UNRAMIFIED CORRESPONDENCES

by

Fedor Bogomolov and Yuri Tschinkel

ABSTRACT. — We study correspondences between algebraic curves defined over algebraic closures of $\mathbb{Q}$ and $\mathbf{F}_p$.

Contents

Introduction

A class $\mathcal{C}(\overline{\mathbb{Q}})$ of complete algebraic curves over $\overline{\mathbb{Q}}$ will be called *dominating* if for every algebraic curve C' over $\overline{\mathbb{Q}}$ there exist a curve $\tilde{C} \in \mathcal{C}(\overline{\mathbb{Q}})$ and a surjective map $\tilde{C} \to C'$. A curve C will be called *universal* if the class $\mathcal{U}_C(\overline{\mathbb{Q}})$ of its unramified covers is dominating.

THEOREM 1.1 (Belyi). — *Every algebraic curve C defined over a number field admits a surjective map onto $\mathbb{P}^1$ which is unramified outside* $(0, 1, \infty)$.

In 1978 Manin pointed out that Belyi's theorem implies the following

Proposition 1.2. — *The class $\mathcal{MU}(\overline{\mathbb{Q}})$ consisting of modular curves and their unramified covers is dominating.*

There are many other classes of curves with the same property, for example:

1. hyperelliptic curves and their unramified coverings;
2. the class $\mathcal{CU}(\overline{\mathbb{Q}}) := \cup_{n\in\mathbb{N}}\mathcal{C}_n(\overline{\mathbb{Q}})$, with $\mathcal{C}_n(\overline{\mathbb{Q}})$ consisting of curves with function field $\overline{\mathbb{Q}}(z, \sqrt[n]{z(1-z)})$ and their unramified coverings.
3. the class $\mathcal{CN}(\overline{\mathbb{Q}}) := \cup_{n\in\mathbb{N}}\mathcal{CN}_n(\overline{\mathbb{Q}})$ where $\mathcal{CN}_n(\overline{\mathbb{Q}})$ consists of all unramified covers of any curve C_n with the property that $C_n \to \mathbb{P}^1$ is ramified in $(0, 1, \infty)$ only and all local ramification indices of C_n over 0 are divisible by 3, over 1 divisible by 2 and over ∞ divisible by n. In particular, we could take C_n to be the modular curve $X(n)$.

Proof. — (Sketch) Let us consider the class of hyperelliptic curves and their unramified covers. Let C' be an arbitrary curve and $\sigma : C' \to \mathbb{P}^1$ a *generic* map, branched over the points $q_1, ..., q_n$ (generic means that there is only one ramification point over each branch point and all local ramification indices are equal to 2). Denote by C a hyperelliptic curve whose ramification contains $q_1, ..., q_n$. Then $\tilde{C} := C \times_{\mathbb{P}^1} C'$ is an unramified cover of C which surjects onto C'. For the classes $\mathcal{CU}(\overline{\mathbb{Q}})$ and $\mathcal{CN}(\overline{\mathbb{Q}})$ we use Belyi's theorem. □

Question 1.3. — Does there exist a universal algebraic curve C (over $\overline{\mathbb{Q}}$)?

Question 1.4. — Does there exist a number $n \in \mathbb{N}$ such that every curve defined over $\overline{\mathbb{Q}}$ admits a surjective map onto $\mathbb{P}^1$ with ramification over $(0, 1, \infty)$ such that all local ramification indices are $\leq n$?

Question 1.5. — Is every curve C (over $\overline{\mathbb{Q}}$) of genus $g(C) \geq 2$ universal?

Remark 1.6. — It is clear that an affirmative answer to Question 1.4 implies a (constructive) affirmative answer to Question 1.3.

In this note we answer these questions in a simple model situation: instead of $\overline{\mathbb{Q}}$ we consider the (separable) closure $\overline{F}_p$ of the finite field $\mathbf{F}_p$.

THEOREM 1.7. — *Let $p \geq 5$ be a prime and C a hyperelliptic curve over $\overline{\mathbf{F}}_p$ of genus $g(C) \geq 2$. Then C is universal: for any projective curve C' there exist a finite étale cover $\tilde{C} \to C$ and a surjective regular map $\tau : \tilde{C} \to C'$.*

In Section 4 we prove the following geometric fact (over arbitrary algebraically closed fields of characteristic $\neq 2, 3$):

PROPOSITION 1.8. — *Every hyperelliptic curve C of genus ≥ 2 has a finite étale cover $\tilde{C}$ which surjects onto the genus 2 curve C_0 given by $\sqrt[6]{z(1-z)}$. In particular, if C_0 is universal then every hyperelliptic curve of genus ≥ 2 is universal.*

REMARK 1.9. — Applying the Chevalley-Weil theorem we conclude that the Mordell conjecture (Faltings' theorem) for C_0 implies the Mordell conjecture for every hyperelliptic curve of genus ≥ 2.

