LAGRANGIAN SUBVARIETIES OF ABELIAN FOURFOLDS

by

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Dedicated to Professor K. Kodaira

1. Introduction

Let $(\mathcal{W}, \omega)$ be a smooth projective algebraic variety of dimension $2n$ over $\mathbb{C}$ together with a holomorphic $(2,0)$-form of maximal rank $2n$. A subvariety $X \subset \mathcal{W}$ is called *weakly lagrangian* if $\dim X \leq n$ and if the restriction of ω to X is trivial (notice that X can be singular). An n-dimensional subvariety $X \subset \mathcal{W}$ with this property is called *lagrangian.* For example, any curve C contained in a K3 or abelian surface S is lagrangian. Further examples of lagrangian subvarieties are obtained by taking a curve $C \subset S$ and by considering the corresponding symmetric products. Alternatively, one could look at a product of different curves (of genus > 1) inside a product of abelian varieties. We will say that a variety $X \subset \mathcal{W}$ is *fibered* if it admits a dominant map onto a curve of genus > 1. In this note we construct examples of nonfibered lagrangian surfaces in abelian varieties. Our motivation comes from the following

Problem 1.1. — Find examples of projective surfaces with a nontrivial fundamental group. In particular, find examples where the fundamental group has a nontrivial nilpotent tower.

If X is fibered over a curve C of genus > 1 then the fundamental group $\pi_1(X)$ surjects onto a subgroup of finite index in $\pi_1(C)$ and consequently both $\pi_1(X)$ and its nilpotent tower are big. Therefore, we are interested in examples of surfaces where the nontriviality of $\pi_1(X)$ is not induced from curves. Consider the map to the Albanese variety $\mathrm{alb} : X \to \mathrm{Alb}(X)$. One is interested in situations where the natural map

$$H^2(\mathrm{Alb}(X), \mathbb{C}) \to H^2(X, \mathbb{C})$$

has a nontrivial kernel – the triviality of the kernel implies the triviality of the nilpotent tower (tensor $\mathbb{Q}$). Such examples were given by Campana ([3], Cor. 1.2) and Sommese-Van de Ven ([7]). However, there the kernel was found in the map

$$\mathrm{Pic}(\mathrm{Alb}(X)) \to \mathrm{Pic}(X).$$

In their construction the fundamental group of the variety is a central extension of an abelian group (and the lower central series has only two steps). The lagrangian property of $\mathrm{alb}(X) \subset \mathrm{Alb}(X)$ implies that there is a nontrivial kernel in $H^{2,0}$ (rather than on the level of the Picard groups). We produce an infinite series of surfaces X of different topological types which are contained in abelian varieties and are lagrangian (or weakly lagrangian) with respect to a nondegenerate $(2,0)$-form. The weakly lagrangian property for a subvariety X of an abelian variety A is related to the nontriviality of the fundamental group $\pi_1(X)$ as follows: the number of linearly independent generators of the second quotient of the central series of $\pi_1(X)$ is bounded from below by the dimension of the rational envelope of the space of those $(2,0)$-forms on A which restrict trivially to X.

Our construction uses dominant maps between K3 surfaces. Let S be a K3 surface and g_1, g_2 dominant rational maps of S to Kummer K3 surfaces S_1, S_2 (blowups of quotients of abelian surfaces A_1, A_2 by standard involutions). Then the (birational) preimage X of $(g_1, g_2)(S) \subset S_1 \times S_2$ in $A_1 \times A_2$ is a lagrangian surface (in general, singular). For special choices of g_1, g_2 we can compute some basic invariants of X and, in particular, show that X is nonfibered. For example, let A be an abelian surface which is not isogenous to a product of elliptic curves and S the associated Kummer surface. Assume that $g_1 : S \to S$ is an isomorphism and $g_2 : S \to S$ is not an isomorphism. Then $X \subset A \times A$ is not fibered, $\mathrm{Alb}(X)$ is isogenous to $A \times A$ and X is lagrangian with respect to exactly one 2-form on $A \times A$. We analyze other constructions, with S an elliptic Kummer surface and g_j induced from the group law. We don't determine the actual structure of $\pi_1(X)$ (it presumably depends on X), but it seems quite plausible that for some X from our list $\pi_1(X)$ has a rather nontrivial nilpotent tower.

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2. Preliminaries

Let $V_{\mathbb{Q}}$ be a finite dimensional $\mathbb{Q}$-vector space and $V_{\mathbb{C}}$ its complexification. Let $w \in V_{\mathbb{C}}$ be a vector. We denote by $L(w) \subset V_{\mathbb{Q}}$ its rational envelope, i.e., the smallest linear subspace such that $w \in L(w)_{\mathbb{C}}$. More generally, if $W \subset V_{\mathbb{C}}$ is any set, we will denote by $L(W) \subset V_{\mathbb{Q}}$ the smallest linear subspace such that $L(w) \subset L(W)$ for all $w \in W$. An element $w \in V_{\mathbb{C}}$ will be called *k-generic* if the dimension of $L(w)$ is $\dim V_{\mathbb{Q}} - k$.

REMARK 2.1. — The set of rationally defined subspaces in $V_{\mathbb{C}}$ is countable. Therefore, for any linear subspace $W \subset V_{\mathbb{C}}$ we have $L(W) = L(w)$ for all $w \in W$ which are not contained in a countable number of linear subspaces.

PROPOSITION 2.2. — *Let (A, ω) be an abelian variety of dimension $2n$ together with a nondegenerate holomorphic $(2,0)$-form ω. Assume that ω defines a k-generic element in $H^{2,0}(A, \mathbb{C})$ with $k < 3$. Then there are no weakly lagrangian surfaces in A.*

Proof. Assume that we have a weakly lagrangian surface $i : X \hookrightarrow A$. Consider the induced $\mathbb{Q}$-rational homomorphism

$$i^* : H^2(A, \mathbb{Q}) \to H^2(X, \mathbb{Q}).$$

The class of ω is contained in the ($\mathbb{Q}$-rationally defined) kernel of i^*. Since ω is k-generic in $H^{2,0}(A, \mathbb{C})$ for $k < 3$ the image of $H^2(A, \mathbb{C})$ in $H^2(X, \mathbb{C})$ has rank at most 2. If we had a holomorphic form $w' \in H^{2,0}(A, \mathbb{C})$ restricting nontrivially to X then the image of $H^2(A, \mathbb{C})$ in $H^2(X, \mathbb{C})$ would contain at least three linearly independent forms $w', \bar{w}'$ and the image of a polarization from $\mathrm{Pic}(A)$. Thus the triviality of ω on X implies that *all* $(2,0)$-forms restrict trivially to X. Consequently, the dimension of X is ≤ 1, contradiction. □

We end this section with a simple description of maps of fibered surfaces onto curves (see also [4]).

Proposition 2.3. — *For every smooth projective algebraic surface V there exists a universal algebraic variety $\mathcal{U}(V)$ such that every dominant rational map $V \to C$ onto a smooth curve of genus ≥ 2 factors through $\mathcal{U}(V)$.*

Proof. First observe that the set of dominant rational maps of V onto smooth curves with generically irreducible fibers is countable. Indeed, the class $f^*(L)$ (where L is some polarization on the image-curve) defines f. If two such classes $[f]$ and $[f']$ differ by an element in $\mathrm{Pic}^0(V)$, then the maps must be the same. Otherwise, some fiber of f would surject onto the image of f' and therefore, the degrees of the two classes $[f]$ and $[f']$ on this fiber would be different.

