ is an algebra over $H^0(Y, \mathcal{O}_Y) = \oplus_{d \geq 0} V_{2d}$, it contains V_1 as a submodule. We obtain a PGL_2-equivariant surjective map

$$\mathbb{P}(\mathcal{V}) \to \mathbb{P}(V_1) = \mathbb{P}^1.$$

Since the stabilizer of a point in $\mathbb{P}^1$ is solvable, we get a slice $S \subset \mathbb{P}(\mathcal{V})$, as claimed.

Assume that there is an orbit $Y \sim \mathrm{PGL}_2$ such that $\mathcal{V}_Y$ contains only even weight GL_2-submodules. Then the central $\mathfrak{C}_2 \subset \mathrm{GL}_2$ acts trivially on $\mathcal{V}_Y$. If follows that $\mathcal{V}_Y$ is a trivial PGL_2-bundle, and $H^0(Y, \mathcal{V}_Y)$ a trivial PGL_2-module. The semi-simplicity of the PGL_2-action implies that $H^0(X, \mathcal{V})$ contains $H^0(Y, \mathcal{V}_Y)$ as a sub-module. Shrinking X if necessary, we can find linearly independent PGL_2-invariant sections, whose specializations to Y generate $H^0(Y, \mathrm{PGL}_2)$. Therefore, $\mathcal{V}$ is lifted from the quotient $\mathrm{PGL}_2 \backslash X$. □

LEMMA 7.22. — *Let* V *be a representation of* G *of dimension* ≥ 2 *(with* G *acting on the left). Then* $V \oplus V$ *is a* $\mathrm{G} \times \mathrm{GL}_2$*-space (with right* GL_2*-action) and*

$$V \oplus V \sim_{\mathrm{G} \times \mathrm{GL}_2} \begin{array}{c} \mathcal{V} \\ \big\downarrow \scriptstyle{\mathrm{M}_2 = V_1 \oplus V_1} \\ \mathrm{Gr}(2, V), \end{array}$$

a vector bundle with fibers 2×2*-matrices (with right* GL_2*-action).*

Proof. — Consider the map

$$\begin{array}{ccc} V \oplus V & \to & \mathrm{Gr}(2, V) \\ (v, v') & \mapsto & \langle v, v' \rangle, \end{array}$$

defined on the open, $\mathrm{G}\times\mathrm{GL}_2$-invariant subset of noncollinear pairs $(v,v')\in V\oplus V$ (with fibers consisting of pairs spanning the same 2-space). The GL_2-action on the fibers is the right multiplication on matrices:

$$(v,v')\mapsto(av+bv',cv+dv').$$

□

Assume that G is reductive and denote by $\mathrm{G}'':=\mathrm{St}_{gen}(\mathrm{G},\mathrm{Gr}(2,V))$ and by $\mathrm{G}':=\mathrm{G}/\mathrm{G}''$ the quotient group of G which acts effectively on $\mathrm{Gr}(2,V)$.

COROLLARY 7.23. — *Assume that the action of* G' *on* $\mathrm{Gr}(2,V)$ *has a slice* S *so that* $\mathrm{Gr}(2,V)\sim S\times\mathrm{G}'$. *Let* $\mathcal{V}_S$ *be the restriction of* $\mathcal{V}$ *to* S *(this makes sense by Corollary 7.16). Then*

$$\mathrm{G}\backslash\mathcal{V}/\mathrm{GL}_2\sim\mathrm{G}'\backslash\mathcal{V}_S.$$

REMARK 7.24. — The group G'' acts as scalars on $\mathcal{V}=\mathcal{V}_1\oplus\mathcal{V}_1$ (it commutes with GL_2).

LEMMA 7.25. — *Assume that we are in the situation of Corollary 7.23,* $\mathrm{G}=\mathrm{GL}_2$ *and* $\mathrm{H}\subset\mathrm{GL}_2$ *has finite image in* PGL_2. *Then* $\mathrm{G}\backslash\mathcal{V}/\mathrm{H}$ *is rational.*

Proof. — By Corollary 7.24, the slice S is 3-stably rational, since

$$S\times\mathrm{PGL}_2\sim\mathrm{Gr}(2,V)$$

and $\mathrm{Gr}(2,V)$ is rational. The quotient of $\mathcal{V}_S$ by a fiberwise linear action is birational to $(\mathrm{M}_2/\mathrm{H})\times S$ (every vector bundle admits an H-equivariant trivialization over an open subset of S). There is a left action of $\mathbb{G}_m^2\subset\mathrm{GL}_2$ on $\mathrm{M}_2=V_1\oplus V_1$ which commutes with H. Thus M_2/H is (birationally) a three-dimensional variety with an af-action of $\mathbb{G}_m$. The quotient (a surface) is unirational, hence rational (by Theorem 7.2), and

$$\mathrm{G}\backslash(V\oplus V)/\mathrm{H}\sim S\times(V_1\oplus V_1)/\mathrm{H}\sim S\times\mathbb{C}^3.$$

□

The group PGL_2 acts on $\mathbb{P}(\mathrm{M}_2)$ on both sides. We will need an explicit description of the action for some of its subgroups.

LEMMA 7.26. — *We have*

$$\mathrm{St}_{gen}(\mathrm{N_T} \times \mathrm{N_T}, \mathbb{P}(\mathrm{M}_2)) = \mathfrak{C}_2.$$

Proof. — Indeed $\mathrm{N_T}$ contains

$$\begin{array}{ccccc} \mathbb{G}_m = \{t\} & : & (x,y) & \mapsto & (tx, t^{-1}y), \\ i & : & (x,y) & \mapsto & (y,x). \end{array}$$

The corresponding actions on $\mathbb{P}(\mathrm{M}_2)$ are

$$(a,b,c,d) \mapsto (t_1t_2a, t_1^{-1}t_2b, t_1t_2^{-1}c, t_1^{-1}t_2^{-1}d)$$

and

$$\begin{array}{ll} i_1 : & a \to c, \quad b \to d \\ i_2 : & a \to b, \quad c \to d, \end{array}$$

respectively. A matrix $(a,b,c,d) \in \mathrm{M}_2$ can be transformed to $(1,1,1,d)$ by a unique element of $\mathbb{G}_m \times \mathbb{G}_m$, the $\mathfrak{S}_2 \times \mathfrak{S}_2$-orbit of which consists of two elements (for d, d^{-1}). □

COROLLARY 7.27. — *The group* $\mathrm{N_T} \times \mathbb{C}^*$ *acts almost freely on* $\mathbb{P}(\mathrm{M}_2)$. *There is an open,* $\mathrm{N_T} \times \mathrm{N_T}$*-stable subvariety* $U \subset \mathbb{P}(\mathrm{M}_2)$ *such that*

$$\begin{array}{c} U \\ \Big\downarrow {\scriptstyle \mathbb{C}^* \times \mathbb{C}^*} \\ \mathbb{C}^* \subset \mathbb{P}^1, \end{array}$$

with a transitive action of $\mathbb{C}^* \times \mathbb{C}^* \subset \mathrm{N_T} \times \mathrm{N_T}$ *on the fibers. The diagonal subgroup*

$$\mathfrak{S}_2^{\Delta} \subset \mathfrak{S}_2 \times \mathfrak{S}_2 = (\mathrm{N_T} \times \mathrm{N_T})/(\mathbb{C}^* \times \mathbb{C}^*)$$

acts on each fiber as an involution $x \mapsto x^{-1}$. *The factor* $\mathfrak{S}_2 = (\mathfrak{S}_2 \times \mathfrak{S}_2)/\mathfrak{S}_2^{\Delta}$ *acts on the base* $\mathbb{C}^* \subset \mathbb{P}^1$ *as an involution without fixed points, on the first factor in the fiber as* $x \to x^{-1}$*, and as identity on the second factor.*

COROLLARY 7.28. — *Let* $\mathfrak{D} \subset \mathrm{N_T}$ *be a dihedral subgroup such that* $\mathfrak{D}\backslash\mathrm{N_T} = \mathbb{C}^*$. *Then the* $\mathbb{C}^*$*-bundle*

$$\mathcal{C} = \mathfrak{D}\backslash\mathbb{P}(\mathrm{M}_2) \to \mathrm{N_T}\backslash\mathbb{P}(\mathrm{M}_2)$$

is induced from the $\mathbb{C}^*$*-bundle*

$$\mathfrak{D}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{N_T} \to \mathrm{N_T}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{N_T} = \mathbb{P}^1$$

and is hence birationally trivial.

Proof. — Indeed, the left and the right actions of $\mathrm{N_T}$ commute. By Lemma 7.26, $\mathrm{St}_{gen}(\mathrm{N_T} \times \mathrm{N_T}, \mathbb{P}(\mathrm{M}_2)) = \mathfrak{C}_2$, which implies that the bundle is induced. $\square$

LEMMA 7.29. — *For every dihedral group* $\mathfrak{D}$ *and every* $\mathrm{H} \subset \mathrm{N_T}$ *the conic bundle*

$$\mathcal{C}_{\mathrm{H}} = \mathfrak{D}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} \to \mathrm{N_T}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H},$$

has a section.

Proof. — The quotient $\mathfrak{D}\backslash U/\mathrm{H}$ from Corollary 7.27 admits a fibration

$$\begin{array}{c} \mathfrak{D}\backslash U/\mathrm{H} \\ \Big\downarrow {\scriptstyle \mathbb{C}^*_{\mathfrak{D}} \times \mathbb{C}^*_{\mathrm{H}}/\mathfrak{S}_2} \\ \mathbb{P}^1/\mathfrak{S}_2. \end{array}$$

Here $\mathbb{C}^*_{\mathfrak{D}} \times \mathbb{C}^*_{\mathrm{H}}$ is the quotient of the fiber $\mathbb{C}^* \times \mathbb{C}^*$ of $U \to \mathbb{C}^*$ by the intersection of $\mathfrak{D}, \mathrm{H}$ with the diagonal $\mathbb{C}^*_{\Delta} \subset \mathbb{C}^* \times \mathbb{C}^*$. Isomorphisms $\mathbb{C}^*_{\mathrm{H}} \to \mathbb{C}^*$ and $\mathbb{C}^*_{\mathfrak{D}} \to \mathbb{C}^*$ induce a birational fiberwise isomorphism

$$\begin{array}{ccccc} \mathcal{C}_{\mathrm{H}} = & \mathfrak{D}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} & \sim & \mathfrak{S}_2\backslash\mathbb{P}(\mathrm{M}_2)/\mathfrak{S}_2 & = \mathcal{C}_0 \\ & \downarrow & & \downarrow & \\ & \mathrm{N_T}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} & & \mathrm{N_T}\backslash\mathbb{P}(\mathrm{M}_2)/\mathfrak{S}_2 & \end{array}$$

and it suffices to consider $\mathfrak{D} = \mathfrak{S}_2, \mathrm{H} = \mathfrak{S}_2$. In this case, an alternative equivariant completion of U is given by

$$\begin{array}{ccc} U \subset & \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3 \,, \\ & \downarrow \\ & \mathbb{P}^1_3 \end{array}$$

with an action of $\mathfrak{S}_2 \times \mathfrak{S}_2$, where the first $\mathfrak{S}_2$ acts as an involution on the first two factors and identity on the base while the complementary $\mathfrak{S}_2$ acts only on the base. Thus the quotient is a conic bundle over the complement in

$$\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_2 \times \mathfrak{S}_2 = \mathbb{P}^1 \times \mathbb{P}^1$$

to the branch locus of the quotient map. Here the left (resp. right) $\mathfrak{S}_2$ acts as an involution on the left (resp. right) $\mathbb{P}^1$ and the branch locus is exactly the union of four lines. By Lemma 7.3, this conic bundle has a section (it is nonsingular on a pencil of lines minus at most two points). □

LEMMA 7.30. — *Let* G *be a subgroup of* SL_2*, not equal to* $\tilde{\mathfrak{A}}_5$*, and* V *a linear representation of* G*. Then* $\mathrm{G}\backslash\mathbb{P}(V)$ *is rational.*

Proof. — For $\mathrm{G} = \mathrm{SL}_2$ this is a theorem of Katsylo [17]. We now consider proper subgroups $\mathrm{G} \subsetneq \mathrm{SL}_2$. If G is solvable and connected then rationality for the quotient follows from a theorem of Vinberg [36]. For compact G the proof is similar to the dihedral case described below. Assume now that G is finite and not equal to $\tilde{\mathfrak{A}}_5$. Then G is either

1. a finite subgroup of $\mathbb{C}^*$,
2. a dihedral group or
3. $\tilde{\mathfrak{A}}_4, \tilde{\mathfrak{S}}_4$.