The fact that there is some interaction between the arithmetic of different curves has been noted previously. Moret-Bailly and Szpiro showed (see [6], [5]) that the proof of an *effective* Mordell conjecture for *one* (hyperbolic) curve (for example, C_0) implies the ABC-conjecture, which in turn implies an effective Mordell conjecture for *all* (hyperbolic) curves (Elkies [4]). Here *effective* means an explicit bound on the height of a K-rational point on the curve for all number fields K. Here again, Belyi's theorem is used in an essential way.

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2. Main construction

NOTATIONS 2.1. — Let $\tau : C \to C'$ be a surjective map of algebraic curves. We denote by $\mathrm{Ram}(\tau) \subset C$ the ramification locus of τ and by $\mathrm{Bran}(\tau) = \tau(\mathrm{Ram}(C)) \subset C'$ the branch locus of τ. For a point $q \in C$ we denote by $e_q(\tau)$ the local ramification index at q. We denote by

$$e(\tau) := \max_{q \in C} e_q(\tau)$$

the maximum local ramification index of τ. We say that τ has *simple* ramification if $e(\tau) \leq 2$ and that τ is *generic* if in addition there is only one ramification point over each branch point.

Remark 2.2. — Every curve admits a generic map onto $\mathbb{P}^1$, at least after a separable extension of the ground field.

Let $p \geq 5$ be a prime number. In this section we work over a separable closure $\overline{\mathbf{F}}_p$ of the finite field $\mathbf{F}_p$. First we show that there exists at least one curve satisfying the conclusion of Theorem 1.7.

Let $\pi_0 : E_0 \to \mathbb{P}^1$ be the elliptic curve given by

$$\sqrt[3]{z(z-1)}.$$

Let $\sigma_0 : C_0 \to \mathbb{P}^1$ be the genus 2 curve given by

$$\sqrt[6]{z(z-1)},$$

and $\iota_0 : C_0 \to E_0$ the corresponding 2-cover. Clearly, ι_0 has simple ramifications over the preimages of $0, 1$. Let C be an arbitrary curve. Choosing a generic function on C we get a generic covering $\sigma : C \to \mathbb{P}^1$ (such covering is defined over $\overline{\mathbf{F}}_p$). Assume further that $\mathrm{Bran}(\sigma) \subset \mathbb{P}^1$ does not contain $(0, 1, \infty)$.

Consider the diagram

$$\begin{array}{ccccc} C & \longleftarrow & C_1 & \longleftarrow & C_2 \\ {\scriptstyle\sigma}\downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \longleftarrow & E_0 & & \downarrow \\ & & {\scriptstyle\varphi}\downarrow & & \downarrow \\ & & E_0 & \longleftarrow & C_0 \end{array}$$

Here $C_1 = C \times_{\mathbb{P}^1} E_0$ (it is irreducible since $E_0 \to \mathbb{P}^1$ is a 2-cover). Then $C_1 \to E_0$ has simple ramification over a finite number of points in E_0. Recall that E_0 has a group scheme structure, and *all* $\overline{\mathbf{F}}_p$-points of E_0 are torsion points. This implies that there exists an étale map

$E_0 \to E_0$ such that all ramification points of C_1 over E_0 are mapped to 0. More precisely, any finite set of $\overline{\mathbf{F}}_p$-points of E_0 is contained in the group subscheme $E_0^{et}[n] \subset E_0$ - the maximal étale subgroup of the multiplication by n-kernel $E_0[n]$ (for some $n \in \mathbb{N}$). For every positive integer n there exists a positive multiple of m of n and an étale map $E_0 \to E_0$ with kernel $E_0^{et}[m]$.

Taking the composition of $C_1 \to E_0$ with the multiplication by a suitable m, we get a (possibly new) surjective regular map $C_1 \to E_0$ which is ramified only over the zero point in E_0 and has the property that all the local ramification indices are at most 2. Using this map let us define $C_2 := C_0 \times_{E_0} C_1$. Consequently, any component of C_2 surjects onto C_1 and is an étale covering of C_0 (ramification cancels ramification). This component satisfies the conclusion of Theorem 1.7.