Further, dominant maps onto smooth curves of genus ≥ 2 define a linear subspace $W \subset H^0(V, \Omega^1)$ by the property that wedge-products of linearly independent forms $w, w' \in W$ are trivial. The set of maximal subspaces W in $H^0(V, \Omega^1)$ with this property is an algebraic variety. Remark that for two forms ω, ω' with $\omega \wedge \omega' = 0$ their ratio ω/ω' is a nonconstant rational function, which is constant on the fibers of the foliations on V defined by ω, resp. ω'. Thus, any such subspace W defines a foliation $\mathcal{F}_W$ on V with compact nonintersecting fibers (locally the form defines a holomorphic map, its fibers are the fibers of $\mathcal{F}_W$). Therefore, $\mathcal{F}_W$ defines a dominant rational map onto a curve C_W (which lifts to a map of V onto the normalization of C_W). Different W define different foliations and different morphisms (the sum of two spaces W and W' with the same foliations $\mathcal{F}_W = \mathcal{F}_{W'}$ has the same property, contradicting the maximality of W and W'). Since the set of such spaces W is on the one hand algebraic (a finite union of subvarieties of a Grassmannian) and on the other hand countable, it must be finite. □

Corollary 2.4. — *For every smooth projective algebraic surface V there exists a universal algebraic variety $\mathcal{U}(V)$ such that every dominant rational map $V \to V'$, where V' is a product of smooth curves of genus ≥ 2, factors through $\mathcal{U}(V)$.*

Proof. We have shown in Lemma 2.3 that there is a finite number of dominant rational maps onto curves of genus ≥ 2. The product of these curves and maps is $\mathcal{U}(V)$. Universality follows. □

3. Construction

Let A be an abelian surface and τ the involution $\tau(a) = -a$ (for $a \in A$). This involution acts on A with 16 fixed points. We denote by $\tilde{S}$ the (singular) quotient A/τ. Let A^* be the blowup of A in the 16 points. The involution τ extends to a fixed point free action on A^* and the quotient A^*/τ is a K3 (Kummer) surface S, a blowup of $\tilde{S}$. It contains 16 exceptional curves and we will denote by $\Delta = \cup_{k=1}^{16} \Delta^k$ their union. We shall denote by $\delta : A \to \tilde{S}$ the double cover. Every K3 surface has a unique (up to constants) nondegenerate holomorphic $(2,0)$-form.

We start with two *simple* abelian surfaces A_1 and A_2 (in particular, A_j don't contain elliptic curves - this assumption simplifies the discussion in Section 4). We consider the corresponding Kummer surfaces S_j and we assume that there exists another K3 surface S together with two dominant rational maps $g_j : S \to S_j$. We denote by $\tilde{g}_j$ the induced maps $S \to \tilde{S}_j$. Now consider the map

$$\tilde{\varphi} = (\tilde{g}_1, \tilde{g}_2) : S \to \tilde{S}_1 \times \tilde{S}_2.$$

We have two double covers

$$(\delta_1, 1) : A_1 \times \tilde{S}_2 \to \tilde{S}_1 \times \tilde{S}_2$$

and

$$(1, \delta_2) : \tilde{S}_1 \times A_2 \to \tilde{S}_1 \times \tilde{S}_2.$$

Denote by $\tilde{Y}_1 = (\delta_1, 1)^{-1}(\tilde{\varphi}(S))$ and by $\tilde{Y}_2 = (1, \delta_2)^{-1}(\tilde{\varphi}(S))$. Let

$$\tilde{X} = \tilde{X}_{\tilde{\varphi}} := (\delta_1, \delta_2)^{-1}(\tilde{\varphi}(S))$$

be the preimage of S in $A_1 \times A_2$. We see that the $(2,2)$-covering $\tilde{X} \to \tilde{\varphi}(S)$ is

$$\tilde{X} = \tilde{Y}_1 \times_{\tilde{\varphi}(S)} \tilde{Y}_2 \subset A_1 \times A_2.$$

The surface $\tilde{X}$ is, in general, singular.

Lemma 3.1. — *There exists a nondegenerate holomorphic 2-form ω' on $A_1 \times A_2$ such that $\tilde{X}$ is lagrangian with respect to ω'.*

Proof. The singular surface $\tilde{S}_j$ carries a nondegenerate holomorphic 2-form, which we again denote by ω_j. Evidently, $\tilde{g}_j^*\omega_j = \lambda_j\omega$ for some nonzero numbers λ_j. Consider the form

$$\omega' = \lambda_2\omega_1 - \lambda_1\omega_2$$

on $\tilde{S}_1 \times \tilde{S}_2$. The form ω' is identically zero on the image $\tilde{\varphi}(S)$ (since it is trivial on the open part of S where $\tilde{\varphi}$ is smooth). Since both forms ω_j lift to nondegenerate forms on the abelian surfaces A_j the form ω' lifts to a nondegenerate form on $A_1 \times A_2$. The restriction of the lift of ω' to $\tilde{X}$ is identically zero on the smooth points of $\tilde{X}$. □

EXAMPLE 3.2. — If $A_1 = A_2 = A$ and the maps $\tilde{g}_j = \mathrm{id}$ then the resulting surface $\tilde{X}$ is the abelian surface A, embedded diagonally into $A \times A$.

EXAMPLE 3.3. — Let S be a K3 surface, which is simultaneously Kummer and a double cover of $\mathbb{P}^2$. Denote by θ the covering involution on S. Put $S_1 = S_2 = S$, $\tilde{g}_1 = \mathrm{id}$ and $\tilde{g}_2 = \theta$. Then the corresponding surface $\tilde{X}$ is lagrangian. A special case of this construction is obtained as follows: Let C be a curve of genus 2 and σ a hyperelliptic involution on C. Consider $C \times C$, together with the involutions: σ_{12} interchanging the factors and

$$\sigma_a : (c_1, c_2) \to (\sigma(c_2), \sigma(c_1)).$$

Denote by $Y_1 = C \times C/\sigma_{12}$ and $Y_2 = C \times C/\sigma_a$. We see that both Y_1 and Y_2 are isomorphic to a symmetric square of C, which is birational to the same abelian surface A. Denote by $\tilde{S}$ the quotient of A by the standard involution τ. Observe that S is realized as a double cover of $\mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1/\sigma_{12}$. Then

$$(\tau, \tau)^{-1}((1, \theta)(S)) = C \times C \subset A \times A.$$

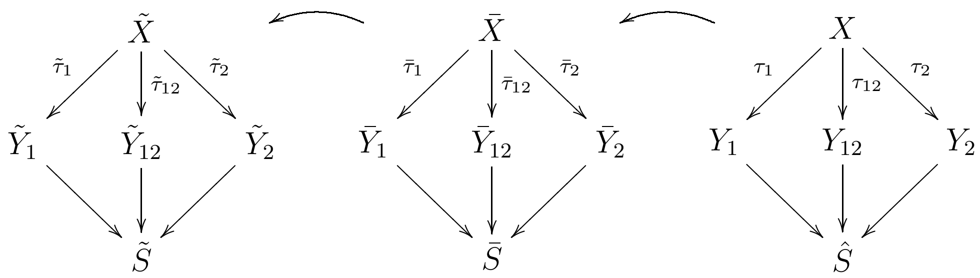
In the following sections we will show that we can arrange a situation where $\tilde{X}$ is not contained in any abelian subvariety of $A_1 \times A_2$ and where it does not admit any dominant morphisms onto a curve of genus ≥ 2.

4. Details and proofs

4.1. Desingularization. — To analyze the surface $\tilde{X}$ constructed in Section 3 we will need its sufficiently explicit partial desingularization. It will be constructed in two steps.

Let Σ be the minimal finite set of points in S such that both maps g_1, g_2 are well defined on the complement $S^0 = S \setminus \Sigma$. On S^0 the maps g_1 and g_2 are local isomorphisms. Consider the divisors $\Delta_j \subset S_j$. The preimages D_j^0 of these divisors (under the maps g_j) in S^0 are smooth and the components of D_j^0 don't intersect in S^0. However, the divisors D_1^0 and D_2^0 do intersect in S^0, and we denote by $p_1, ..., p_l \in S^0$ their intersection points. Let D_j be the closure of D_j^0 in S. We fix a blowup $\bar{S}$ of S with centers supported in Σ such that the preimage of the intersection $D_1 \cap D_2$ in the neighborhood of every point in Σ is a normal crossing divisor. Of course, $\bar{S}$ has a map to $\tilde{S}_j$ and we denote by $\bar{Y}_j$ the fibered product $\bar{S} \times_{\tilde{S}_j} A_j$. The surface $\bar{Y}_j$ is a double cover of $\bar{S}$ and it has at most $A1$-singularities (double points), since D_j^0 is smooth in S^0. Consider the fibered product $\bar{X} = \bar{Y}_1 \times_{\bar{S}} \bar{Y}_2$. All singular points of $\bar{X}$ which are not of type A_1 lie over the intersection points of D_1^0 and D_2^0 in S^0. The surface $\bar{X}$ has a natural action of $\mathbb{Z}/2 + \mathbb{Z}/2$. It admits equivariant surjective maps $\bar{\delta}_j : \bar{X} \to \bar{Y}_j$, where $\bar{Y}_j$ is the quotient of $\bar{X}$ by the involution $\bar{\tau}_j$. We denote by $\bar{Y}_{12}$ the quotient of $\bar{X}$ by $\bar{\tau}_{12} = \bar{\tau}_1 \bar{\tau}_2$, it is still singular.