The first case is easy. For dihedral groups all irreducible representations of G have dimension ≤ 2 and the corresponding quotients are rational by Theorem 7.2. Let V be a faithful representation of a dihedral group $\mathfrak{D}$ (otherwise, we are reduced to a quotient group). Thus $V = W \oplus W'$, where $\dim W = 2$ and $\dim W' \ge 1$. Denote by $\mathrm{G}' = \mathrm{G}/\mathfrak{C}'$ the quotient acting faithfully on W' ($\mathfrak{C}'$ is a cyclic group). We have $W \sim_{\mathfrak{D}} \mathbb{C}^* \times \mathbb{P}^1$, with trivial action of $\mathfrak{D}$ on $\mathbb{C}^*$ and trivial action of $\mathfrak{C}'$ on $\mathbb{P}^1$. By Lemma 7.15,

$$\mathbb{C}^* \times \mathbb{P}^1 \times W' \sim_{\mathrm{G}'} (\mathbb{C}^* \times \mathbb{P}^1) \times W',$$

with trivial action of G' on $\mathbb{C}^* \times \mathbb{P}^1$. Thus

$$\mathfrak{D}\backslash V \sim (\mathfrak{D}'\backslash W') \times (\mathbb{C}^* \times \mathbb{P}^1)$$

and we can apply induction.

We turn to the last case. An irreducible representation of $\tilde{\mathfrak{A}}_4$ is either a character, or a faithful two-dimensional representation, or a three-dimensional representation, trivial on the center (a faithful representation of $\mathfrak{A}_4$). An irreducible representation of $\tilde{\mathfrak{S}}_4$ is either a faithful two-dimensional representation, a faithful four-dimensional representation $W := \mathrm{Sym}^3(V_1)$ or a representation of $\mathfrak{S}_4$ (of dimension ≤ 3).

For irreducible representations of dimension ≤ 3 rationality for the quotient follows from Theorem 7.2. We turn to W. Recall that

$$W = \mathrm{Sym}^3(V_1) = V_1^{\chi} \oplus V_1^{-\chi},$$

as a $\tilde{\mathfrak{A}}_4$-representation, where $V_1^{\chi} = V_1 \otimes \chi$, $V^{-\chi} = V_1 \otimes \chi^{-1}$ and

$$\chi : \mathfrak{A}_4 \to \mathbb{Z}/3 \subset \mathbb{C}^*$$

is the cubic character. A pair of (generic) points

$$p_{\chi} \in \mathbb{P}^1 = \mathbb{P}(V^{\chi}), p_{-\chi} \in \mathbb{P}^1 = \mathbb{P}(V^{-\chi})$$

defines a line $\mathbb{P}^1 \subset \mathbb{P}(W)$. This shows that

$$\begin{array}{ccc} \mathbb{P}(W) \sim_{\tilde{\mathfrak{S}}_4} & & \\ & \Big\downarrow L & , \\ & \mathbb{P}^1 \times \mathbb{P}^1 & \end{array}$$

where $\mathfrak{S}_4$ acts on the base, $\mathfrak{A}_4$ acts linearly on the fiber L and $\mathfrak{S}_2 = \tilde{\mathfrak{S}}_4/\mathfrak{S}_4$ acts as an involution on the fiber L. Thus $\tilde{\mathfrak{S}}_4\backslash\mathbb{P}(W)$ is a conic bundle over the rational surface $\mathfrak{S}_4\backslash(\mathbb{P}^1 \times \mathbb{P}^1)$. We now analyze the geometry of this bundle in more detail. Consider the action $\mathfrak{D}_2 \subset \mathfrak{S}_4$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and on $\mathbb{P}^2 = \mathrm{Sym}^2(\mathbb{P}^1)$. Every involution $i \in \mathfrak{D}_2$ has two invariant points x_i, y_i. Consider the graphs $\mathbb{P}^1$ connecting the points $(x_i, y_i) - (y_i, x_i)$. Their set is equal to $\mathbb{P}^1$ and there is a graph:

$$l_i : (x_i, y_i) - (y_i, x_i) \subset \mathbb{P}^1 \times \mathbb{P}^1$$

consisting of points $(x, i(x))$. The line l_i is exactly the subset of i-invariant points in $\mathbb{P}^1 \times \mathbb{P}^1$. The action of $\mathfrak{D}_2$ is free outside the three lines $l_i, i \in \mathfrak{D}_2, i \neq 1$. There are exactly 6 points which are invariant under $\mathfrak{D}_2$.

The corresponding action on $\mathbb{P}^2$ can be described as follows. There are three points corresponding to (x_i, y_i) which are stable under $\mathfrak{D}_2$ and three lines (images of l_i) so that the action is free on the torus $\mathbb{C}^* \times \mathbb{C}^*$ (the complement in $\mathbb{P}^2$ to the union of l_i). The group $\mathfrak{D}_2$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ as a translation by the subgroup of points of order 2.

The quotient $\mathbb{P}^2_q := \mathfrak{D}_2\backslash\mathbb{P}^2$ is a nonsingular variety isomorphic to $\mathbb{P}^2$ (indeed the only possible singularities come from the three $\mathfrak{D}_2$-invariant points in $\mathbb{P}^2$ but the quotient by the local action is nonsingular). The diagonal $\mathbb{P}^1_{\Delta} \subset \mathbb{P}^1 \times \mathbb{P}^1$ projects onto a conic $C \subset \mathbb{P}^2$, which is invariant under $\mathfrak{D}_2$. The conic C intersects the "vertical" and "horizontal" subgroups in $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^2$ in two points and does not intersect the line at infinity.

Thus in $\mathbb{P}^2_q = \mathfrak{D}_2\backslash\mathbb{P}^2$, the image of $\mathbb{P}^1_{\Delta}$ intersects $\mathbb{C}^*$ in one point. Therefore, the images of $\mathbb{P}^1_{\Delta}$ and of l_i are lines (since pairwise intersections of the l_i are equal to 1) and the $(\mathfrak{C}_2)^3$-covering $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2_q$ is ramified exactly over a union of four lines. If suffices to observe that every conic bundle over $\mathbb{P}^2_q$ has a section. Indeed, let p be the intersection point of two lines l_i and $l_{i'}$ and consider the pencil of lines in $\mathbb{P}^2_q$ through p. Each line in this pencil intersects the ramification locus in at most three points and we can apply Lemma 7.3.

Now we turn to reducible representations $V = \oplus_{\alpha\in\mathcal{A}} V^\alpha$ of $\tilde{\mathfrak{A}}_4$. If V is faithful for $\tilde{\mathfrak{A}}_4$ then there is an $\alpha_0 \in \mathcal{A}$ such that V^{α_0} is a three-dimensional irreducible *faithful* representation of $\tilde{\mathfrak{A}}_4$ and

$$V \sim_{\tilde{\mathfrak{A}}_4} V^{\alpha_0} \times (\oplus_{\alpha\neq\alpha_0} V^\alpha)$$

with trivial action of $\tilde{\mathfrak{A}}_4$ on $\oplus_{\alpha\neq\alpha_0} V^\alpha$ (by Lemma 7.15). If V is faithful for $\mathfrak{A}_4$ then V contains a faithful irreducible three-dimensional representation of $\mathfrak{A}_4$ and we can apply the same argument. In all other cases V is a sum of one-dimensional representations and we are reduced to Case 1.

Finally, consider reducible representations V of $\tilde{\mathfrak{S}}_4$. If V is faithful then it contains either a faithful irreducible two-dimensional representation or the faithful representation W. Again, we apply Lemma 7.15 as before. If V is faithful for $\mathfrak{S}_4$ then it contains a faithful irreducible representation of dimension ≤ 3 and we conclude as above. In all other cases V is a sum of one-dimensional representations. □

LEMMA 7.31. — *Let V be a representation of $\mathrm{G} \subsetneq \mathrm{SL}_2$, with $\mathrm{G} \neq \tilde{\mathfrak{A}}_5$. Then $\mathrm{G}\backslash \mathrm{Gr}(2, V)$ is rational.*

Proof. — The relevant groups G can be subdivided as follows:

1. G is a subgroup of the normalizer of a maximal torus;
2. G an infinite subgroup of a Borel subgroup;
3. $\mathrm{G} = \mathfrak{A}_4, \tilde{\mathfrak{A}}_4$;
4. $\mathrm{G} = \mathfrak{S}_4, \tilde{\mathfrak{S}}_4$.