Lemma 2.3. — *Let C be any smooth complete algebraic curve and E any curve of genus 1. There exists a curve C_1 which surjects onto C and E such that the ramification of the map $C_1 \to E$ lies entirely over a single point of E and its local ramification indices are all equal to* 2.

Proof. — Consider a generic map $\sigma : C \to \mathbb{P}^1$ with $e(\sigma) \leq 2$. Choose a double cover $\pi : E \to \mathbb{P}^1$ such that the branch loci $\mathrm{Bran}(\sigma)$ and $\mathrm{Bran}(\pi)$ on $\mathbb{P}^1$ are disjoint. Then the product $C_1 := C \times_{\mathbb{P}^1} E$ is an irreducible curve which is a double cover of C. The curve admits a surjective map $\iota_1 : C_1 \to E$ with $e(\iota_1) \leq 2$. Similarly to the previous construction we can find an unramified cover $\varphi : E \to E$ such that the composition $\varphi \circ \iota_1 : C_1 \to E$ is ramified only over one point in E and the local ramification indices are still equal to 2. □

Corollary 2.4. — *Assume that some unramified covering $\tilde{C}$ of C surjects onto an elliptic curve E. Assume further that there exists a point q on E such that* all *local ramification indices of the map $\tilde{C} \to E$ over q are divisible by 2. Then C is universal.*

Proof. — It is sufficient to take the product of $\tilde{C} \times_E C_1$. Any irreducible component of the resulting curve will be an unramified covering of $\tilde{C}$ (and hence C) and will admit a surjective map onto C_1 and C. □

Corollary 2.5 (Theorem 1.7). — *Every hyperelliptic curve C over $\overline{\mathbf{F}}_p$ (with $p \geq 5$) of genus ≥ 2 is universal.*

Proof. — Consider the standard projection $\sigma : C \to \mathbb{P}^1$ (of degree 2). Its branch locus $\mathrm{Bran}(\sigma)$ consists of $2g+2$ points. Let $\pi : E \to \mathbb{P}^1$ be a double cover such that $\mathrm{Bran}(\pi)$ is contained in $\mathrm{Bran}(\sigma)$. Then the product $\tilde{C} = C \times_{\mathbb{P}^1} E$ is an unramified double cover of C. Moreover, $\tilde{C}$ is a double cover of E with ramification at most over the preimages in E of the points in $\mathrm{Bran}(\sigma) \setminus \mathrm{Bran}(\pi)$. We now apply Corollary 2.4. □

In *finite* characteristic, there are many other (classes of) universal curves. For example, cyclic coverings with ramification in 3 points, hyperbolic modular curves, etc. Thus it seems plausible to formulate the following

Conjecture 2.6. — Any smooth complete curve C of genus $g(C) \geq 2$ defined over $\overline{\mathbf{F}}_p$ (for $p \geq 2$) is universal.

3. The case of characteristic 0

In this section we work over $\overline{\mathbb{Q}}$. We show that the method outlined in Section 2 can employed in characteristic zero to produce natural infinite sets of algebraic points on $\mathbb{P}^1$ which occur as ramification points of surjective maps from $\mathbb{P}^1_2$ to $\mathbb{P}^1_1$ branched over $(0, 1, \infty) \in \mathbb{P}^1_1$ only and having an *a priori* bound on the ramification index (here $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$ are two different copies of the projective line $\mathbb{P}^1$).

Notice that, in principle, it is easy to produce *some* sets of points (of any finite cardinality) with this property: Take an $n \geq 6$ and any triangulation of $\mathbb{P}^1_2$ with vertices of index $\leq n$. A barycentric subdivision of each such triangulation defines a function from $\mathbb{P}^1_2$ to $\mathbb{P}^1_1$ with local ramification indices $\leq 2n$ (for more details see [3]). Therefore, any curve with bounded ramification over this set of vertices will have bounded ramification over $\mathbb{P}^1_1$. However, we have no explicit control over the coordinates of the ramification points on $\mathbb{P}^1_2$.