Let $\hat{S}$ be the minimal blowup of $\bar{S}$ with support in the points $p_1, ..., p_l \in S^0 \subset \bar{S}$ such that proper transforms of the irreducible components D_1^0 and D_2^0 are disjoint in the preimage of S^0 in $\hat{S}$. Now we define Y_j as the induced (from the open part) double covers of $\hat{S}$. Their ramification is contained in the full transforms of the divisors D_j. Define X as the fibered product $Y_1 \times_{\hat{S}} Y_2$. We have the induced involutions (again denoted by τ_j, τ_{12}) on X and we define Y_{12} as the quotient of X under τ_{12} (it admits a map onto $\hat{S}$). By construction, the surfaces X, Y_j and Y_{12} all have at most $A1$-singularities. One has surjective regular maps $X \to \tilde{X}, Y_j \to \tilde{Y}_j$ etc.



LEMMA 4.2. — *For all* $k = 1, ..., l$ *every irreducible component of the preimage of* p_k *in* Y_{12} *is a rational curve.*

Proof. Notice that D_j^0 are smooth in S^0 and that their components don't intersect in S^0. Every point p_k is a point of intersection of an irreducible component of D_1^0 with an irreducible component of D_2^0. In the neighborhood of p_k these two divisors have a canonical form: D_1 is given by $x = 0$ and D_2 is given by $y = x^n$, where n is the order of tangency. There is a standard chain of blowups separating the proper transforms of D_1 and D_2 over p_k (which intersect one of the ends of the chain in two distinct points). The induced double cover on every component of the preimage of p_k is ramified in at most three points (hence in fact, two) and is therefore a rational curve. □

COROLLARY 4.3. — *The map* alb : $Y_{12} \to \mathrm{Alb}(Y_{12})$ *factors through* $\bar{Y}_{12}$.

Proof. Indeed, the natural map $Y_{12} \to \bar{Y}_{12}$ contracts connected graphs of rational curves to distinct points in $\bar{Y}_{12}$. Since these connected graphs of rational curves map into points in $\mathrm{Alb}(Y_{12})$ our claim follows. □

4.4. Elliptic fibrations. — Let $\mathcal{E} \to \mathbb{P}^1$ be a Jacobian elliptic fibration and M_1, M_2 two irreducible horizontal divisors on $\mathcal{E}$. Let $M_{12} \subset \mathcal{E}$ be the divisor of pairwise differences: $M_{12} \cap \mathcal{E}_b$ is the set of all points of the form $p_1 - p_2$, where $p_j \in M_j \cap \mathcal{E}_b$. The divisor M_{12} may have several irreducible components. We shall say that the divisor M_{12} is torsion if every irreducible component of $M_{12} \subset \mathcal{E}$ consists of torsion points.

LEMMA 4.5. — *Consider the restriction map to the generic fiber*

$$\eta^* : \mathrm{Pic}(\mathcal{E}) \to \mathrm{Pic}(\mathcal{E}_\eta).$$

If the divisor M_{12} is torsion then there exists a positive integer N such that the class $\eta^*([M_{12}]) \in \mathrm{Pic}^{(0)}(\mathcal{E}_\eta)$ is annihilated by N.

Proof. For every irreducible component of M_{12} there exists a positive integer N' such that all points p in this component are annihilated by N'. Since the divisor M_{12} has only a finite number of irreducible components we can find an N annihilating all points in M_{12}. Consider the class $N[M_{12}]$. The corresponding divisor is trivial upon restriction to the generic fiber $\mathcal{E}_\eta$. □

COROLLARY 4.6. — *The kernel of the map $\eta^* : \mathrm{Pic}(\mathcal{E}) \to \mathrm{Pic}(\mathcal{E}_\eta)$ is isomorphic to the subgroup of* $\mathrm{Pic}(\mathcal{E})$ *generated by the components of the singular fibers.*

LEMMA 4.7. — *Let $\mathcal{E} \to \mathbb{P}^1$ be a Jacobian elliptic fibration with singular fibers of simple multiplicative type (irreducible nodal curves). Let $M_1, M_2, M_3 \subset S$ be three different irreducible horizontal divisors which are linearly independent in* $\mathrm{Pic}(\mathcal{E})$. *Then at most one of the divisors M_{ij}, $(i, j = 1, 2, 3,$ and $i \neq j)$ is torsion.*

Proof. Indeed if two of the above divisors, for example M_{12} and M_{13}, are torsion then there is a positive integer N which annihilates both $\eta^*([M_{12}])$ and $\eta^*([M_{13}])$. Hence the kernel of η^* has rank ≥ 2. By assumption, the singular fibers of $\mathcal{E}$ generate a subgroup of rank 1. Contradiction. □

Let $\mathcal{E} \to \mathbb{P}^1$ be a nonisotrivial elliptic fibration and r the order of $\mathcal{E}$ in the Tate-Shafarevich group of the corresponding Jacobian elliptic fibration $\mathcal{J}(\mathcal{E})$. Recall that for each integer r' we have a principal homogeneous fibration $\mathcal{J}^{(r')}(\mathcal{E})$ of (relative) zero cycles of degree r', together with natural maps

$$\mathcal{J}^{(r')}(\mathcal{E}) \times \mathcal{J}^{(r'')}(\mathcal{E}) \to \mathcal{J}^{(r'+r'')}(\mathcal{E})$$

and an identification of $\mathcal{J}^{(0)}(\mathcal{E})$ and $\mathcal{J}^{(r)}(\mathcal{E})$, depending on the choice of a global section of the Jacobian fibration $\mathcal{J}(\mathcal{E}) = \mathcal{J}^{(0)}(\mathcal{E})$. After fixing the identification, we get for every integer $m \equiv 1 \mod r$ a rational map

$$\phi_m : \mathcal{E} \to \mathcal{E}$$

of degree m^2, well defined up to the action of $H^0(\mathbb{P}^1, \mathcal{E})$. This map is regular étale on the open (grouplike) part $\mathcal{E}^0$ (complement to the singular points of the singular fibers), but highly nonregular on $\mathcal{E}$.

Lemma 4.8. — *Let $\mathcal{E} \to \mathbb{P}^1$ be a nonisotrivial elliptic fibration and M a horizontal irreducible divisor. Then for all but finitely many primes (congruent to 1 mod r) the preimage $\phi_p^{-1}(M)$ is also irreducible.*

Proof. After base change, we can assume that M is a nonzero section. The global monodromy group of $\mathcal{E}_M$ over M is a subgroup of a finite index in $\mathrm{SL}(2, \mathbb{Z})$. Let $\mathbb{F}$ be the fundamental group of the complement $M \setminus \mathrm{Sing}$ where Sing is a subset of M corresponding to singular fibers of $\mathcal{E} \to M$. Every section $M' \in \mathcal{E}_M$ defines a cocycle $s_{M'} \in H^1(\mathbb{F}, (\mathbb{Q}/\mathbb{Z})^2)$ (corresponding to the principal $(\mathbb{Q}/\mathbb{Z})^2$-fibration, whose fiber over $b \in M \setminus \mathrm{Sing}$ is the set of points differing from $M' \cap \mathcal{E}_b$ by torsion). This cocycle gives an affine action of $\mathbb{F}$ on $(\mathbb{Q}/\mathbb{Z})^2$ and the orbits of the action of $\mathbb{F}$ correspond to the irreducible components of the preimages of M' under the maps $\phi_{m,M} : \mathcal{E}_M \to \mathcal{E}_M$ for all m.

If the section M is not divisible in the group of sections $H^0(M, \mathcal{E}_M)$ by a prime p and the monodromy map $\mathbb{F} \to \mathrm{SL}(2, \mathbb{Z}/p)$ is surjective then the preimage of M under the map $\phi_{p,M} : \mathcal{E}_M \to \mathcal{E}_M$ is irreducible. Indeed in this case either the action is affine and its orbit is $(\mathbb{Z}/p)^2$ or the action is linear. In the first case the preimage of M is irreducible. In the second case there is a section M' with $pM' = M$ and hence M is divisible as a section (see also [2]). □

Lemma 4.9. — *Let $\mathcal{E} \to C$ be a nonisotrivial Jacobian elliptic fibration over an affine connected and smooth curve C. Let M_1, M_2 be two irreducible multisections on $\mathcal{E}$ with the property that there exists a smooth fiber $\mathcal{E}_b$ and a pair of points $p_j \in \mathcal{E}_b \cap M_j$ such that $p_1 - p_2$ is nontorsion in $\mathcal{E}_b$. Then there exists a positive integer N (depending on $\mathcal{E}, M_1, M_2$) such that for all positive integers n_1, n_2 with $n_1 + n_2 > N$ the preimage $\phi_{n_1}^{-1}(M_1)$ intersects $\phi_{n_2}^{-1}(M_2)$.*

Proof. The proof runs in the analytic category. It uses the following universal construction.