Let $V = W \oplus W'$ be a reducible representation of G. Then (birationally)

$$\begin{array}{c} \mathrm{Gr}(2,V) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(\mathbb{C}^2_x, W)} \\ \mathrm{Gr}(2,W) \end{array}$$

(where x is a point on the base). In particular, if $\dim W \leq 2$ then

$$\mathrm{Gr}(2, V) \sim_{\mathrm{G}} \mathrm{Hom}(W', W),$$

with linear G-action on $\mathrm{Hom}(W', W)$. This reduction suffices for the relevant infinite groups (for example, for connected solvable G we can apply Lemma 7.6). Further,

- if $\mathrm{St}_{gen}(\mathrm{G}, \mathrm{Gr}(2, V)) = 1$ then (birationally)
$$\mathrm{G}\backslash \mathrm{Gr}(2, V) \to \mathrm{G}\backslash \mathrm{Gr}(2, W),$$
a vector bundle.
- if $\mathrm{St}_{gen}(\mathrm{G}, \mathrm{Gr}(2, V)) = \mathfrak{C} \subset \mathrm{Z}_{\mathrm{G}}$ (a cyclic subgroup) then (birationally)
$$\begin{array}{c} \mathrm{G}\backslash \mathrm{Gr}(2, V) \\ \big\downarrow {\scriptstyle \mathfrak{C}\backslash \mathrm{Hom}(\mathbb{C}^2_x, W)} \\ \mathrm{G}\backslash \mathrm{Gr}(2, W). \end{array}$$

We now consider $\mathfrak{A}_4, \tilde{\mathfrak{A}}_4, \mathfrak{S}_4$. The rationality of $\mathrm{G}\backslash \mathrm{Gr}(2, V)$ for *irreducible* representations of these groups follows from the fact that all of them have dimension ≤ 3. Assume now that $V = W \oplus W'$, with W irreducible of dimension 3. The classification of these representations implies that the action of the center must be trivial. Then, birationally,
$$\begin{array}{c} \mathrm{Gr}(2, V) \\ \big\downarrow {\scriptstyle \mathrm{Hom}(\mathbb{C}^2_x, W')} \\ \mathbb{P}^2 = \mathbb{P}(W^*). \end{array}$$
The G-action is equivalent to a G-action on a vector bundle
$$\begin{array}{c} \mathrm{G}\backslash \mathrm{Gr}(2, V) \\ \big\downarrow \\ \mathrm{G}\backslash \mathrm{Gr}(2, W) = \mathrm{G}\backslash \mathbb{P}^2. \end{array}$$

Finally, let us consider the case of $\tilde{\mathfrak{S}}_4$. Let W be its unique irreducible representation of dimension four (as in Lemma 7.30). We claim that $\tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, W)$ is rational. Indeed, as $\tilde{\mathfrak{A}}_4$-modules, we have
$$W = W^{\chi} \oplus W^{-\chi},$$
where $W^{\chi}, W^{-\chi}$ are two copies of the standard representation of $\tilde{\mathfrak{A}}_4$ of dimension 2 and χ (resp. $-\chi$) indicates the eigenspace decomposition for the nontrivial character
$$\chi : \mathfrak{A}_4 \to \mathbb{Z}/3 \subset \mathbb{C}^*.$$

Further,

$$\mathrm{Gr}(2, W) \sim \mathrm{Hom}(W^{\chi}, W^{-\chi}),$$

with a linear $\mathfrak{A}_4$-action (since the center acts trivially) and a permutation $\mathfrak{S}_2$ inverting the map $w \in \mathrm{Hom}(W^{\chi}, W^{-\chi})$. More precisely, $W^{-\chi} = (W^{\chi})^*$ and

$$\mathrm{Hom}(W^{\chi}, W^{-\chi}) = \mathrm{Sym}^2(W^{-\chi}) \oplus C_1,$$

where C_1 corresponds to skew symmetric maps and $\mathfrak{A}_4$ acts on C_1 by χ. The involution $\mathfrak{S}_2 = \mathfrak{S}_4/\mathfrak{A}_4$ acts on C_1 and on $\mathrm{Sym}^2(W^{-\chi})$ as $t \mapsto t^{-1}$. In particular, if $\mathbb{C}^* \times \mathbb{C}^*$ is the diagonal group acting on $\mathrm{Sym}^2(W^{-\chi}) \oplus C_1$ then $\mathfrak{S}_2$ acts as

$$X \to s^{-1}X,$$

where $s \in \mathbb{C}^* \times \mathbb{C}^*$ and $X \in \mathrm{Sym}^2(W^{-\chi}) \oplus C_1$. There is an equivariant map

$$\begin{array}{rcccl} f & : & \mathrm{Hom}(W^{\chi}, W^{-\chi}) & \to & C_1, \\ & & s & \mapsto & (x, s(y)) - (s(x), y), \end{array}$$

with an effective action of $\mathfrak{S}_3 = \mathfrak{S}_4/\mathfrak{D}_2$ on the target C_1, which to a subspace $s \in \mathbb{C}^2 \subset W^{\chi} \oplus W^{-\chi}$ assigns the value of the 2-form $(x, s(y)) - (s(x), y)$. The fiber of f is $\mathfrak{D}_2$-birational to $\mathrm{Sym}^2(W^{\chi}) = \mathbb{P}^2$. We have already seen in the proof of Lemma 7.30 that $\mathfrak{D}_2\backslash\mathbb{P}^2 = \mathbb{P}^2$. Thus $\tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, W)$ is a $\mathbb{C}^*$- bundle over a $\mathbb{P}^2$-fibration over $\mathfrak{S}_2\backslash C_1$. It is clear that this $\mathbb{P}^2$-fibration is trivial. The quotient conic bundle is nondegenerate over a product of $\mathbb{P}^2$ with an open subvariety in $C_1/\mathfrak{S}_3$. Hence it has a section. Rationality of $\tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, W)$, and more generally, $\tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, W \oplus \cdots \oplus W)$, follows (the latter is a vector bundle over the former).

Assume now that $V = nW \oplus V'$, where $\dim V' \geq 1$, and $n \in \mathbb{N}$. Since the $\mathfrak{S}_4$-action on $\mathrm{Gr}(2, nW)$ is af there is a $\tilde{\mathfrak{S}}_4$-equivariant homogeneous rational map $f : \mathrm{Gr}(2, nW) \to V'$ sending the generic $\tilde{\mathfrak{S}}_4$-orbit in W to the generic $\tilde{\mathfrak{S}}_4$-orbit in V'. Notice that the center $\mathfrak{C}_2$ acts as a scalar on $\mathrm{Hom}(W, V')$. We have (birationally)

(7.1)
$$\begin{array}{ccccc} \tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, V) & \sim & \mathbb{C}^* \times & & \\ \downarrow {\scriptstyle \mathfrak{S}_2\backslash \mathrm{Hom}(\mathbb{C}^2_x, V')} & & & & \downarrow {\scriptstyle \mathbb{P}(\mathrm{Hom}(\mathbb{C}^2_x, V'))} \\ \tilde{\mathfrak{S}}_4\backslash \mathrm{Gr}(2, nW) & & & & \mathfrak{S}_4\backslash \mathrm{Gr}(2, nW) \end{array}$$

(with rational bases). The projective bundle on the right has a section. Indeed,

(7.2)
$$\begin{array}{c} \downarrow {\scriptstyle \mathrm{Hom}(\mathbb{C}^2_x, V')} \\ \mathrm{Gr}(2, nW) \end{array}$$

is an equivariant quotient bundle of the trivial bundle with fiber $\mathrm{Hom}(W, V')$. The map f defines an $\mathfrak{S}_4$-equivariant section $s(f)$ in the projective bundle in (7.1). The (equivariant) linear projection

$$\mathrm{Hom}(W, V') \to \mathrm{Hom}(\mathbb{C}^2_x, V')$$

maps $s(f)$ to an equivariant section of the bundle in (7.2). Thus $s(f)$ projects onto a section of the bundle on the right in (7.1), making it birationally trivial. □

We proceed to describe possible SL_2, resp. PGL_2-actions on Grassmannians. (If all weights in V are of the same parity then $\mathrm{Gr}(2, V)$ carries the PGL_2-action, otherwise the SL_2-action.)

LEMMA 7.32. — *Let V be a faithful SL_2-representation of dimension ≥ 3. Then*

V	St_{gen}
$\dim \geq 5$	1
V_4	$\mathfrak{C}_2$
V_3	$\mathfrak{D}_2$
$V_2 \oplus V_0$	$\mathfrak{C}_2$
V_2	N_{T}
$V_1 \oplus V_1$	$\mathbb{C}^*$
$V_1 \oplus V_0$	$\tilde{\mathrm{B}}$

Further,

- $\mathrm{Gr}(2, V_4)$ *has a* $(\mathrm{PGL}_2, \mathrm{N}_{\mathrm{T}})$*-slice* $S = \mathrm{Sym}^2(\mathbb{P}^2)$ *with an* af*-action of* $\mathrm{N}_{\mathrm{T}}/\mathfrak{C}_2$*, (where* $\mathfrak{C}_2$ *is the center of* N_{T}*);*
- $\mathrm{Gr}(2, V_3)$ *has a* $(\mathrm{PGL}_2, \mathfrak{A}_4)$*-slice birational to* $\mathbb{P}^1$*, with* $\mathfrak{A}_4$ *acting on* $\mathbb{P}^1$ *as* $\mathfrak{C}_3$*.*

Proof. — Consider first irreducible representations $V = V_d = \mathrm{Sym}^d(V_1)$ and assume that the stabilizer of a generic line $\mathbb{P}^1 \subset \mathbb{P}(V)$ contains a nontrivial cyclic group $\mathfrak{C}$. Then $\mathfrak{C}$ fixes at least two points in this $\mathbb{P}^1$. Any orbit of $\mathfrak{C}$ on $\mathbb{P}^1$ is a union of a zero-cycle $\mathfrak{C} \cdot x$ and a zero-cycle supported in the fixed points. In particular, the subvariety of points in $\mathbb{P}^(V_d)$ which are fixed by $\mathfrak{C}$ has dimension $\leq d/|\mathfrak{C}|$. The dimension of the variety of $\mathfrak{C}$-fixed lines in $\mathbb{P}(V)$ is therefore $\leq 2d/|\mathfrak{C}|$. The subvariety of distinct cyclic subgroups $\mathfrak{C} \subset \mathrm{PGL}_2$ has dimension 2 and $\dim \mathrm{Gr}(2, V_d) = 2d - 2$. Since $d/|\mathfrak{C}| \leq d/2$ the inequalities

$$2d - 4 > 2d/2 \text{ and } d - 4 > 0$$

imply the result.

Assume that $V = \oplus_{j \in J} V_{d_j}$, $|J| \geq 2$ and that $\mathrm{St}_{gen} \neq 1$. Then $d_j \leq 2$, for all $j \in J$. Indeed, the stabilizer of a generic $\mathbb{P}^1$ through a generic point $p \in \mathbb{P}(V_d)$ is a subgroup of the stabilizer of p, which stabilizes some generic line in the tangent space at p. This group is trivial for $d > 2$ and equal to $\mathfrak{C}_2$ for $d = 2$.

If $V = V_2 \oplus V'$, with $\dim V' > 2$, then $\mathrm{Gr}(2, V)$ is (birationally) a fibration over $\mathrm{Gr}(2, V_2)$, with fibers $\mathrm{Hom}(\mathbb{C}^2, V')$ so that $\mathrm{St}_{gen} = 1$ if $\dim V' > 3$. If $V = V_2 \oplus V_1$ then St_{gen} is the same as the (generic) stabilizer of the $\mathrm{N}_{\tilde{\mathrm{T}}}$-action on $\mathrm{Hom}(V', V_1), V' \in \mathrm{Gr}(2, V_2) = \mathbb{P}^2$, hence trivial. For $V = V_2 \oplus V_0$, $\mathrm{St}_{gen} = \mathfrak{C}_2$.

In the remaining cases $d_j = 0$ or 1, for all $j \in J$. If V contains at least three copies of V_1 then the argument above shows that the action is af. Similarly, if $V = V_1 \oplus V_1$ then $\mathrm{St}_{gen} = \mathbb{C}^*$ and if $V = V_1 \oplus V_1 \oplus V_0$ then $\mathrm{St}_{gen} = 1$. For $V_1 \oplus 3V_0$, the generic stabilizer is the same as for three linear functionals - which is zero. □

LEMMA 7.33. — *The quotient* $\mathrm{PGL}_2 \backslash \mathrm{Gr}(2, V)$ *is 2-stably rational.*

REMARK 7.34. — For *even* $d \geq 10$, $\mathrm{PGL}_2 \backslash \mathrm{Gr}(2, V_d)$ is rational by [32].