An (obvious) analogous way to control ramification indices is to consider the following diagram

$$\begin{CD} E @>{\pi}>> \mathbb{P}^1_2 \\ @V{\phi_n}VV @VV{\varphi_{n,E}}V \\ E @>>{\pi}> \mathbb{P}^1_1, \end{CD}$$

where the map ϕ_n is the quotient by the subscheme of n-torsion points and the maps $E \to \mathbb{P}^1$ are the standard double covers, ramified over $(0, 1, \infty, \lambda)$. Clearly, all the ramification points of $\varphi_{n,E}$ (in $\mathbb{P}^1_2$) are over $0, 1, \infty$ and λ (in $\mathbb{P}^1_1$) and $e(\varphi_{n,E}) = 2$. Belyi's theorem gives a map $\beta : \mathbb{P}^1_1 \to \mathbb{P}^1_0$, which ramifies only over the points $(0, 1, \infty) \in \mathbb{P}^1_0$, maps $\{0, 1, \infty, \lambda\} \subset \mathbb{P}^1_1$ into $\{0, 1, \infty\} \subset \mathbb{P}^1_0$ and has local ramification indices $\leq n$. Moreover, it provides an explicit bound on $\deg(\beta)$ and, consequently, on $e(\beta)$ (in terms of the absolute height of λ). Let $\beta_\lambda : \mathbb{P}^1_1 \to \mathbb{P}^1_0$ be a map such that

$$e(\beta_\lambda) = \inf_\beta \{e_\beta\}$$

over the set of all maps as above. Then the map $\beta_\lambda \circ \varphi_{n,E} : \mathbb{P}^1_2 \to \mathbb{P}^1_0$ ramifies over three points only and has index $e(\beta_\lambda \circ \varphi_{n,E}) \leq 2n$. Let

$$R_E := \pi(E(\overline{\mathbb{Q}})_{\mathrm{tors}}) \subset \mathbb{P}^1_2(\overline{\mathbb{Q}})$$

be the image of the torsion points of E. Let $\sigma : C \to \mathbb{P}^1_2$ be any map ramified only in a subset of R_E. Let $\pi := \beta_\lambda \circ \varphi_{n,E} \circ \sigma$. Then

$$e(\pi) \leq 2e(\sigma) \cdot e(\beta_\lambda).$$

A natural application of the construction in Section 2 is as follows:

Example 3.1. — Let $\pi : E \to \mathbb{P}^1$ be a triple cover with $\mathrm{Bran}(\pi) = \{0, 1, \infty\}$ (E is a CM elliptic curve with j-invariant 0). Consider the following diagram

$$\begin{CD} @. E @>{\pi}>> \mathbb{P}^1_2 \\ @. @V{\phi_n}VV @VV{\varphi_{n,E}}V \\ C_0 @>>> E @>>{\pi}> \mathbb{P}^1_1, \end{CD}$$

where C_0 is a curve of genus $g(C_0) = 2$ given by $\sqrt[6]{z(z-1)}$, ϕ_n is the quotient map by the subscheme of torsion points of order n, and $\varphi_{n,E}$ the corresponding map from $\mathbb{P}^1_2$ to $\mathbb{P}^1_1$ ramified only over $(0, 1, \infty)$. Let $\mathcal{X}_g = \{X\}$ be the subset of curves of genus g admitting a map $\sigma_X : X \to \mathbb{P}^1_2$ such that

- $e(\sigma_X) = 2$;
- $\mathrm{Bran}(\sigma_X) \subseteq \pi(E(\overline{\mathbb{Q}})_{\mathrm{tors}})$.

Then, for any $X \in \mathcal{X}_g$ the map

$$\varphi_{n,E} \circ \sigma_X : X \to \mathbb{P}^1_1$$

has index $e(\varphi_{n,E} \circ \sigma_X) \leq 6$ and there exists an unramified cover $\tilde{C} \to C_0$ surjecting onto X. Moreover, $\mathcal{X}_g$ is *dense* (in real and p-adic topologies) in the natural Hurwitz scheme $\mathcal{H}_g$ parametrizing curves of genus g.

The set of curves dominated by unramified covers of C_0 is much larger than $\mathcal{X}_g$. Indeed, consider any 4-tuple of points in

$$\pi(E(\overline{\mathbb{Q}})_{\mathrm{tors}}) \subseteq \mathbb{P}^1_2$$

and an elliptic curve E' obtained as a double cover of $\mathbb{P}^1_2$ ramified in those 4 points. Then E' is also dominated by unramified covers of C_0 and we can iterate the above construction for E'.