Construction. Consider the universal Jacobian elliptic curve $u : \mathcal{E}_{\mathcal{U}} \to \mathcal{H}$ which is obtained as a quotient $\mathcal{H} \times \mathbb{C}/(1, \lambda)$, where λ is a coordinate function in the upper halfplane $\mathcal{H}$. This fibration is topologically trivial and has a natural trivialization map

$$\kappa : \mathcal{H} \times \mathbb{C}/(1, \lambda) \to \mathrm{T} = \mathbb{C}/(1, i),$$

(where $i = \sqrt{-1}$). The map κ is not complex analytic; however, the preimages of points $t \in \mathrm{T}$ are analytic sections of the fibration $u : \mathcal{E}_{\mathcal{U}} \to \mathcal{H}$ (this gives a nonanalytic family of analytic sections). Indeed, the preimage of the point $t = a + bi$ is a section of u which is given in a parametric form $(\lambda, b\lambda + a) \in \mathcal{H} \times \mathbb{C}$ (which descends to $\mathcal{E}_{\mathcal{U}}$). The torsion sections of u map into points in T (since κ is a continuous homomorphism of algebraic groups). The construction is equivariant with respect to $\mathrm{SL}(2, \mathbb{Z})$. Indeed, the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathcal{E}_{\mathcal{U}}$ transforms torsion sections into torsion sections, (which are dense in the family $\kappa^{-1}(\mathrm{T})$). For any subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ of finite index which doesn't contain the center $\mathbb{Z}/2$ we have the induced map

$$u_\Gamma : \mathcal{E}_{\mathcal{U}}/\Gamma \to \mathcal{H}/\Gamma$$

and a factorization map π_Γ. We have the diagram:

$$\begin{array}{ccccc}
\mathcal{E}_{\mathcal{U}}/\Gamma & \xleftarrow{\pi_\Gamma} & \mathcal{E}_{\mathcal{U}} & \xrightarrow{\kappa} & \mathrm{T} \\
\downarrow{\scriptstyle u_\Gamma} & & \downarrow{\scriptstyle u} & & \\
\mathcal{H}/\Gamma & \xleftarrow{\pi_\Gamma} & \mathcal{H} & &
\end{array}$$

For every $t \in \mathrm{T}$, which is not a torsion point in T the orbit $\cup_{\gamma \in \Gamma} \gamma(t)$ is dense (in the usual topology) in T. Thus if t is not torsion in T the intersection of the set $\pi_\Gamma(\kappa^{-1}(\cup_{\gamma \in \Gamma} \gamma(t)))$ with every fiber $\mathcal{E}_b \in \mathcal{E}_{\mathcal{U}}/\Gamma$ is dense.

Now we return to the proof of Lemma 4.9. Consider the nonisotrivial Jacobian elliptic fibration $\mathcal{E} \to C$. We have a diagram

$$\begin{array}{ccccccccc}
\mathcal{E} & \longleftarrow & \mathcal{E}' & \longleftarrow & \mathcal{E}'_U & \longrightarrow & \mathcal{E}_{\mathcal{U}} & \longrightarrow & \mathrm{T} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C & \longleftarrow & C' & \longleftarrow & C' \times_{\mathcal{H}/\Gamma} \mathcal{H} & \longrightarrow & \mathcal{H} & &
\end{array}$$

Changing the base, we reduce to the case when M_1, M_2 are sections of a (topologically trivial) elliptic fibration $\mathcal{E}' \to C'$ over some analytic curve C'. Moreover, we can assume that the fibration $\mathcal{E}' \to C'$ is induced from $\mathcal{E}_{\mathcal{U}}/\Gamma \to \mathcal{H}/\Gamma$ under a dominant map $C' \to \mathcal{H}/\Gamma$. We can identify M_1' (the pullback of M_1) with the zero section of $\mathcal{E}'$ (changing the zero section amounts to changing the argument of κ by a fiberwise translation). Then the image $M_{2,U}$ of M_2 in $\mathcal{E}_{\mathcal{U}}/\Gamma$ is algebraic, and therefore not an orbit of Γ on (the preimage of) some nontorsion point $t \in \mathrm{T}$. This means that $\kappa(\pi_\Gamma^{-1}(M_{2,U}))$ covers some open (in the usual topology) subset V of T. If $n_1 + n_2$ is sufficiently big, then the translations of the set V by n_2 torsion points in T contain n_1 torsion points. □

4.10. Kummer surfaces. — Let A be an abelian surface, τ the standard involution, A^* the blowup of A in the 16 fixed points of τ, S the associated Kummer K3 surface and Δ the union of the 16 exceptional (-2)-curves on S. We will say that A is *generic* if $\mathrm{Pic}(A)/\mathrm{Pic}^0(A) = \mathbb{Z}$ and if the endomorphism ring $\mathrm{End}(A) = \mathbb{Z}$. In particular, A is not isogenous to a product of elliptic curves.

Lemma 4.11. — *We can choose A such that*

1. *A is generic;*
2. *the elliptic fibration $S \to \mathbb{P}^1$ is Jacobian;*
3. *all singular fibers of the elliptic fibration $S \to \mathbb{P}^1$ are irreducible (consequently, the exceptional divisor Δ is horizontal in $S \to \mathbb{P}^1$).*

Proof. The Picard lattice of a polarized Kummer surface S is given by a sublattice (of rank at least 17) in $3H \oplus (-E_8) \oplus (-E_8)$, where H is the standard hyperbolic lattice. Let h_A be the generator (polarization) of the Neron-Severi group of our generic abelian surface A. This class is invariant under the involution. It descends to the singular surface $\tilde{S}$ and lifts to a class $h_S \in \mathrm{Pic}(S)$. For any even positive integer $2k$ there exists a generic A such that $h_S^2 = 2k$. For generic A the Picard group of S is a direct sum $\mathbb{Z}h_S \oplus \Pi$, where Π sits in the exact sequence

$$0 \to (-2)\mathrm{Id}_{16} \to \Pi \to \Lambda^2((\mathbb{Z}/2)^4) \to 0.$$

The last projection extends naturally to a projection $\mathrm{Pic}(S) \to \Lambda^2((\mathbb{Z}/2)^4)$ (h_S is mapped to zero). Given a lattice vector of square zero we can find a translate of this vector (under involutions of $\mathrm{Pic}(S)$ with respect to (-2)-classes) representing an elliptic fibration (see [6]).

Remark that though the lattice Π is not unimodular, for any primitive element $e \in \mathrm{Pic}(S)$ with a nontrivial projection to $\Lambda^2((\mathbb{Z}/2)^4)$ we can find an element $x \in \Pi$ such that $(e, x) = 1$. Let $e' \in \mathrm{Pic}(S)$ be a class obtained from e by reflexion with respect to a (-2)-class. Then there still exists a class $x' \in \mathrm{Pic}(S)$ with $(e', x') = 1$.

Choose a class e of square zero giving an elliptic fibration $f_e : S \to \mathbb{P}^1$, such that the projection of e to $\Lambda^2((\mathbb{Z}/2)^4) = (\mathbb{Z}/2)^6$ is nonzero. The lattice $N_e := \{n \in \Pi \,|\, (n, e) = 0\}$ is negative semi-definite. We will choose N_e such that it has no elements of square -2. This implies that the singular fibers of f_e are irreducible. Simultaneously, we will choose the (affine) lattice $L_e := \{l \in \mathrm{Pic}(S) \,|\, (l, e) = 1\}$ such that L_e has (-2)-vectors. Since N_e has no (-2)-classes every class (modulo translations by e) of square (-2) in L_e corresponds to a section of $f_e : S \to \mathbb{P}^1$.