Proof. — By Lemma 7.32, if $\dim V \geq 5$ then the $\mathrm{St}_{gen} = 1$ and we can apply Lemma 7.19 and Corollary 7.20 to conclude that

$$\mathrm{PGL}_2 \backslash \mathrm{Gr}(2, V) \times \mathbb{C}^2 \sim_{\mathrm{G}} \mathrm{N}_{\mathrm{T}} \backslash \mathrm{Gr}(2, V).$$

The claim follows from Lemma 7.31. It remains to consider:

1. $\mathrm{Gr}(2, V_4)$,
2. $\mathrm{Gr}(2, V_3)$,
3. reducible V.

In the first case, $\mathrm{St}_{gen}(\mathrm{PGL}_2, \mathrm{Gr}(2, V_4)) = \mathfrak{S}_2$, with normalizer $\mathrm{N}_{\mathrm{T}} \subset \mathrm{PGL}_2$. We claim that the subset $X \subset \mathrm{Gr}(2, V_4)$ of $\mathfrak{S}_2$-invariant points is a $(\mathrm{PGL}_2, \mathrm{N}_{\mathrm{T}})$-slice. Indeed, there is a Zariski open subset $U \subset X$ such that the stabilizer of each point in U is exactly $\mathfrak{S}_2$. In particular, $g \cdot U$ intersects U only if $g \in \mathrm{N}_{\mathrm{T}}$. Consider the $\mathbb{P}^2 \subset \mathbb{P}(V_4)$ consisting of $\mathfrak{S}_2$-invariant subschemes containing 4 points. Any line in U joins a pair of points in this $\mathbb{P}^2$. Therefore, we have a (birational) $\mathrm{N_T}$-isomorphism of U and $\mathrm{Sym}^2(\mathbb{P}^2)$. The stabilizer of a generic point in X is a central subgroup in $\mathrm{N_T}$ whose action on $\mathbb{P}^2$ is equivalent to a linear action on $\mathbb{C}^2$. (Indeed, $\mathrm{Sym}^2(V_1) = \mathbb{C} \oplus W_2$, where $\mathbb{C}$ is the trivial representation - the invariant symmetric form - and W_2 is a faithful two-dimensional representation of $\mathrm{N_T}/\mathfrak{S}_2$). Thus instead of X with the $\mathrm{N_T}$-action we can consider $\mathbb{C}^2 \times \mathbb{C}^2$ with the $(\mathrm{N_T}/\mathfrak{S}_2) \times \mathfrak{S}_2$-action (where the second $\mathfrak{S}_2$ interchanges the factors). In particular, (by linearity)

$$\mathrm{N_T}\backslash X \sim \mathbb{C}^* \times \mathrm{N_T}\backslash \mathbb{P}^3,$$

and is hence rational.

In the second case, $\mathrm{Gr}(2, V_3)$ has a surjection of degree 2 onto $\mathbb{P}(V_4)$. The connected component of the preimage of the $(\mathrm{PGL}_2, \mathfrak{S}_4)$-slice $\mathbb{P}^1$ in $\mathbb{P}(V_4)$ is a $(\mathrm{PGL}_2, \mathfrak{A}_4)$-slice, isomorphic to $\mathbb{P}^1$. The quotient is rational.

If V is reducible and the PGL_2-action on the Grassmannian has nontrivial stabilizer then $\dim V < 5$. Rationality follows since $\dim \mathrm{Gr}(2, V) \leq 4$ and the generic orbit has dimension at least 2. □

PROPOSITION 7.35. — *Let* G, H *be finite solvable subgroups of* PGL_2. *Then*

$$\mathrm{G}\backslash \mathrm{PGL}_2/\mathrm{H}$$

is rational.

Proof. — The action is birational to the (projective) action of $\mathrm{G} \times \mathrm{H}$ on $\mathbb{P}(\mathrm{M}_2)$, where G acts on the right and H on the left. The groups G, H are either:

- cyclic;
- dihedral or
- $\mathfrak{A}_4$, $\mathfrak{S}_4$.

The case of *primitive* solvable groups is covered by Theorem 7.10, [26]. If V is reducible then there is a nontrivial action of $\mathbb{C}^*$ on $\mathrm{G}\backslash\mathbb{P}(V)/\mathrm{H}$, leading to rationality. This covers the case when either G or H is cyclic.

We claim that if V is irreducible and imprimitive (for the $\mathrm{G}\times\mathrm{H}$-action) then either G or H is dihedral. By definition, $V := \mathrm{M}_2 = \oplus_\alpha V^\alpha$, such that $\tilde{g}V^\alpha = V^{\alpha'}$ for all $\tilde{g} \in \mathrm{G} \times \mathrm{H}$. Moreover, by irreducibility, all V^α must have the same dimension, $= 1$ or 2. Notice that imprimitivity for an action of a group G' implies imprimitivity for the induced action of every subgroup $\mathrm{G}'' \subset \mathrm{G}'$ (with the same decomposition of V). We now claim that the actions of $\mathfrak{A}_4 \times \mathfrak{A}_4$, and consequently of $\mathfrak{A}_4 \times \mathfrak{S}_4$ and $\mathfrak{S}_4 \times \mathfrak{S}_4$ are primitive. Indeed, $\mathfrak{A}_4 \times \mathfrak{A}_4$ contains $\mathfrak{D}_2 \times \mathfrak{D}_2$ as a normal subgroup, for which the imprimitive structure is either a sum of two subspaces of dimension

2 or four subspaces of dimension 1, corresponding to the choice of a subgroup $\mathfrak{S}_2 \subset \mathfrak{D}_2$. The first possible imprimitive structure for $\mathfrak{D}_2 \times \mathfrak{D}_2$ does not extend to one for $\mathfrak{A}_4 \times \mathfrak{A}_4$ (which has no index 2 subgroups). The second structure is also impossible: $\mathfrak{A}_4$ rotates the subgroups $\mathfrak{S}_2 \subset \mathfrak{D}_2$, hence there is no $\mathfrak{A}_4$-invariant imprimitive structures for $\mathfrak{D}_2 \times \mathfrak{D}_2$.

It remains to consider the case when both G and H are dihedral. On V_1 there is a unique imprimitive structure, corresponding to the eigenspaces C_1, C_2 of the elements of G. In particular, there is an imprimitive structure on

$$\mathrm{M}_2 = V_1 \oplus V_1' = (C_1 \oplus C_1') \oplus (C_2 \oplus C_2)'.$$

We claim that (birationally)

$$\begin{array}{c} \mathrm{G}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} \\ \Big\downarrow \\ \mathbb{P}^2 = \mathrm{G}\backslash\mathrm{Sym}^2(\mathbb{P}^1) \end{array}$$

is a conic bundle degenerating precisely over the image of the diagonal and the subvarieties in $\mathbb{P}^2$ with nontrivial stabilizers.

Indeed, since $\mathrm{H} \subset \mathrm{N}_{\mathrm{T}}$ (a $\mathfrak{C}_2$-extension of $\mathbb{C}^*$), (birationally)

$$\begin{array}{c} \mathrm{G}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} \\ \Big\downarrow {\scriptstyle \mathbb{C}^* = \mathrm{N}_{\mathrm{T}}/\mathrm{H}} \\ \mathrm{N}_{\mathrm{T}}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H}. \end{array}$$

The quotient $\mathbb{C}^*\backslash\mathbb{P}(\mathrm{M}_2)$ is (birationally) a fibration over $\mathbb{P}^1 \times \mathbb{P}^1$, with $\mathfrak{S}_2$ acting by permutation, where the coordinate $\mathbb{P}^1$s are the projectivizations of the two-dimensional eigenspaces for the $\mathbb{C}^*$-action on M_2. Thus

$$\begin{array}{c} \mathbb{P}(\mathrm{M}_2)/\mathrm{H} \\ \Big\downarrow \\ \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_2 \end{array}$$

is a conic bundle nondegenerate outside a conic (the image of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$). The G-action commutes with the N_{T}-action and is effective on the base. This proves the claim.

We have $\mathrm{G} \subset \mathrm{N}_{\mathrm{T}}$ and

$$\mathrm{G}\backslash\mathbb{P}^2 \to \mathrm{N}_{\mathrm{T}}\backslash\mathbb{P}^2$$

is a conic bundle. Since the left and right actions of N_{T} commute, $\mathrm{G}\backslash\mathbb{P}^2$ contains an open subvariety $U \times \mathbb{C}^*$ where the restriction of the conic bundle is nondegenerate. Here $\mathbb{C}^* = \mathrm{G}\backslash\mathrm{N}_{\mathrm{T}}$ and U is a subset of $\mathbb{P}^1 = \mathrm{N}_{\mathrm{T}}\backslash\mathbb{P}^2$. Therefore the conic bundle has at most 2 singular fibers on any completion of the fiber $\mathbb{C}^* \subset U \times \mathbb{C}^*$. Rationality follows.

We can now describe some open subvariety in the quotient $\mathrm{G}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H}$ explicitly. Consider the action of $\mathbb{C}^* \subset \mathrm{N}_{\mathrm{T}}$ on both sides $\mathbb{C}^*\backslash\mathbb{P}(\mathrm{M}_2)/\mathbb{C}^*$. With respect to this action $\mathbb{P}(\mathrm{M}_2)$ is birationally equivalent to a trivial $\mathbb{C}^* \times \mathbb{C}^*$-fibration over $\mathbb{P}^1$. Now we add the action of $\mathfrak{S}_2$ on both sides. The product $\mathfrak{S}_2 \times \mathfrak{S}_2$ acts on the base $\mathbb{P}^1$. The group $\mathfrak{S}_4$ contains a normal subgroup $\mathfrak{D}_2 \subset \mathrm{N}_{\mathrm{T}}$ and the action of each $\mathfrak{S}_2 \subset \mathfrak{D}_2$ inverts the respective $\mathbb{C}^*$ action. Thus (birationally)

$$\begin{array}{c} \mathrm{N}_{\mathrm{T}}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{NT} \\ \Big\downarrow{\scriptstyle \mathrm{N}_{\mathrm{T}}\times\mathrm{N}_{\mathrm{T}}} \\ \mathbb{P}^1 - 3\,pts, \end{array}$$

where the deleted points are the ramification points of the map $\mathbb{P}^1 \to \mathbb{P}^1/\mathfrak{D}_2$. In particular, there is an open U such that

$$\begin{array}{c} \mathrm{G}\backslash\mathbb{P}(\mathrm{M}_2)/\mathrm{H} \\ \Big\downarrow{\scriptstyle \mathbb{C}^*} \\ U \\ \Big\downarrow{\scriptstyle \mathbb{C}^*} \\ \mathbb{P}^1 - 3\,pts. \end{array}$$

By Lemma 7.3, the conic bundles are trivial.

Finally, the conic bundles on $\mathbb{P}^2/\mathfrak{S}_4$ and $\mathbb{P}^2/\mathfrak{A}_4$ have sections. Indeed, both $\mathfrak{A}_4$ and $\mathfrak{S}_4$ contain dihedral subgroups of index 3 ($\mathfrak{D}_2$, resp. $\mathfrak{D}_4$). The image of the section in the conic bundle over $\mathfrak{D}_2\backslash\mathbb{P}^2$ (resp. $\mathfrak{D}_4\backslash\mathbb{P}^2$), has odd degree in the conic bundles over $\mathfrak{A}_4\backslash\mathbb{P}^2$ and $\mathfrak{S}_4\backslash\mathbb{P}^2$, respectively. We apply Lemma 7.4. □

PROPOSITION 7.36. — *Let V be an irreducible GL_2-representation and $\mathrm{H} \subset \mathrm{SL}_2$ a finite group, not equal to $\tilde{\mathfrak{A}}_5$. Then*

$$\mathrm{GL}_2\backslash(V \oplus V)/\mathrm{H}$$

is rational.