4. Geometric constructions

Let (E, q_0) be an elliptic curve, q_1 a torsion point of order two on E and $\pi : E \to \mathbb{P}^1$ the quotient with respect to the involution induced by q_1. Let n be an odd positive integer and $\varphi_{n,E} : \mathbb{P}^1_2 \to \mathbb{P}^1_1$ the map induced by

$$\begin{array}{ccc} E & \xrightarrow{\pi} & \mathbb{P}^1_2 \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \varphi_{n,E}} \\ E & \xrightarrow[\pi]{} & \mathbb{P}^1_1. \end{array}$$

Any quadruple $r = \{r_1, ..., r_4\}$ of four distinct points in $\varphi^{-1}_{n,E}(\pi(q_0))$ defines a genus 1 curve E_r (the double cover of $\mathbb{P}^1$ ramified in these four points).

Proposition 4.1. — *Let $\iota : C \to E$ be any finite cover such that all local ramification indices over q_0 are even. Then there exists an unramified cover $\tau_r : C_r \to C$ which dominates E_r and has only even local ramification indices over some point in E_r.*

Proof. — Assume that $n \geq 3$ and consider the following diagram

$$\begin{array}{ccccc} C & \xleftarrow{\tau_2} & C_2 & \xleftarrow{\tau_r} & C_r \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_r} \\ E & \xleftarrow{\varphi_n} & E & & E_r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_r} \\ \mathbb{P}^1_1 & \xleftarrow{\phi_{n,E}} & \mathbb{P}^1_2 & & \mathbb{P}^1_2, \end{array}$$

where E_r is a double cover of $\mathbb{P}^1_2$ ramified in any quadruple of points in the preimage $\phi_{n,E}^{-1}(\pi(q_0))$ and C_r is any irreducible component of $C_2 \times_{\mathbb{P}^1_2} E_r$. Any point $q_r \in E_r$ such that $q_r \notin \mathrm{Ram}(\pi_r)$ (that is, its image in $\mathbb{P}^1_2$ is distinct from $r_1, ..., r_4$) has the claimed property. □

Remark 4.2. — Iterating this procedure (and adding isogenies) we obtain many elliptic curves E' which are dominated by curves having an unramified cover onto E. It would be interesting to know if for any two elliptic curves over $\overline{\mathbb{Q}}$ there exists a cycle connecting them (at least modulo isogenies). We will now show that *any* elliptic curve can be connected in this way to E_0.

Let $E_0 \subset \mathbb{P}^2 = \{(x : y : z)\}$ be the elliptic curve

$$x^3 + y^3 + z^3 = 0,$$

and

$$E_0[3] = \mathsf{T} := \left\{ \begin{array}{lll} (1:0:-1), & (1:0:-\zeta), & (1:0:-\zeta^2), \\ (0:1:-1), & (0:1:-\zeta), & (0:1:-\zeta^2), \\ (1:-1:0), & (1:-\zeta:0), & (1:-\zeta^2:0) \end{array} \right\}$$

its set of 3-torsion points (where ζ is a primitive cubic root of 1). Denote by $\mathcal{E}_\lambda = \{E_\lambda\}$ the family of elliptic curves on $\mathbb{P}^2$ passing through T given by

$$E_\lambda \ : \ x^3 + y^3 + z^3 + \lambda xyz = 0.$$

It is easy to see that for each λ the set $E_\lambda[3]$ of 3-torsion points of E_λ is precisely T. Let

$$\begin{array}{cccc} \pi & : & \mathbb{P}^2 & \rightarrow & \mathbb{P}^1 \\ & & (x:y:z) & \mapsto & (x+z:y) \end{array}$$

be the projection respecting the involution $x \rightarrow z$ on $\mathbb{P}^2$. Denote by $E_\lambda^0 = E_\lambda \setminus (1:0:-1)$ and by π_λ the restriction of π to E_λ^0. Clearly, π_λ exhibits each E_λ^0 as a double cover of $\mathbb{P}^1$. and π_λ has only simple double points for all λ. Moreover,

$$\pi(\mathsf{T} \setminus (1:0:-1)) = \{(1:-\zeta), (1:-\zeta^2), (1:-1), (1:0)\}$$

and for all λ there exists a (non-empty) set $S_\lambda \subset \mathrm{Bran}(\pi_\lambda) \subset \mathbb{P}^1$ such that $\pi_\lambda^{-1}(S_\lambda) \subset \mathsf{T}$. Let $\pi_0' : E_0' \rightarrow \mathbb{P}^1$ be a double cover ramified in the 4 points in $\pi(\mathsf{T} \setminus (1:0:-1))$.