The lattice Π contains a finite number of (-2)-vectors. Thus a generic vector in this lattice is not orthogonal to any (-2)-vector. Take such a primitive vector x and choose a polarization h_S such that $h_S^2 = -x^2$. Now we can choose a (generic) abelian surface A (with endomorphisms $\mathbb{Z}$) such that the square of the generator h_A of $\mathrm{NS}(A)$ equals $2h_S^2$. On the corresponding Kummer K3 surface S we have $(h_S - x)^2 = 0$. It follows that S has an elliptic fibration $f : S \to \mathbb{P}^1$ and that the lattice N_f is isomorphic to $N_x \oplus \mathbb{Z}(h_S - x)$, and therefore has no classes of square (-2) (for any x as above). Let z be a vector Π such that $(x, z) = 1$ (this is possible since x is primitive). Every vector in L_x is equal to $z+n+c(h_S-x)$ where $n \in N_x$ and $c \in \mathbb{Z}$. Its square is $z^2+n^2+2(z,n)-2c$. Since the lattice Π is even and c an arbitrary integer it follows that L_x always contains classes of square -2. $\square$

Remark 4.12. — The same proof shows that one can construct Jacobian elliptic Kummer surfaces with all singular fibers of simple multiplicative type without requiring that the associated abelian surface is generic.

We return to the general setup of Section 3 and Section 4.1. Let $\mathcal{E} \to \mathbb{P}^1$ be a Jacobian elliptic fibration, M a horizontal divisor and $M_1, ..., M_k$ its irreducible components. Let $\Gamma(M)$ be the "torsion" graph of M - each M_l defines a vertex and two vertices M_l and $M'_{l'}$ are connected by an edge if the difference divisor $M_{ll'}$ is not torsion.

Lemma 4.13. — *Let S be a Kummer surface as in the statement of Lemma 4.11. Then the torsion graph $\Gamma(\Delta)$ is connected.*

Proof. The divisor Δ consists of 16 irreducible components, their classes are linearly independent in $\mathrm{Pic}(S)$. The rank of the subgroup of $\mathrm{Pic}(S)$ generated by the components of singular fibers is 1. Applying Lemma 4.7 we see that every component of Δ is connected to at least 14 other components of Δ. In particular, every two vertices in $\Gamma(\Delta)$ are connected by a path of length at most 2. □

Lemma 4.14. — *Consider $D_1^0, D_2^0 \subset S^0$, obtained as preimages of Δ under the maps ϕ_{q_1}, ϕ_{q_2}. For almost all pairs of distinct prime numbers q_1, q_2 the intersection graph of the irreducible components of $D_1^0 \cup D_2^0$ (in S^0) is connected.*

Proof. By Lemma 4.8, the preimages of the irredicible components of Δ under ϕ_{q_j} remain irreducible for almost all pairs of primes q_1, q_2. Moreover, by Lemma 4.9, the *intersection* graphs of the (irreducible) divisors $\phi_{q_j}^{-1}(\Delta_k)$ is connected for q_j big enough. Now take an irreducible component $\Delta_1 \subset \Delta$ and a component Δ_k, which is connected to Δ_1 in the *torsion* graph $\Gamma(\Delta)$. Applying Lemma 4.9 with $n_1 = q_1$ and $n_2 = q_2$ (and $q_1 + q_2$ big enough) we see that $\phi_{q_1}^{-1}(\Delta_1)$ intersects $\phi_{q_2}^{-1}(\Delta_k)$ (in S^0). □

Remark 4.15. — The difficulty was to show that $D_1^0 \cup D_2^0$ is a connected divisor in the *open* surface S^0. The corresponding fact in the closed surface S is trivial.

4.16. Surfaces in abelian fourfolds. — We use the existence of nontrivial maps between elliptic Kummer surfaces (see Sections 4.4 and 4.10) to construct interesting examples of surfaces in abelian fourfolds.

Fix a generic abelian surface A such that the associated Kummer surface S admits a Jacobian elliptic fibration (this is possible by Lemma 4.11). Choose a pair (q_1, q_2) of positive integers and consider the map

$$\tilde{\phi}_{q_1,q_2} = (\tilde{\phi}_{q_1}, \tilde{\phi}_{q_2}) \,:\, S \to \tilde{S} \times \tilde{S}.$$

Put $\tilde{g}_1 = \tilde{\phi}_{q_1}, \tilde{g}_2 = \tilde{\phi}_{q_2}$ and denote by $\tilde{X} = \tilde{X}_{q_1,q_2}$ and by Y_1, Y_2, Y_{12} the surfaces obtained through the construction in Sections 3 and 4.1. The surface $\tilde{X}$ is lagrangian with respect to some nondegenerate 2-form on $A \times A$. We will show that for appropriate choices of integers q_1, q_2 the surface X (and consequently $\tilde{X}$) is not fibered.

4.17. Uniqueness. —

Proposition 4.18. — *Let A_1, A_2 be simple abelian surfaces and $\tilde{X}$ a lagrangian surface in $A_1 \times A_2$. Assume that $\tilde{X}$ is not isomorphic to an abelian surface, that $\tilde{X}$ projects dominantly onto A_1 and A_2, that it is stable under the involutions $\tilde{\tau}_1, \tilde{\tau}_2$ and that it is lagrangian with respect to at least two nonproportional nondegenerate forms, one of which is invariant with respect to both involutions $\tilde{\tau}_j$. Then $\tilde{X}$ is a finite unramified cover of a product of two curves.*

Proof. By assumption, every $(1,0)$-form on $A_1 \times A_2$ restricts nontrivially to $\tilde{X}$. If there are two nonproportional nondegenerate $(2,0)$-forms which restrict trivially to $\tilde{X}$ then there exists a holomorphic 2-form ω of rank 2 on $A_1 \times A_2$ which restricts trivially to X. (Indeed, the $(2,0)$-forms of rank 2 on the abelian variety $A_1 \times A_2$ correspond to points on a quadric in the projective space $\mathbb{P}^5 = \mathrm{Proj}(H^{2,0}(A_1 \times A_2, \mathbb{C}))$. Any line in $\mathbb{P}^5$ intersects this quadric.) Any such ω is equal to $w' \wedge w''$, where w', w'' are nonproportional $(1,0)$-forms on $A_1 \times A_2$. Thus we have a pencil of curves $\mathcal{P}_\omega$ on $\tilde{X}$ such that both w' and w'' are equal to zero on the fibers of this pencil. Therefore, we have a family of abelian surfaces $A_t \subset A_1 \times A_2$ (where t is a point in the surface $B_\omega = (A_1 \times A_2)/A_0$) such that the intersection of $\tilde{X}$ with A_t is either empty or a fiber of the pencil $\mathcal{P}_\omega$.

Hence, the image of $\tilde{X}$ under the projection to the base of $\mathcal{P}_\omega$ coincides with the image of $\tilde{X}$ in the abelian surface B. The image of $\tilde{X}$ in B is a curve C_ω of genus ≥ 2. The form ω is induced from the holomorphic volume form on the abelian surface B. Consider the action of $\tilde{\tau}_1, \tilde{\tau}_2$ on ω. Since this action transforms $(2,0)$-forms of rank 2 on $A_1 \times A_2$ onto themselves, we have the following possibilities: either ω is invariant under $\mathbb{Z}/2 + \mathbb{Z}/2$ (modulo multiplication by a constant) or there is another 2-form $\tilde{\omega}$ of rank 2, which is trivial on $\tilde{X}$. The first case is excluded by the assumptions (that both projections on A_1, A_2 are dominant). In the second case we have another projection of $\tilde{X}$ onto an abelian surface, with fibers transversal to the fibers of the first projection. Thus we have an isogeny

$$A_1 \times A_2 \to B_\omega \times B_{\tilde{\omega}},$$

which exhibits $\tilde{X}$ as a finite abelian covering of a product $C_\omega \times C_{\tilde{\omega}}$ of two curves of genus ≥ 2. □

Corollary 4.19. — *Let $\tilde{X}$ be a surface obtained as a $\mathbb{Z}/2 + \mathbb{Z}/2$-cover of a (singular) K3-surface as in Section 3 (in particular, we do not assume that S is elliptic). Assume that the conditions of Proposition 4.18 hold for $\tilde{X}$. Then $\tilde{X}$ is isomorphic to $C \times C$, where C is a curve of genus 2 and the quotient map*

$$\tilde{X} \to S = (C \times C)/(\mathbb{Z}/2 + \mathbb{Z}/2)$$

is described in Example 3.3.