Proof. — First of all, $V_1 \oplus V_1/\mathrm{H}$ is rational. Next, by Lemma 7.22,

$$V \oplus V \sim_{\mathrm{GL}_2 \times \mathrm{GL}_2} \begin{array}{c} \mathcal{V} \\ \Big\downarrow {\scriptstyle \mathrm{M}_2 = V_1 \oplus V_1} \\ \mathrm{Gr}(2, V). \end{array}$$

First we assume that V has odd weight. The Grassmannian $\mathrm{Gr}(2, V)$ carries the action of PGL_2. If we restrict the bundle $\mathcal{V}$ to a generic PGL_2-orbit O in $\mathrm{Gr}(2, V)$ then the corresponding module $H^0(O, \mathcal{V}_O)$ contains V_1 as a submodule. By Lemma 7.18, this gives an equivariant map

$$\mathcal{V} \to V_1 \oplus V_1$$

with a 1-transitive action of GL_2 on the target. Thus

$$\mathrm{GL}_2\backslash\mathcal{V}/\mathrm{H} \sim \mathrm{H}\backslash \mathrm{Gr}(2, V) \tag{7.3}$$

(with the *same* subgroup $\mathrm{H} \subset \mathrm{GL}_2$ appearing on the left). Indeed, $\mathrm{GL}_2 \subset (V_1 \oplus V_1) = \mathrm{M}_2$ and multiplication by H on the right gives an orbit $x \cdot \mathrm{H}$. This orbit is a $(\mathrm{GL}_2 \times \mathrm{H}, \mathrm{H}^x \times \mathrm{H})$-slice (with $\mathrm{H}^x = x\mathrm{H}x^{-1}$) and it is stabilized exactly by $\mathrm{H}^x \times \mathrm{H}$, acting doubly transitively on the set $\mathrm{H}^x \cdot x$. It follows that every point $x' \in x \cdot \mathrm{H}$ is a $(\mathrm{H}^x \times \mathrm{H}, \mathrm{H}^x)$-slice of the orbit $x \cdot \mathrm{H}$. The quotient $\mathrm{H}\backslash \mathrm{Gr}(2, V)$ is rational by 7.31.

Assume that V has even weight. If the PGL_2-action is af then

$$\mathrm{GL}_2\backslash\mathcal{V}/\mathrm{H} \sim (\mathrm{PGL}_2\backslash \mathrm{Gr}(2, V)) \times (\mathbb{C}^*\backslash(V_1 \oplus V_1)/\mathrm{H}).$$

If it is not af, then, by Lemma 7.32, $V = V_4$ or V_2.

For $V = V_4$ we have the $(\mathrm{PGL}_2, \mathrm{N}_\mathrm{T})$-slice $X = \mathrm{Sym}^2(\mathbb{P}^2)$ with the N_T-action which we can replace by $\mathbb{C}^2 \times \mathbb{C}^2$ with a $(\mathrm{N}_\mathrm{T}/\mathfrak{C}_2) \times \mathfrak{C}_2$-linear action. In particular, we identify the quotient with a quotient of $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus V_1 \oplus V_1$ by a linear action of $\mathrm{N}_{\tilde{\mathrm{T}}} \times \mathfrak{S}_2 \times \mathrm{H}$ (where $\mathrm{N}_{\tilde{\mathrm{T}}} \subset \mathrm{GL}_2$). The action of $\mathrm{N}_{\tilde{\mathrm{T}}} \times \mathrm{H}$ on V_1 is transitive with stabilizer $\mathfrak{C}_2 \times \mathrm{H}$. Hence it is equivalent to the action of $\mathfrak{D}_2 \times \mathrm{H}$ on $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus V_1$, which is a $\mathbb{C}^2$-vector bundle (permutation of the anti-invariant part of $\mathfrak{S}_2$-action) over $\mathbb{C}^2 \times V_1$, with $\mathfrak{D}_2 \times \mathrm{H}$ action. The latter quotient is rational. For $V = V_2$ the action is transitive on $\mathrm{Gr}(2, V) = \mathbb{P}^2$ and the quotient has dimension 2 - rationality follows. $\square$

We will also need a more general result for $\mathrm{H} = \mathfrak{S}_2$.

Proposition 7.37. — *Let*

$$X \xrightarrow{L} Y = \prod_{j \in J} \mathbb{P}(V_{d_j})$$

be a GL_2*-homogeneous line bundle. If at least one* $d_j \neq 2$ *then* $\mathrm{GL}_2 \backslash X \times X / \mathfrak{S}_2$ *is rational.*

Proof. — **Case 1.** $|J| = 1$. If $d = d_1$ is even or if d is odd and the line bundle has odd degree on $\mathbb{P}(V_d)$ then

$$X \times X \sim_{\mathrm{GL}_2 \times \mathfrak{S}_2} V_d \oplus V_d$$

and we apply Proposition 7.36. If the line bundle has even degree then it is trivial and GL_2 acts as $\mathrm{PGL}_2 \times \mathbb{C}^*$. If the PGL_2-action on $\mathbb{P}(V_d)$ is af we have

$$\mathbb{P}(V_d) \sim_{\mathrm{PGL}_2} S \times \mathrm{PGL}_2,$$

for a rational slice S (with trivial PGL_2-action). We have a $\mathrm{PGL}_2 \times \mathbb{C}^* \times \mathfrak{S}_2$-action on

$$\mathbb{C} \times \mathrm{PGL}_2 \times S \times \mathbb{C} \times \mathrm{PGL}_2 \times S.$$

The quotient variety is a vector bundle over $\mathrm{PGL}_2 \backslash \mathrm{PGL}_2 \times \mathrm{PGL}_2 / \mathfrak{S}_2$ (rational by Lemma 7.12). The claim follows. If the PGL_2-action is not af, then $V = V_3$ or V_1. For V_1 the quotient is rational by dimensional reasons. For V_3 we have a projection

$$\begin{array}{c} \mathbb{C} \times \mathbb{P}^3 \times \mathbb{C} \times \mathbb{P}^3 \\ \Big\downarrow {\scriptstyle \mathbb{P}^1 \times \mathbb{P}^1} \\ \mathrm{Gr}(2, V_3) \end{array}$$

commuting with both actions. Recall that $\mathrm{Gr}(2, V_3)$ has $\mathbb{P}^1$ as a $(\mathrm{PGL}_2, \mathfrak{A}_4)$-slice, with $\mathfrak{A}_4$ effectively acting as a cyclic group $\mathfrak{C}_3 = \mathfrak{A}_4 / \mathfrak{D}_2$ on $\mathbb{P}^1$ (the group $\mathfrak{D}_2$ acts trivially on the $(\mathrm{PGL}_2, \mathfrak{A}_4)$-slice $\mathbb{P}^1 \subset \mathbb{P}^4$ and similarly for $\mathrm{Gr}(2, V_3)$, see Lemma 7.32). Thus the quotient is the same as for the bundle

$$\begin{array}{c} \\ \Big\downarrow {\scriptstyle \mathbb{P}^1 \times \mathbb{C} \times \mathbb{P}^1 \times \mathbb{C}} \\ \mathbb{P}^1 \end{array}$$

under the action of $\mathfrak{A}_4 \times \mathfrak{S}_2$. In particular, it is a vector bundle over a $\mathbb{P}^2 = \mathfrak{D}_2\backslash\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_2$-fibration over $\mathbb{P}^1 = \mathbb{P}^1/\mathfrak{C}_3$, hence is rational.

Case 2. $|J| \geq 2$. If at least one d_j is odd and > 1 or if all $d_j = 1$ and $|J| > 2$, then there is a slice S and the PGL_2-action is af. We can write Y as (the total space of the) line bundle:

$$\begin{array}{c} X \\ \Big\downarrow{\scriptstyle L} \\ S \times \mathrm{PGL}_2 \end{array}$$

and, using Lemma 7.21, reduce to either a vector bundle over

$$\mathrm{PGL}_2\backslash\mathrm{PGL}_2 \times \mathrm{PGL}_2/\mathfrak{S}_2,$$

when L is trivial on PGL_2, or to

$$\mathrm{GL}_2\backslash\mathrm{GL}_2 \times \mathrm{GL}_2/\mathfrak{S}_2$$

otherwise. In both cases the base is rational by Lemma 7.12.

If $d_j = 1$ for every $j \in J$ and $|J| = 2$ then the there is a map

$$(\mathbb{P}^1)^4 \to \mathbb{P}(V_4) = \mathrm{Sym}^4(\mathbb{P}^1) = \mathbb{P}^4$$

(of degree 24, mapping 4 points to a form of degree 4). The preimage in $(\mathbb{P}^1)^4$ of the $(\mathrm{PGL}_2, \mathfrak{S}_4)$-slice $\mathbb{P}^1_s = \mathbb{P}^1$ of $\mathbb{P}^4$, will be a set of six lines $\mathbb{P}^1_{g,h}$, labeled by a pair of generators $g, h \in \mathfrak{D}_2$ (which act trivially on $\mathbb{P}^1_s \subset \mathbb{P}^4$). More precisely, the line $\mathbb{P}^1_{g,h}$ is the set given by $(x : gx : hx : ghx) \in (\mathbb{P}^1)^4$, for $x \in \mathbb{P}^1$. The map $\mathbb{P}^1_{g,h} \to \mathbb{P}^1_s = \mathbb{P}^1_{t,s}/\mathfrak{D}_2$ has degree 4. Thus $\mathbb{P}^1_{g,h}$ is a $(\mathrm{PGL}_2, \mathfrak{D}_2)$-slice of $(\mathbb{P}^1)^4$ and the quotient of a vector bundle $\stackrel{L\oplus L}{\longrightarrow} \mathbb{P}^1$ by a linear action of $\mathfrak{D}_2$ is rational.

Assume that all d_i are even. Then L is (birationally) trivial. Unless $|J| = 2$ and $d_1 = d_2 = 2$, there is a decomposition of

$$Y \times Y = \mathbb{P}(V_d) \times Y' \times \mathbb{P}(V_d) \times Y'$$

such that the PGL_2-action is af and

$$\mathbb{P}(V_d) \times Y' \times \mathbb{P}(V_d) \times Y' \sim_{\mathrm{PGL}_2 \times \mathfrak{S}_2} (Y' \times Y') \times (\mathbb{P}(V_d) \times \mathbb{P}(V_d))$$

(with trivial PGL_2-action on $\mathbb{P}(V_d)$), by Lemma 7.19. The quotient is birational to a vector bundle over $\mathrm{PGL}_2 \times \mathbb{C}^*\backslash X' \times X'/\mathfrak{S}_2$, where X' is the trivial line bundle over Y'.