Lemma 4.3. — *Let $\iota : C \rightarrow E_\lambda$ be a double cover such that over at least one point in* $\mathrm{Bran}(\iota)$ *the local ramification indices are even. Then there exists an unramified cover $\tilde{C} \rightarrow C$ and a surjective morphism $\tilde{\iota} : \tilde{C} \rightarrow E_0'$ such that over at least one point in* $\mathrm{Bran}(\tilde{\iota}) \subset E_0'$ *all local ramification indices of $\tilde{\iota}$ are even.*

Proof. — Consider the diagram

$$\begin{array}{ccc} E_\lambda & \xleftarrow{\ \iota\ } & C_1 \\ {\scriptstyle \varphi_3}\downarrow & & \downarrow \\ E_\lambda & \longleftarrow & C \\ {\scriptstyle \pi_\lambda}\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

Then $C_1 \rightarrow \mathbb{P}^1$ has even local ramification indices over all points in $\pi(\mathsf{T})$. It follows that

$$\tilde{C} := C_1 \times_{\mathbb{P}^1} E_0' \rightarrow E_0'$$

has even local ramification indices over the preimages of the fifth point in $\pi(\mathsf{T})$, as claimed. $\square$

NOTATIONS 4.4. — Let $\mathcal{C}$ be the class of curves such that there exists an elliptic curve E, a surjective map $\iota : C \to E$ and a point $q \in \mathrm{Bran}(\iota)$ such that all local ramification indices at points in $\iota^{-1}(q)$ are even.

EXAMPLE 4.5. — Any hyperelliptic curve of genus ≥ 2 belongs to $\mathcal{C}$. More generally, $\mathcal{C}$ contains any curve C admitting a map $C \to \mathbb{P}^1$ with even local ramification indices over at least 5 points in $\mathbb{P}^1$.

PROPOSITION 4.6. — *For any $C \in \mathcal{C}$ there exists an unramified cover $\tilde{C} \to C$ surjecting onto C_0 (with $C_0 \to \mathbb{P}^1$ given by $\sqrt[6]{z(1-z)}$).*

Proof. — Consider $C_1 = C \in \mathcal{C}$ with $\iota_1 : C_1 \to E = E_\lambda$ as in 4.4. Define C_2 as an irreducible component of $C_1 \times_E E$:

$$\begin{array}{ccc} C_1 & \xleftarrow{\tau_2} & C_2 \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ E & \xleftarrow[\varphi_3]{} & E \\ & & \downarrow{\scriptstyle \pi_\lambda} \\ & & \mathbb{P}^1 \end{array}$$

Define $C_3 := C_2 \times_{\mathbb{P}^1} E_0$ by the diagram

$$\begin{array}{ccc} C_2 & \xleftarrow{\tau_3} & C_3 \\ {\scriptstyle \sigma_2}\downarrow & & \downarrow{\scriptstyle \iota_3} \\ \mathbb{P}^1 & \xleftarrow[\pi_0]{} & E_0. \end{array}$$

Observe that for $q \in \mathrm{Bran}(\pi_0)$ the local ramification indices in the preimage $(\pi_\lambda \circ \iota_2)^{-1}(q)$ are all even. It follows that the map $\tau_3 : C_3 \to C_2$ is *unramified* and that $\iota_3 : C_3 \to E_0$ has even local ramification indices over (the preimage of) $q_5 \in \{\pi(\mathsf{T}) \setminus \mathrm{Bran}(\pi_0)\}$ (the 5th point). Define C_4 as an irreducible component of $C_3 \times_{E_0} E_0$ in the diagram

$$\begin{array}{ccc} C_3 & \xleftarrow{\tau_4} & C_4 \\ {\scriptstyle \iota_3}\downarrow & & \downarrow{\scriptstyle \iota_4} \\ E_0 & \xleftarrow[\varphi_3]{} & E_0. \end{array}$$

The map ι_4 is ramified over the preimages $(\pi_0 \circ \varphi_3)^{-1}(q_5)$, with even local ramification indices. Finally, $C_5 = C_4 \times_{E_0} C_0$ from the diagram

$$\begin{array}{ccc} C_4 & \xleftarrow{\tau_5} & C_5 \\ {\scriptstyle \iota_4}\downarrow & & \downarrow \\ E_0 & \xleftarrow{\iota_0} & C_0. \end{array}$$

has a dominant map onto C_0 and is unramified over C_4 (and consequently, C_1). □

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