Proof. Let $\tilde{X}$ be an unramified abelian cover of the product $C_1 \times C_2$ of curves of genus ≥ 2 as in the proof of Proposition 4.18. Our assumption implies that there is a unique $(\mathbb{Z}/2 + \mathbb{Z}/2)$-equivariant $(2,0)$-form on $\tilde{X}$. The involutions τ_1 and τ_2 interchange the projections of X to $C_1 \times C_2$ (since we have two projections of X onto abelian surfaces B_1, B_2 and A_1, A_2 both map surjectively onto B_1, B_2 and the involution on A_1 interchanges the two projections). In particular, $C_1 = C_2$ and $C_1 \times C_1 = C \times C$. The involution $\tau_{12} = \tau_1 \tau_2$ induces an involution on the abelian cover of $C \times C$. Therefore, $\tau_{12} = (\sigma, \sigma)$, where σ is some involution on C. Thus we have a map

$$S = \tilde{X}/(\mathbb{Z}/2 + \mathbb{Z}/2) \to \mathrm{Sym}^2(C)/\sigma.$$

The condition $h^{1,0}(S) = 0$ implies that σ is a hyperelliptic involution and the condition $h^{2,0}(S) = 1$ implies that $g(C) = 2$. Moreover, the map $\tilde{X} \to C \times C$ is in fact an isomorphism (unramified covers increase the Euler characteristic and the number of invariant 2-forms has to increase as well). □

Corollary 4.20. — *We keep the notations of Section 3. Consider*

$$\tilde{\varphi} = (\tilde{g}_1, \tilde{g}_2) : S \to \tilde{S}_1 \times \tilde{S}_2$$

(and the associated maps $g_j : S \to S_j$). Assume that at least one of the maps g_1, g_2 is not an isomorphism or (if it is) the map $g_1 g_2^{-1}$ doesn't lift to an automorphism of $C \times C$. Then X is lagrangian with respect to exactly one (up to multiplication by constants) nondegenerate form on $A_1 \times A_2$.

Proof. Indeed, the proof of Corollary 4.19 shows that in the opposite case, X is isomorphic to $C \times C$, with $g(C) = 2$ and the action of $\mathbb{Z}/2 + \mathbb{Z}/2$ is the product of the hyperelliptic involution and interchanging of coordinates, modulo automorphisms of $C \times C$. □

4.21. 1-forms. —

Lemma 4.22. — *Let $\hat{S}$ be birational to a K3 surface and $Y \to \hat{S}$ its double cover. Assume that Y has at most $A1$-singularities (double points). Then either $h^0(Y, \Omega^1) \le 2$ or $\mathrm{alb}(Y) \subset \mathrm{Alb}(Y)$ is a hyperelliptic curve C and the (covering) involution on Y transforms into a hyperelliptic involution on C.*

Proof. Indeed, since there are no 1-forms on S, the involution acts by multiplication with -1 on $H^0(Y, \Omega^1)$. Thus the product of any two forms is invariant under the involution. Therefore, the map

$$\alpha : \wedge^2 H^0(Y, \Omega^1) \to H^0(Y, \Omega^2)$$

has image of dimension 1 or 0. If it is 0, then $\mathrm{alb}(Y)$ has dimension 1. (Indeed, it means that any $(2,0)$-form on $\mathrm{Alb}(Y)$ pulls back to a 0-form on Y and hence $\mathrm{alb}(Y)$ has dimension 1.) In this case the involution acts on $C = \mathrm{alb}(Y) \subset \mathrm{Alb}(Y)$ as well as on $H^0(C, \Omega^1)$ as (-1). It follows that

C is hyperelliptic and that the map $Y \to C$ transforms the involution on Y into a hyperelliptic involution on C.

If the dimension of the image of α is 1, then there is a pair of nonproportional forms ω, ω' such their product is nonzero. Assume that there is another form ω'', linearly independent of ω, ω'. Then $\omega' \wedge \omega'' = f\omega \wedge \omega'$, where f is a nonconstant function, contradiction. □

Corollary 4.23. — *Let Y_j be the double covers of $\hat{S}$ as above. Then $h^0(Y_j, \Omega^1) = 2$ (for $j = 1, 2$). Moreover, Y_j admits no dominant rational maps onto curves of genus > 0.*

Proof. Indeed, both surfaces Y_j admit a dominant map onto an abelian surface A, which has two linearly independent 1-forms. Now we apply Lemma 4.22. This proves the first statement. By assumption, the abelian surface A contains no elliptic curves. By the previous lemma, $\mathrm{Alb}(Y_j)$ is isogenous to A. The second statement follows. □.

A priori, we don't know that Y_{12} does not admit dominant maps onto curves of genus > 0. We have to consider the following (mutually exclusive) possibilities:

$h^0(Y_{12}, \Omega^1) = 0$;

$h^0(Y_{12}, \Omega^1) = 2$ and the wedge product of the two forms is nontrivial;

There is a projection $Y_{12} \to C$, where C is a curve of genus ≥ 1.

We will show that the surfaces X_{q_1,q_2} constructed in Section 4.16 are not of the last two types for almost all pairs of primes q_1, q_2. The surfaces of the first type are nonfibered; and there are simple examples such surfaces:

Lemma 4.24. — *Assume that the map $g_1 : S \to S_1$ is an isomorphism. Then $h^0(Y_{12}, \Omega^1) = 0$.*

Proof. The covering Y_1 is (birational to) the abelian surface A_1. The variety X is (birational to) a fiber product of A_1 and some surface Y_2 over $\hat{S}$. Consider the product

$$H^0(Y_2, \Omega^1) \times H^0(Y_{12}, \Omega^1) \to H^0(X, \Omega^2).$$

The image is at most 1-dimensional, since it consists of τ_1-invariant forms and since there is exactly one such form which is induced from $Y_1 = A_1$.

The space $H^0(Y_2, \Omega^1)$ has dimension 2 and hence for any nontrivial form ω_{12} in $H^0(Y_{12}, \Omega^1)$ its product with some form $\omega_2 \in H^0(Y_2, \Omega^1)$ is zero. This means that ω_2 is trivial on the fibers of some pencil of curves on X. This pencil covers (birationally) A_2. The image of a generic fiber of the pencil must be an elliptic curve in A_2 (being the zero set of a 1-form on A_2). This leads to a contradiction, since A_2 doesn't contain elliptic curves by assumption. Thus there are no holomorphic 1-forms on Y_{12}. □

Corollary 4.25. — *Assume that g_1 is an isomorphism and that either g_2 is not an isomorphism or (if it is) $g_1 g_2^{-1} : S \to S$ is not an involution on S. Then the associated surface X is not fibered and $\tilde{X}$ is lagrangian with respect to exactly one 2-form on $A \times A$.*

Proof. By Corollary 4.19, $\tilde{X}$ is lagrangian with respect to exactly one nondegenerate 2-form, unless $\tilde{X} = C \times C$, where C is a curve of genus 2. Thus it suffices to prove that X is not fibered. Notice that $h^0(X, \Omega^1) = 4$. By Lemma 4.24, the Albanese $\mathrm{Alb}(X)$ is isogenous to $A_1 \times A_2$. The image $\mathrm{alb}(X)$ in the Albanese is also lagrangian with respect to exactly one nondegenerate 2-form. If X were fibered over a curve of genus > 1 there would be another (a degenerate) 2-form on $\mathrm{Alb}(X)$ which would be trivial on $\mathrm{alb}(X)$. This is a contradiction. Notice also that X doesn't admit any dominant maps onto elliptic curves since $\mathrm{Alb}(X)$ doesn't map onto elliptic curves. □

In the following sections we will give further (more complicated) examples of nonfibered lagrangian surfaces in $A_1 \times A_2$.

4.26. Weakly lagrangian structures. — Let $\tilde{X} \subset A_1 \times A_2$ be as in Section 3 and X its (partial) desingularization as in Section 4.1. Consider the set L^X of (possibly degenerate) holomorphic $(2,0)$-forms on $\mathrm{Alb}(X)$ which restrict trivially to $\mathrm{alb}(X)$. It contains the set of weakly lagrangian structures on $\mathrm{alb}(X)$. Since the condition is linear, L^X is a linear subspace of $H^{2,0}(\mathrm{Alb}(X), \mathbb{C})$, The $\mathbb{Z}/2 + \mathbb{Z}/2$-action on X lifts to an action on $\mathrm{Alb}(X)$, leaving the space L^X invariant.

From now on we assume that $\tilde{X}$ is not isomorphic to $C \times C$, where C is a curve of genus 2 (since in this case $\dim L^X = 2$, see Proposition 4.18).

Furthermore, we will assume that $h^0(Y_{12}, \Omega^1) \neq 0$ (if $\tilde{X} \neq C \times C$ and $h^0(Y_{12}, \Omega^1) = 0$ then $\dim L^X = 1$).