We have reduced to $|J| = 1$ treated in Case 1 or to $|J| = 2$ and $d_1 = d_2 = 2$, treated in Lemma 7.38. □

LEMMA 7.38. — *The quotient*

$$X := \mathrm{PGL}_2\backslash(\mathbb{P}_1(V_2)\times\mathbb{P}_2(V_2)\times\mathbb{P}_1(V_2)\times\mathbb{P}_2(V_2))/\mathfrak{S}_2$$

is rational, where $\mathbb{P}_1(V_2)$ *and* $\mathbb{P}_2(V_2)$ *are different copies of* $\mathbb{P}^2=\mathbb{P}(V_2)$ *and* $\mathfrak{S}_2$ *acts by permutation.*

Proof. — Consider the projection

$$X\to \mathrm{PGL}_2\backslash\mathbb{P}_1(V_2)\times\mathbb{P}_1(V_2)/\mathfrak{S}_2$$

and the $\mathrm{PGL}_2\times\mathfrak{S}_2$-equivariant map of degree 6

$$\begin{array}{rccc} pr: & \mathbb{P}(V_2)\times\mathbb{P}(V_2) & \to & \mathbb{P}(V_4)\\ & (Q_1,Q_2) & \mapsto & Q_1\cdot Q_2. \end{array}$$

The space $\mathbb{P}(V_4)$ has a $(\mathrm{PGL}_2,\mathfrak{S}_4)$-slice $\mathbb{P}^1_s$ (the $\mathfrak{D}_2$-invariant polynomials). The zeroes of a (polynomial) $p\in\mathbb{P}^1_s$ form an orbit under $\mathfrak{D}_2$. The preimage $pr^{-1}(\mathbb{P}^1_s)\subset\mathbb{P}^2\times\mathbb{P}^2$ consists of 3 lines, each invariant under $\mathfrak{D}_2$. Indeed, the ordered pair (Q_1,Q_2) corresponds to a choice of a generator $g\in\mathfrak{D}_2$ such that $x,g(x)$ are zeroes of Q_1 and $h(x),hg(x)$ are zeroes of Q_2. Thus the line $\mathbb{P}^1_g\subset\mathbb{P}^2\times\mathbb{P}^2$ consists of tupels $\{(x,gx),(hx,ghx)\}$, where x is an arbitrary point in $\mathbb{P}^1$ and $(x,gx)=Q_1,(hx,ghx)=Q_2$. The map $\mathbb{P}^1_g\to\mathbb{P}^1_s$ has degree two and its fibers coincide with orbits of h (since g acts trivially on $\mathbb{P}^1_g$). The action of h is given by

$$h\ :\ \{(x,gx),(hx,ghx)\}\mapsto\{(hx,ghx),(x,gx)\}.$$

Thus $h(Q_1,Q_2)=(Q_2,Q_1)$ and the action of h coincides with the restriction of the permutation action on $\mathbb{P}^2\times\mathbb{P}^2$ to $\mathbb{P}^1_g$. The line $\mathbb{P}^1_g$ is invariant under $\mathfrak{D}_4\times\mathfrak{S}_2$ (considered as a subgroup of $(\mathrm{PGL}_2\times\mathfrak{S}_2)$). The group $\mathfrak{S}_4$ permutes the lines in $pr^{-1}(\mathbb{P}^1_s)$. Each $\mathbb{P}^1_g$ is a $(\mathrm{PGL}_2\times\mathfrak{S}_2,\mathfrak{D}_4\times\mathfrak{S}_2)$-slice of $\mathbb{P}^2\times\mathbb{P}^2$. Therefore,

$$X\sim\mathfrak{D}_4\backslash\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2/\mathfrak{S}_2.$$

The space $\mathbb{P}^2\times\mathbb{P}^2$ contains a subspace $\mathbb{C}^2\times\mathbb{C}^2$ with a *linear* action of $\mathfrak{D}_4\times\mathfrak{S}_2$. Indeed, the action of $\mathfrak{D}_4$ on $\mathbb{P}^1$ corresponds to the irreducible representation of $\tilde{\mathfrak{D}}_4$ on $\mathbb{C}^2=V$. Under the $\mathfrak{D}_4$-action, one has a decomposition $\mathrm{Sym}^2(V)=V'\oplus V''$, where $\dim V'=2,\dim V''=1$ and the action of $\mathfrak{D}_4$ on $\mathbb{P}^2$ is equivalent to the linear action on V'. The additional $\mathfrak{S}_2$ permutes the $\mathbb{P}^2$ and hence acts by permutation on $V'\oplus V'$. Thus

$$\begin{array}{c} \mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2\sim_{\mathfrak{D}_4\times\mathfrak{S}_2} \\ \qquad\qquad\qquad\qquad\Big\downarrow{\scriptstyle V'\oplus V'} \\ \qquad\qquad\qquad\qquad\mathbb{P}^1 \end{array}$$

(a vector bundle).

Consider the effective action of (the nonabelian group) $\mathfrak{D}_4 \times \mathfrak{S}_2$ on $\mathbb{P}^1$. It has a normal subgroup $\mathfrak{D}_2 \times \mathfrak{S}_2$ with generators g, h, k and an element $i, i^2 = 1$ which commutes with g, k and acts on h as $ihi = gh$. The stabilizer of a generic point on $\mathbb{P}^1_g$ is a normal abelian subgroup generated by g, hk. Thus $\mathfrak{D}_4 \times \mathfrak{S}_2$ acts on $\mathbb{P}^1$ effectively through the quotient $\mathfrak{D}_4/\langle g, hk\rangle = \mathfrak{D}_2$. The action of this $\mathfrak{D}_2$ on $\mathbb{P}^1$ is almost free. Indeed, the action of k coincides with the action of h and permutes Q_1, Q_2. Thus the orbits of h and k on $\mathbb{P}^1_g$ coincide with fibers of the map $\mathbb{P}^1_g \to \mathbb{P}^1_s$. On the other hand, i acts nontrivially on $\mathbb{P}^1_s$. We claim that

$$\begin{array}{c} \mathfrak{D}_4\backslash(V' \oplus V') \times \mathbb{P}^1/\mathfrak{S}_2 \\ \downarrow \\ \mathfrak{D}_4\backslash(V' \times \mathbb{P}^1) \end{array}$$

is a vector bundle. Indeed, consider the subspace $V'_{inv} \subset V' \oplus V'$ of invariant vectors (under the permutation). The action of $\mathfrak{D}_4 \times \mathfrak{S}_2$ on $((V' \oplus V')/V'_{inv}) \times \mathbb{P}^1$ is almost free. Hence

$$\begin{array}{c} \mathfrak{D}_4\backslash(V' \oplus V') \times \mathbb{P}^1/\mathfrak{S}_2 \\ \downarrow \\ \mathfrak{D}_4\backslash((V' \oplus V')/V'_{inv}) \times \mathbb{P}^1/\mathfrak{S}_2 \end{array}$$

is a vector bundle with base a quotient of the vector bundle $(V' \oplus V'/V'_{inv}) \to \mathbb{P}^1_g$ by $\mathfrak{D}_4 \times \mathfrak{S}_2$. The variety $(V' \oplus V'/V'_{inv}) \times \mathbb{P}^1$ has a fiberwise (scalar) $\mathbb{C}^*$-action commuting with the $\mathfrak{D}_4 \times \mathfrak{S}_2$-action. Since every $\mathbb{C}^*$-action has a slice,

$$X' := \mathfrak{D}_4\backslash((V' \oplus V')/V'_{inv}) \times \mathbb{P}^1/\mathfrak{S}_2,$$

is rational by dimensional reasons: $X'/\mathbb{C}^*$ is a unirational, therefore, rational surface and

$$X' \sim (X'/\mathbb{C}^*) \times \mathbb{C}^*.$$

□

PROPOSITION 7.39. — *Let X be $V \oplus V$, where $V = V_d$ is an irreducible* GL_2-*representation, $\ell > 0$ and $\mathrm{H} \subset \mathrm{SL}_2$ with $\mathrm{H} \neq \tilde{\mathfrak{A}}_5$. Then*

$$\mathrm{GL}_2\backslash X \times \mathbb{P}(V_\ell)/\mathrm{H}$$

is rational (where H acts trivially on $\mathbb{P}(V_\ell)$).

Proof. — If ℓ is even and the action of GL_2 or a quotient of GL_2 by a central subgroup is af then we apply Lemma 7.19 combined with Proposition 7.36, resp. 7.37.

If ℓ is odd and the action is af then there exists a slice, which is a rational variety, by Lemma 7.31 resp. 7.30. Rationality follows.

Now we assume that the action is not af. This means that $d \leq 4$. The subcases with $d \leq 2$ are trivial since the action on the corresponding Grassmannian is transitive. If ℓ is odd, then the PGL_2-action on $\mathrm{Gr}(2, V) \times \mathbb{P}(V_\ell)$ has a rational slice and our claim follows.

If $d = 3$, the action of PGL_2 on $\mathrm{Gr}(2, V_3)$ has a $(\mathrm{PGL}_2, \mathfrak{A}_4)$-slice $\mathbb{P}^1$. For even $\ell > 0$ the action of $\mathfrak{A}_4$ on $\mathbb{P}^\ell$ is faithful and it lifts to a linear representation of of $\mathfrak{A}_4$. Further, $\mathfrak{A}_4$-acts on $\mathbb{P}^1$ is through a cyclic quotient. Thus

$$(\mathbb{P}^1 \times \mathbb{P}(V_\ell) \sim_{\mathfrak{A}_4} \mathbb{P}^1 \times \mathbb{P}(V_\ell)$$

with trivial $\mathfrak{A}_4$-action on the $\mathbb{P}^1$ on the right. This implies that the quotient is equivalent to

$$\mathbb{P}^1 \times (\mathbb{P}^\ell/\mathfrak{A}_4) \times (V_1 \oplus V_1)/\mathbb{C}^* \times \mathrm{H},$$

a product of rational varieties.

If $d = 4$, the action of PGL_2 on $\mathrm{Gr}(2, V_4)$ has a $(\mathrm{PGL}_2, \mathrm{N}_\mathrm{T})$-slice X'. The action of N_T on $\mathbb{P}(V_\ell)$ is linear and the quotient of $X \times \mathbb{P}^\ell$ is a vector bundle over the quotient of X, which is rational. □

PROPOSITION 7.40. — *Let* $X = (\xrightarrow{L} Y)^2$, *where* $Y = \prod_{j \in J} \mathbb{P}(V_{d_j})$ *and* $\ell > 0$. *Then*

$$\mathrm{GL}_2 \backslash X \times \mathbb{P}(V_\ell)/\mathfrak{S}_2$$

is rational (where $\mathfrak{S}_2$ *acts trivially on* $\mathbb{P}(V_\ell)$ *and by permutation on* X*).*

Proof. — The same argument as in the proof of Proposition 7.39 shows that it suffices to assume that the action on X is not af. This happens only if $Y = \mathbb{P}^2$ or $\mathbb{P}^1$. The case $Y = \mathbb{P}^2$ reduces to Proposition 7.39 (Grassmannian). If $Y = \mathbb{P}^1$ then the action of PGL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ is transitive and

$$\mathrm{GL}_2 \backslash X \times \mathbb{P}(V_\ell)/\mathfrak{S}_2 \sim (\mathbb{C}^* \backslash \mathbb{P}(V_\ell)) \times (\mathbb{C}^2/\mathbb{C}^* \times \mathfrak{S}_2),$$

a rational variety. □

8. Special rationality results

In this section we collect rationality results for spaces of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ with prescribed (special) ramification over exactly three distinguished points $(0, 1, \infty)$ and unspecified ramifications over other points.

Let $\mathcal{R}(r_0, r_1, r_\infty)$ be the space of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ with local ramification data (vectors) r_0, r_1, r_∞ over the points $0, 1, \infty$.