We have to consider the following cases:

1. $h^0(Y_{12}, \Omega^1) \neq 0$ and the wedge product is degenerate;
2. $h^0(Y_{12}, \Omega^1) = 2$ and the wedge product
$$\wedge : H^0(Y_{12}, \Omega^1) \times H^0(Y_{12}, \Omega^1) \to H^0(Y_{12}, \Omega^2)$$
is nondegenerate.

Remark 4.27. — Notice however, that examples of surfaces X such that
$$h^0(Y_{12}, \Omega^1) \neq 0$$
are somewhat pathological. In particular, almost all surfaces X_{q_1,q_2} are not of this type.

The space L^X is invariant under $\mathbb{Z}/2 + \mathbb{Z}/2$ and hence decomposes into the direct sum
$$L^X = L^X_0 \oplus L^X_1 \oplus L^X_2 \oplus L^X_{12},$$
where L^X_0 stands for a subspace of $\mathbb{Z}/2 + \mathbb{Z}/2$-invariant forms in L^X, the space L^X_j is the space of τ_j-invariant forms and L^X_{12} is the space of τ_{12} invariant forms.

This decomposition arises from the decomposition of the space $H^{2,0}$ of $(2,0)$-forms on $\mathrm{Alb}(X)$ under the $\mathbb{Z}/2 + \mathbb{Z}/2$-action. We will denote these spaces using the same indices. We know that
$$\begin{array}{rcl} H^{1,0}(\mathrm{Alb}(X))_0 & = & 0, \\ H^{1,0}(\mathrm{Alb}(X))_1 & = & H^{1,0}(A_2), \\ H^{1,0}(\mathrm{Alb}(X))_2 & = & H^{1,0}(A_1), \\ H^{1,0}(\mathrm{Alb}(X))_{12} & = & H^{1,0}(Y_{12}). \end{array}$$
Thus by taking the exterior product we obtain that
$$H^{2,0}_0 = \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_1 + \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_2 + \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_{12}.$$
We also have the decompositions
$$\begin{array}{rcl} H^{2,0}(\mathrm{Alb}(X))_1 & = & H^{1,0}(\mathrm{Alb}(X))_{12} \times H^{1,0}(A_2), \\ H^{2,0}(\mathrm{Alb}(X))_2 & = & H^{1,0}(\mathrm{Alb}(X))_{12} \times H^{1,0}(A_2), \\ H^{2,0}(\mathrm{Alb}(X))_{12} & = & H^{1,0}(A_1) \times H^{1,0}(A_2). \end{array}$$

Proposition 4.28. — *If* $h^0(Y_{12}, \Omega^1) \neq 0$ *and the wedge product is degenerate then*

$$\dim L^X = \dim \Lambda^2 H^0(Y_{12}, \Omega^1) + 1.$$

Proof. We subdivide the proof into a sequence of lemmas.

Lemma 4.29. — *Under the conditions in Proposition 4.28 we have*

$$L_1^X = L_2^X = L_{12}^X = 0.$$

Proof. Since the space L_{12}^X is induced from $A_1 \times A_2$ it has dimension 0 (by Proposition 4.18 there is a unique lagrangian structure on $\tilde{X} \subset A_1 \times A_2$). We have

$$L_1^X \subset H^{2,0}(\mathrm{Alb}(X))_1 = H^{1,0}(\mathrm{Alb}(X))_{12} \times H^{1,0}(A_2).$$

Since $\dim H^{1,0}(A_2) = 2$ for any element $l \in L_1^X$ there are two forms $w_1, w_2 \in H^{1,0}(A_2)$ and two forms $u_1, u_2 \in H^{1,0}(\mathrm{Alb}(X))_{12}$ with the property

$$w_1 \wedge u_1 + w_2 \wedge u_2 = l.$$

We know that the forms u_j are induced from a map $f : X \to C$, where C is some hyperelliptic curve.

If w_1, w_2 are linearly dependent then w_1 is trivial on the images of the fibers of the projection $f : X \to C$. Since X surjects upon A_1 and A_2 this means that w_1 is trivial on a family of curves in A_2. This can happen only if A_2 is isogenous to a product of elliptic curves. This contradicts the assumption that A_2 is simple.

Consider the case when w_1, w_2 are linearly independent. Since the wedge product on $H^{0,1}(\mathrm{Alb}(X))_{12}$ is trivial we can write $u_2 = f u_1$ for some rational function f on X and hence $w_1 + f w_2 \wedge u_1 = 0$. The function f is constant on the fibers of the family $X \to C$ and the fibers map into A_1. Since the generic fiber is not an elliptic curve the forms $w_1, w_2 \in H^{1,0}(A_2)$ are not linearly dependent on the fibers which contradicts the assumption.

Similar argument yields $L_2^X = 0$. □

Corollary 4.30. — *Under the conditions of Proposition 4.28 we have*

$$L^X = L_0^X = \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_{12} + L_A,$$

where $\dim L_A = 1$. *The subspace* $L_A = 1$ *is induced from* $A_1 \times A_2$. *A generic element in* L^X *is nondegenerate if* $\dim \mathrm{Alb}(X)$ *is even and is degenerate of corank* 1 *if* $\dim \mathrm{Alb}(X)$ *is odd.*

Proof. If

$$v = v_1 + v_2,\ v_1 \in L_A,\ v_2 \in \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_{12}$$

then $\operatorname{rk} v = \operatorname{rk} v_1 + \operatorname{rk} v_2$ (since the spaces L_A, L_0^X belong to the second exterior powers of complementary linear subspaces in $H^{1,0}(\mathrm{Alb}(X))$).

The nonzero element of L_A is induced from a nondegenerate form on $A_1 \times A_2$ and has rank 4. We are in the case when Y_{12} (and consequently X) admits a dominant map onto a curve C of genus ≥ 1. If $g(C)$ is even the generic element of L^X is a nondegenerate form on $\mathrm{Alb}(X)$. The dimension of $\mathrm{Alb}(X)$ is even iff $g(C)$ is even. Similarly when $g(C)$ is odd. □

This finishes the proof of Proposition 4.28.

Lemma 4.31. — *Under the conditions of Proposition 4.28 any dominant map* $h : X \to C'$ *of* X *onto a curve with* $g(C') > 0$ *is a composition*

$$h = sf,\ f : X \to C,\ s : C \to C'.$$

Proof. Repeating the previous argument we see that the rank of a nonzero form in L_A is 2 and hence $\operatorname{rk} v \geq 2$ for $v = v_1 + v_2$, $v_1 \in L_A$, $v_2 \in \Lambda^2 H^{1,0}(\mathrm{Alb}(X))_{12}$ with $v_1 \neq 0$. Thus any form of rank 1 in L^X belongs to $\Lambda^2 H^{1,0}(\mathrm{Alb}(X))_{12}$. Since any map $h : X \to C'$, $g(C') \geq 2$ gives a nontrivial form of rank 1 we obtain the result in this case.

If $g(C') = 1$ then there is a map $\mathrm{Alb}(X) \to C'$. The variety $\mathrm{Alb}(X)$ is isogenous to the product $\mathcal{J}(C) \times A_1 \times A_2$ and any map from $A_1 \times A_2$ to a curve of genus 1 is trivial. Hence the map $\mathcal{J}(C) \times A_1 \times A_2 \to C'$ is induced from the projection onto $\mathcal{J}(C)$. It means that the holomorphic $(1,0)$-form lifted from C' on X is induced from $f : X \to C$. This yields the result for $g(C') = 1$. □

Corollary 4.32. — *Under the conditions of Proposition 4.28 every map* $X \to C'$ *factors through the map* $X \to Y_{12} \to C \to C'$.

We consider the case when $h^0(Y_{12}, \Omega^1) = 2$ and the wedge product on 1-forms is nondegenerate on Y_{12}. It follows that the Albanese variety

$\mathrm{Alb}(X)$ is isogenous to a product of three abelian surfaces to $A_1 \times A_2 \times A_{12}$. Indeed, for all 3 quotients Y_1, Y_2, Y_{12} of X by the nontrivial involutions the corresponding Albanese varieties have dimension 2. These are very strong conditions; though we are not sure that such examples exist, we would like to analyze this potential possibility:

In this case $\bar{Y}_{12}$ admits a $\mathbb{Z}/2$-equivariant map $\bar{Y}_{12} \to A_{12}$, which descends to a map $\bar{S} \to \tilde{S}_{12}$ (here $\tilde{S}_{12}$ is the K3 surface obtained as the quotient $A_{12}/\mathbb{Z}/2$). The ramification over S^0 is the union of $D_1^0 \cup D_2^0$. It follows that the preimage D_{12}^0 of Δ_{12} in S^0 (under the map $S^0 \to S_{12}$) is equal to $D_1^0 \cup D_2^0$. It follows that D_{12}^0 is a smooth divisor and hence D_1^0 and D_2^0 have no intersection in S^0.