PROPOSITION 8.1. — *Assume that* (r_0, r_1, r_∞) *satisfies one of the following:*

- *all entries of the vectors* r_0, r_∞ *are even and some fixed number of entries of* r_1 *is even;*
- *all entries of the vectors* r_0, r_∞ *are even and a fixed number of entries of* r_1 *is divisible by* 3;
- *all entries of the vectors* r_0, r_∞ *are divisible by* 3 *and all entries of* r_1 *are even.*

Then $\mathcal{R}(r_0, r_1, r_\infty)$ *is a finite union of irreducible rational varieties.*

Proof. — In these cases the map $f = f_0/f_\infty$ is given by coprime polynomials satisfying the equations:

- $f_0^2 - f_\infty^2 = g_1^2 g_1'$;
- $f_0^2 - f_\infty^2 = g_1^3 g_1'$;
- $f_0^3 - f_\infty^3 = g_1^2 g_1'$,

where g_1' is an arbitrary polynomial. The first equation leads to

$$(f_0 - f_\infty)(f_0 + f_\infty) = g_1^2 g_1'$$

and, by coprimality, to

$$\begin{array}{rcl} f_0 - f_\infty & = & g_{11}^2 g_{11}', \\ f_0 + f_\infty & = & g_{12}^2 g_{12}', \end{array}$$

with arbitrary $g_{11}, g_{11}', g_{12}, g_{12}'$ (satisfying the obvious degree conditions) — a union of rational varieties.

The second case is analogous. Consider the third case: since $f_0^3 - f_\infty^3$ is a square we obtain

$$\begin{array}{rcl} f_0 - f_\infty & = & g_1^2 \\ f_0 - \zeta f_\infty & = & g_2^2 \\ f_0 - \zeta^2 f_\infty & = & g_3^2 \end{array}$$

(where $\zeta^3 = 1$) and we need to solve

$$\frac{2\zeta}{1+\zeta} g_1^2 + \frac{1-\zeta}{1+\zeta} g_2^2 = g_3^2.$$

Now we apply the parametrization as above. □

COROLLARY 8.2. — *Let* $\mathcal{R}(r_0, r_1, r_\infty)$ *be as in* 8.1. *Then*

$$\mathrm{PGL}_2\backslash\mathcal{R}(r_0, r_1, r_\infty)$$

is rational.

Proof. — We have established an explicit parametrization of $\mathcal{R}(r_0, r_1, r_\infty)$ as a direct sum of spaces of polynomials (with different weights as irreducible GL_2-representations). By the theorem of Katsylo 7.14, the corresponding quotients are rational. □

REMARK 8.3. — Only the first case with $g_1' = 1$ can admit a nontrivial action of H_Γ (which necessarily is $\mathbb{Z}/3$). But even in this case the action of $\mathbb{Z}/3$ is linear and it commutes with the action of GL_2 on pairs of polynomials. Lemma 7.31 implies rationality.

LEMMA 8.4. — *Every irreducible component of the variety* $\mathcal{R}$ *of rational maps* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* 5 *and prescribed global ramification datum*

$$\mathrm{RD}(f) = [(2,2,1)_0, (2,2,1)_1, (2,2,1)_\infty, (2), (2)]$$

is rational.

Proof. — Changing the variables (fixing two ramification points over $1 \in \mathbb{P}^1$ as $0, \infty$), we can write $f = F_1/F_2$ where

$$\begin{array}{rcl} F_1(x) & = & \hat{f}_1(x)^2\hat{a}_1(x)^2\hat{b}_1(x) \\ F_2(x) & = & \hat{f}_2(x)^2\hat{a}_2(x)^2\hat{b}_2(x) \end{array}$$

where $\hat{f}_1, \hat{f}_2, \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ are linear forms in x. Since the leading coefficients of F_1 and F_2 are equal we can assume that they are both equal to 1 and write $\hat{f}_1(x) = x + f_1, \dots, \hat{b}_2(x) = x + b_2$, with some nonzero constants $f_1, \dots, b_2$. Since we have one free parameter (under the action of PGL_2) we can assume that $b_1 = 1$. Thus

$$\hat{f}_1(x)^2\hat{a}_1(x)^2\hat{a}_2(x) - \hat{f}_2(x)^2\hat{b}_1(x)^2\hat{b}_2(x) = \sum_i g_i x^i = c_1x^2(x + c_2)$$

with arbitrary constants c_1, c_2. We get a system of equations on the coefficients g_j:

$$g_4 = 0, g_1 = 0, g_0 = 0.$$

Remark that the coefficients of g are symmetric functions on pairs (f_1, a_1) and (f_2, a_2). To parametrize $\mathcal{R}$ we introduce the following variables:

$$X_1 = a_1 + f_1,\ Y_1 = a_1f_1,\ X_2 = f_2 + a_2,\ Y_2 = f_2a_2,\ b_1,\ b_2.$$

Write the equations on the coefficients g_j as

$$\begin{aligned} 2X_1 + b_1 &= 2X_2 + b_2 \\ Y_1^2 b_1 &= Y_2^2 b_2 \\ Y_1^2 + 2X_1Y_1b_1 &= Y_2^2 + 2X_2Y_2b_2. \end{aligned}$$

Since $b_1 = 1$, for a fixed b_2 we get

$$\begin{aligned} 2X_1 + 1 &= 2X_2 + b_2 \\ Y_1 &= \pm\sqrt{b_2}Y_2 \\ b_2Y_2 + 2\sqrt{b_2}X_1 &= Y_2 + 2X_2b_2. \end{aligned}$$

This is a union of two (affine) lines. After a rational covering $(\sqrt{b_2})$ our surface is (rationally) a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, a rational surface. □

LEMMA 8.5. — *Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 and ramification datum*

$$\mathrm{RD}(f) = [(2,2)_0, (2,1,1)_1, (2,1,1)_\infty]$$

is a rational surface.

Proof. — Using the PGL_2-action on the preimage $\mathbb{P}^1$ we can assume that the points $(2,2)$ are $+1, -1$, respectively, and that the point of degree 2 (in the local ramification datum $(2,1,1)$) over 0 is ∞. Thus we can write

$$(x^2-1)^2 - c(x+c_1)(x+c_2)(x+c_3)^2 = g_2(x),$$

where g_2 is an arbitrary polynomial of degree 2 and c is some constant. We get two equations

$$c = 1,$$

$$c_1 + c_2 + 2c_3 = 0.$$

Thus we have a (rational) surjection of $\mathbb{P}^2$ onto $\mathcal{R}$. □

LEMMA 8.6. — *Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 with ramification datum*

$$\mathrm{RD}(f) = [(2,2)_0, (3,1)_1, (2,1,1)_\infty, (2), (2)]$$

is a rational curve.

Proof. — A generic map with this ramification datum is given by the equation $f = f_1/f_2$, where

$$f_1 = (x^2-1)^2, \;\; f_2 = (x+c_1)(x+c_2)(x+c_3)^2$$

and

$$f_1 - f_2 = (x^2-1)^2 - c(x+c_1)(x+c_2)(x+c_3)^2 = g_1(x),$$

where $g_1(x)$ is linear. Thus $c=1$ and

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1c_2 + 2c_1c_3 + 2c_2c_3 + c_3^2 = 0,$$

clearly rational. □

LEMMA 8.7. — *The irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 3 with ramification datum*

$$\mathrm{RD}(f) = [(2,1)_0, (2,1)_1, (2,1)_\infty, (2)]$$

is a rational curve.

Proof. — Reduces easily to the rationality of a cuspidal cubic curve. □

9. Rationality of moduli

THEOREM 9.1. — *Any connected component of a moduli space of rational or K3 elliptic surfaces with fixed monodromy group is rational.*

Proof. — In Proposition 3.11 we have identified (Zariski open subsets of) the corresponding moduli spaces $\mathcal{F}_{r,\tilde{\Gamma}}$ as quotients (by the left PGL_2 and right H_Γ-action)

$$\mathrm{PGL}_2 \backslash \mathcal{U}'_{r,\tilde{\Gamma},\ell} / H_\Gamma.$$

Here

$$\mathcal{U}'_{r,\tilde{\Gamma},\ell} \sim_{\mathrm{PGL}_2 \times H_\Gamma} \mathrm{Sym}^\ell(\mathbb{P}^1) \times \mathcal{R}_\Gamma$$

and

$$\mathcal{R}_\Gamma = \{f : \mathbb{P}^1 \to \mathbb{P}^1\}$$

is the space of rational maps (with prescribed ramification). For elliptic rational or K3 surfaces $\ell \le 3$ and H_Γ is either trivial, cyclic, dihedral or a subgroup of $\mathfrak{S}_4$ (see Corollary 3.14). The actions if PGL_2 and H_Γ commute and H_Γ acts only on $\mathcal{R}_\Gamma$.

First we consider *general families*:

$$\mathrm{ET}(\mathcal{E}) - 12\ell = \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma).$$

For $\mathbf{d} = (d_1, ..., d_k) \in \mathbb{N}^k$ we put

$$\mathbb{P}^{\mathbf{d}} := \prod_{j=1}^{k} \mathbb{P}(V_{d_j}).$$

Recall that $\mathcal{R}_\Gamma$ is (birationally) the total space of a line bundle over the space

$$\mathbb{P}^{\mathbf{d}} \times \mathbb{P}^{\mathbf{d}'},$$

where $\sum_{j=1}^{k} d_j = \sum_{j=1}^{k'} d'_j$.

Case 1. $\mathbf{d} \neq \mathbf{d}'$. Then, by 3.14, $H_\Gamma = 1$ and rationality of $\mathrm{PGL}_2 \backslash \mathcal{R}_\Gamma$ (in all cases) follows from the rationality of

$$\mathrm{PGL}_2 \backslash \mathbb{P}^{\mathbf{d}} \times \mathbb{P}^{\mathbf{d}'},$$

which is the theorem of Katsylo 7.14.

Case 2. $\mathbf{d} = \mathbf{d}'$ and $k \geq 2$. By Corollary 3.14, $H_\Gamma = \mathfrak{S}_2$ (permutation of the factors). This case is covered by Proposition 7.37.

Case 3. $\mathbf{d} = \mathbf{d}' = (d)$. This case is covered by Proposition 7.36.

Now we discuss the *special families*:

$$\mathrm{ET}(\mathcal{E}) - 12\ell < \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma).$$

We use the classification of these families established in Section 5. All families listed in Lemma 5.2 are covered by Propositions 7.37 and the Theorem 7.14. Consider the families listed in Lemma 5.3: Lemma 7.30 covers the cases $j_1, j_4, j_5, j_6, j_{13}$. The case j_2, j_8 and j_{12} are covered by Proposition 8.1, j_3 by Lemma 8.6, j_7, j_9, j_{10} by 8.1 and 8.3, j_{11} by Lemma 8.5. The case j_{14} is covered by Lemma 8.7. Finally, the families j_{15} and j_{16} (listed in Lemma 6.2) are covered by Proposition 7.37 and the remaining families $j_{17} - j_{20}$ by Theorem 7.14. □

REMARK 9.2. — Our methods extend to some moduli spaces of elliptic surfaces with higher Euler characteristic. In particular, the results of Section 8 imply that any moduli space of Jacobian elliptic surfaces over $\mathbb{P}^1$ such that a generic surface in this space has only singular fibers of multiplicative type is rational. However, we expect that there are nonrational moduli spaces already for Euler characteristic 36.

10. Pictures

In this section we give a combinatorial description of monodromy groups of elliptic K3 surfaces. More precisely, we describe a simple procedure which allows to enumerate all the possible graphs Γ with given $\mathrm{ET}(\Gamma)$. Let $\mathcal{E} \to \mathbb{P}^1$ be an elliptic K3 surface. We have shown in Section 4 that

$$48 = \mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma)$$

and that $\mathrm{ET}(\Gamma)$ is divisible by 12. Thus $\mathrm{ET}(\Gamma)$ equals $12, 24, 36$ or 48 and all possible $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ are described by connected trivalent graphs T_Γ with ≤ 8 edges embedded into $\mathbb{S}^2$, with an arbitrary bicoloring of the ends.

Case $\mathrm{ET}(\Gamma) = 12$: There is only one tree T_{12} with $\mathrm{ET}(T_{12}) = 12$

FIGURE 1. The tree T_{12}.

The ends of T_{12} can be either A or B-vertices. To obtain all possible graphs T_Γ with $\mathrm{ET}(\Gamma) = 12$ we just need to attach to T_{12} a single loop L.

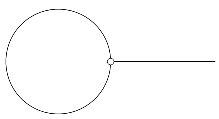


FIGURE 2. The loop L.

This gives the following list of graphs:

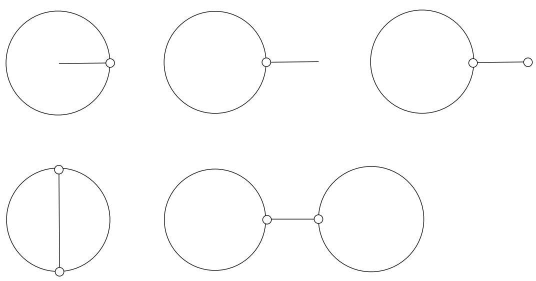


FIGURE 3. The case $\mathrm{ET}(\Gamma) = 12$.

There is only one saturated graph from the list above which has no outer loops (Figure 4).

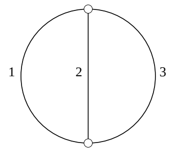


FIGURE 4.

This graph will be a basic building block in the construction of graphs with $\mathrm{ET}(\Gamma) > 12$ - we will attach trees and loops to its edges. The edges are numbered to simplify the count of all possible outcomes.

Case $\mathrm{ET}(\Gamma) = 24$: Again, we have only one topological tree T_{24} with $\mathrm{ET}(T) = 24$:

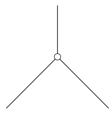


FIGURE 5. The tree T_{24}.

Case $\mathrm{ET}(\Gamma) = 36$: There are only 3 saturated graphs without end-loops (modulo equivalent embedding into the sphere):

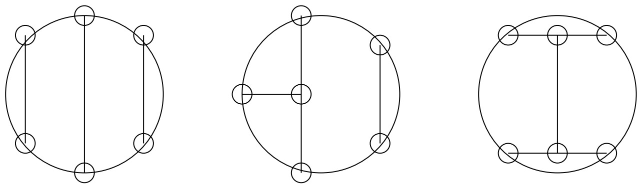


FIGURE 6. The case $\mathrm{ET}(\Gamma) = 36$.

Any other graph is either a tree or a sum of a saturated graph T' with $\mathrm{ET}(T') = 0, 12, 24$ with trees (with complementary ET). There is only one topological tree T_{36} with $\mathrm{ET}(T_{36}) = 36$.

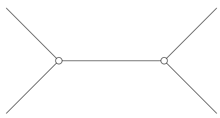
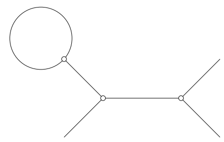


FIGURE 7.

The number of possible markings of the tree or loops at the ends is 81 but due to the symmetry of the graph the actual number of graphs T_Γ corresponding to different placement of loops at the end and markings is smaller: there are 34 different T_Γ of this type.

The number of markings of T_{36} is 16 but due to its symmetry the number of different graphs T_Γ is 7. (Recall that two graphs T_Γ give the same Γ modulo conjugation if they are isotopic in a $\mathbb{S}^2$).

The graphs of tree type with one end loop are topologically equivalent to:



There are 8 possible markings of the above graph and they all give different T_Γ with $\mathrm{ET}(\Gamma) = 36$. We have 12 different T_Γ with 2 end-loops, 6 with 3 end-loops and one with 4 end-loops.

All topological graphs which are sums of a loop and a tree can be obtained by placing a loop into a tree. Thus there are two types:

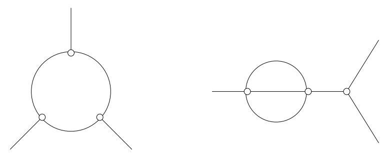


FIGURE 8.

This gives 8 graphs T_Γ in the first case and 4 in the second case.

Case $\mathrm{ET}(\Gamma) = 48$: We have one tree T_{48} with $\mathrm{ET}(\Gamma) = 48$:

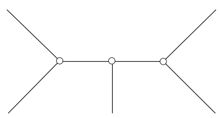


FIGURE 9. The tree T_{48}.

Here is the list of all saturated graphs with $\mathrm{ET}(\Gamma) = 48$.

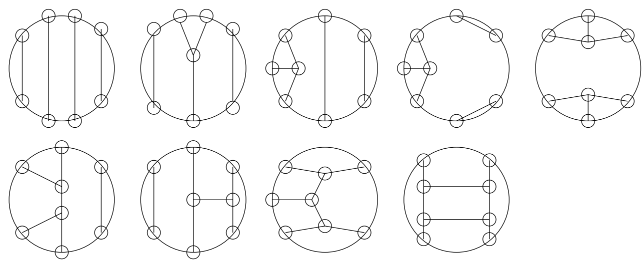


FIGURE 10. Saturated graphs in the case $\mathrm{ET}(\Gamma) = 48$.

References

[1] J. Alexander, *Note on Riemann spaces*, Bull. AMS **26**, 370–372, (1920).

[2] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are nor rational*, Proc. Lond. Math. Soc. **25**, 75–95, (1972).

[3] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Math. und ihrer Grenzgebiete, vol. **4**, Springer-Verlag, 1984.

[4] G. Belyi, *On Galois extensions of the maximal cyclotomic field*, Izv. AN USSR **43:2**, 269–276, (1979).

[5] H. Boden, K. Yokogawa, *Rationality of moduli spaces of parabolic bundles*, Journ. Lond. Math. Soc. (2) **59**, no. 2, 461–478, (1999).

[6] A. Beauville, J.-L. Colliot-Thélèlene, J.-J. Sansuc, P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles*, Ann. of Math. **121**, 238–318, (1985).

[7] F. Bogomolov, *Rationality of the moduli space of hyperelliptic curves of arbitrary genus*, CMS Conf. Proceedings **6**, AMS, 17–37, (1986).

[8] F. Bogomolov, Yu. Tschinkel, *Monodromy of elliptic surfaces*, math.AG/0002168.

[9] F. Bogomolov, P. Katsylo, *Rationality of some quotient varieties*, Mat. Sb. (N.S.) **4**, 584–589, (1985).

[10] H. Clemens, Ph. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95**, 281–356, (1972).

[11] L. Costa, R. Miró-Roig, *On the rationality of moduli spaces of vector bundles on Fano surfaces*, Journ. Pure Appl. Algebra **137**, no. 3, 199–220, (1999).

[12] I. Dolgachev, *Rationality of fields of invariants*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 3–16, Proc. Sympos. Pure Math., **46**, Part 2, AMS 1987.

[13] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer Verlag, 1998.

[14] R. Friedman, J. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Ergebnisse der Math. und ihrer Grenzgebiete, vol. **27**, Springer-Verlag, 1994.

[15] J. Harris, D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** no. 1, 23–88, (1982).

[16] V. Iskovskikh, Yu. I. Manin, *Three-dimensional quartics and counterexamples to Luroth's problem*, Math. Sb. **86**, 140–166, (1971).

[17] P. Katsylo, *Rationality of fields of invariants of reducible representations of the group* SL_2, (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. **5**, 77–79, (1984).

[18] P. Katsylo, *Rationality of the moduli spaces of hyperelliptic curves*, Izv. Akad. Nauk SSR Ser. Math. **48**, 705–710, (1984).

[19] P. Katsylo, *Rationality of moduli varieties of plane curves of degree* $3k$, Math. Sb. (N.S.) **136**, no. 3, 377–383, (1989).

[20] P. Katsylo, *Rationality of the variety of moduli of curves of genus 5*, Math. Sb. **182**, no. 3, 457–464, (1991).

[21] P. Katsylo, *Rationality of the moduli variety of curves of genus 3*, Comm. Math. Helv. **71**, no. 4, 507–524, (1996).

[22] A. King, A. Schofield, *Rationality of moduli of vector bundles on curves*, Indag. Math. (N.S.) **10**, no. 4, 519–535, (1999).

[23] K. Kodaira, *On compact analytic surfaces II, III*, Ann. of Math. (2) **77**, 563–626, (1963); **78**, 1–40, (1963).

[24] Sh. Kondo, *The rationality of the moduli space of Enriques surfaces*, Comp. Math. **91**, no. 2, 159–173, (1994).

[25] P. E. Newstaead, *Rationality of moduli spaces of stable bundles*, Math. Ann. **215**, 251–268, (1975).

[26] I. Kolpakov-Mirochnichenko, Yu. Prokhorov, *Rationality of fields of invariants of some four-dimensional linear groups and an equivariant construction related to the Segre cubic*, Math. Sborn. **182**, 1430–1445, (1991).

[27] W.-P. Li, *Relations between moduli spaces of stable bundles over* $\mathbb{P}^2$ *and rationality*, Journ. Reine Angew. Math. **484**, 201–217, (1997).

[28] P. Lochak, L. Sch, eds., *Geometric Galois Actions*, LMS Lecture Notes, vol. **242**, Cambridge Univ. Press, 1997.

[29] T. Maeda, *An elementary proof of the rationality of moduli spaces for rank 2 vector bundles on* $\mathbb{P}^2$, Hiroshima Journ. of Math. **20**, no. 1, 103–107, (1990).

[30] D. J. Saltman, *Noether's problem over an algebraically closed frield*, Inv. Math. **77**, 71–84, (1984).

[31] N. I. Shepherd-Barron, *Invariant theory for* S_5 *and the rationality of* M_6, Compositio Math. **70**, no. 1, 13–25, (1989).

[32] N. I. Shepherd-Barron, *The rationality of some moduli spaces of plane curves*, Compositio Math. **67**, no. 1, 51–88, (1988).

[33] N. I. Shepherd-Barron, *Rationality of moduli spaces via invariant theory*, Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988), 153–164, Progr. Math., **80**, Birkhäuser Boston, Boston, MA, 1989.

[34] L. Schneps, ed., *The Grothendieck's Theory of Dessins d'Enfants*, LMS Lecture Notes, vol. **200**, Cambridge Univ. Press, 1994.

[35] I. Shafarevich, et al., *Algebraic Surfaces*, Proc. Steklov Inst. Math. **75**, III-VI, 52–84, 162–182, (1965).

[36] E. B. Vinberg, *Rationality of the field of invariants of a triangular group*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **115** no. 2, 23–24, (1982).