Corollary 4.33. — *The intersection graph of the irreducible components of D_{12}^0 in S^0 is totally disconnected.*

4.34. Nonfibered lagrangian surfaces. — In this section we show that for infinitely many pairs of integers q_1, q_2 the surface $X = X_{q_1,q_2}$ is not fibered (in particular, the lagrangian surface $\tilde{X}_{q_1,q_2} \subset A_1 \times A_2$ is also not fibered).

Lemma 4.35. — *Assume that Y_{12} admits a dominant $\mathbb{Z}/2$-equivariant (with respect to the covering $Y_{12} \to \hat{S}$) map ρ onto a hyperelliptic curve C of genus > 0. Then ρ descends to a dominant map $\bar{\rho} : \bar{Y}_{12} \to C$ which is also equivariant with respect to the covering involution $\bar{Y}_{12} \to \bar{S}$.*

Proof. Indeed, the map ρ is a composition of a map $Y_{12} \to \mathrm{Alb}(Y_{12})$ and $\mathrm{Alb}(Y_{12}) \to \mathcal{J}(C)$. Now we apply Corollary 4.3. □

After factorization of $\bar{Y}_{12}$ by the involution $\bar{\tau}$ we obtain a map $\bar{\rho}_\sigma : \bar{S} \to \mathbb{P}^1 = C/\sigma$, (where σ is the hyperelliptic involution on C). Notice that the fibers of $\bar{\rho}_\sigma$ are connected (since the same property holds for ρ). Denote by $R = \{p_1, ..., p_{2g+2}\} \subset C$ the ramification divisor of $C \to \mathbb{P}^1$.

$$\begin{array}{ccc} \bar{Y}_{12} & \xrightarrow{\bar{\rho}} & C \\ {\scriptstyle\bar{\tau}}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \bar{S} & \xrightarrow[\bar{\rho}_\sigma]{} & \mathbb{P}^1 \end{array}$$

Lemma 4.36. — *The irreducible components of $D_1^0 \cup D_2^0$ are contained in the fibers of $\bar{\rho}_\sigma$. Moreover, they lie over $R \subset \mathbb{P}^1$.*

Proof. Since the map $\bar{\rho}$ commutes with the covering involution on $\bar{Y}_{12}$ the image of the ramification divisor in $S^0 \subset \bar{S}$ under the map $\bar{\rho}_\sigma$ is contained in R. The divisors D_j^0 are exactly the ramification divisors of the double cover on S^0. □

Corollary 4.37. — *Let $I_1^0 \subset D_1^0$ and $I_2^0 \subset D_2^0$ be two irreducible components of the divisors D_1^0, D_2^0. Assume that the intersection $I_1^0 \cap I_2^0 \cap S^0 \neq \emptyset$. Then I_1^0 and I_2^0 are mapped to the same point in $R \subset \mathbb{P}^1$.*

Now we describe the structure of those fibers of $\bar{\rho}_\sigma$ which lie over a point $p \in R$.

Lemma 4.38. — *All the components of a fiber in S^0 over a point $p \in R$, apart from components in D_j^0 have even multiplicities.*

Proof. Indeed, the double cover $\bar{Y}_{12} \to \bar{S}$ is induced from the double cover $C \to \mathbb{P}^1$. Therefore, the ramification divisor of the former is the preimage of R. Since we know that the ramification divisor in S^0 is $D_1^0 \cup D_2^0$ all the other components of the preimage of R have even multiplicities. □

Lemma 4.39. — *Assume that the graph of components of $D_1^0 \cup D_2^0$ is connected in S^0. Then there are no dominant maps of Y_{12} onto curves of genus > 0. Moreover,* $\mathrm{Alb}(Y_{12}) = 0$.

Proof. Assume first that $\dim \mathrm{Alb}(Y_{12}) = 2$ and that the wedge product of 1-forms on Y_{12} is nondegenerate. Then, by the Corollary 4.33 - this contradicts the assumptions. Thus we have to consider the remaining case when Y_{12} does admit a dominant $\mathbb{Z}/2$-equivariant (with respect to the covering) map ρ onto a hyperelliptic curve C of genus > 0. Then S^0 has a regular map onto $\mathbb{P}^1$ such that components of D_1 and D_2 are mapped onto ramification points $p_1, ..., p_{2g+2} \subset R$. By Corollary 4.37, all connected (in S^0) components of the union of the divisors D_1^0 and D_2^0 are mapped to one point, which we can assume to be p_1. Consider the singular symmetric tensor on C, which is equal to 0 at $\mathbb{P}^1$ and has the singularity dz^{2n}/z^n at $p_2, ..., p_{2g+2}$. Since the preimages of points $p_2, ..., p_{2g+2}$ consist of components of multiplicity two this tensor lifts into a nonsingular symmetric tensor on S^0. Since the complement to S^0 in the K3 surface S consists of finitely many points we can extend this tensor to a holomorphic symmetric tensor on S. However, there are no such tensors on a K3 surface, contradiction. □

4.40. Maps to curves. —

THEOREM 4.41. — *For almost all pairs of distinct prime numbers* (q_1, q_2) *the surface* $X = X_{q_1,q_2}$ *is not fibered.*

Proof. By Lemma 4.14, we can insure (for almost all pairs of distinct primes q_1, q_2) that the intersection graph $D_1^0 \cup D_2^0$ is connected in S^0. By Lemma 4.39, we have $\mathrm{Alb}(Y_{12}) = 0$. This implies that X has a dominant map onto a curve only if X is isomorphic to $C \times C$, where C is a curve of genus 2. Contradiction. □

4.42. Fundamental groups. —

PROPOSITION 4.43. — *Let* $\tilde{X} \subset A_1 \times A_2$ *be a surface constructed in Section 3. Assume that* $\tilde{X}$ *is not isomorphic to* $C \times C$*, where* C *is a curve of genus* ≥ 2*. Then* $\pi_1(\tilde{X})$ *admits a surjective homomorphism onto a group* G*, where* G *is a nontrivial central extension of* $\mathbb{Z}^8$ *by* $\mathbb{Z}^5$*. If* $\mathrm{Alb}(\tilde{X})$ *is isogenous to* $A_1 \times A_2$ *then*

$$G = \pi_1(\tilde{X})/[[\pi_1(\tilde{X}), \pi_1(\tilde{X})]\pi_1(\tilde{X})]$$

(modulo torsion).

Proof. For generic A_j the $\mathbb{Q}$-linear envelope of $\omega_j \in H^{2,0}(A_j)$ has dimension 5. Same holds for $\lambda_2\omega_1 - \lambda_1\omega_2$ on $A_1 \times A_2$ (for $\lambda_j \in \mathbb{Q}$). Thus the kernel is $\mathbb{Z}^5$. The image is $\mathbb{Z}^8 = H^1(\tilde{X})/\mathrm{torsion}$. It remains to recall a general fact from group cohomology: Consider the exact sequences

$$1 \longrightarrow K_1 \longrightarrow H^2(G/[G,G],\mathbb{Z}) \longrightarrow H^2(G,\mathbb{Z})$$

$$1 \longrightarrow K_2 \longrightarrow H^2(G/[G,G],\mathbb{Z}) \longrightarrow H^2(G/[[G,G],G],\mathbb{Z})$$

Then K_1 and K_2 are isomorphic and, by duality, K_2 is isomorphic to $[G,G]/[[G,G],G]$ (tensor $\mathbb{Q}$).

REMARK 4.44. — There are two other potential cases: when Y_{12} admits a surjective map onto a curve C of genus $g(C) = g$ and when $\mathrm{Alb}(\tilde{X})$ is isogenous to a product of 3 abelian surfaces. In the first case we have an exact sequence

$$0 \to \mathbb{Z}^{1+i+g(2g-1)} \to G \to \mathbb{Z}^{8+2g} \to 0.$$

In the second case

$$0 \to \mathbb{Z}^{4+2i} \to G \to \mathbb{Z}^{12} \to 0.$$

Here in both cases $0 \leq i \leq 3$. We believe that these cases do occur if A_1, A_2 have Picard groups of higher rank or admit nontrivial endomorphisms. Then the rational envelope of $(2,0)$-forms has smaller dimension.

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