POINTS OF BOUNDED HEIGHT ON EQUIVARIANT COMPACTIFICATIONS OF VECTOR GROUPS, I

by

Antoine Chambert-Loir & Yuri Tschinkel

Abstract. — We prove asymptotic formulas for the number of rational points of bounded height on certain equivariant compactifications of the affine plane.

Résumé. — Nous établissons un développement asymptotique du nombre de points rationnels de hauteur bornée sur certaines compactifications équivariantes du plan affine.

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Introduction

Let X be a smooth projective algebraic variety defined over a number field F and $X(F)$ the set of rational points of X. Let $\mathscr{L}$ be a metrized ample line bundle and

$$H_{\mathscr{L}} : X(F) \to \mathbf{R}_{>0}$$

the associated exponential height (cf. [**17**], [**18**]). We are interested in the asymptotic behavior of the counting function

$$N(U, \mathscr{L}, B) := \#\{x \in U(F) \mid H_{\mathscr{L}}(x) \leqslant B\}$$

as $B \to \infty$, where $U \subset X$ is some Zariski open subset. There is a conjectural framework describing such asymptotics for varieties with ample (or, more generally, effective) anticanonical class (cf. [**7**], [**1**], [**14**], [**17**], [**2**] and references therein). In particular, it is expected that

$$N(U, \mathcal{K}_X^{-1}, B) = \frac{\Theta(X)}{r!} B(\log B)^r(1 + o(1)),$$

as $B \to \infty$, for some appropriate $U \subset X$. Here $\mathcal{K}_X^{-1}$ is the metrized anticanonical line bundle on X, $r = \operatorname{rk}\operatorname{Pic}(X) - 1$ and $\Theta(X)$ is a product of a Tamagawa type number $\tau(\mathcal{K}_X)$ (which depends on the metrization), a rational number $\alpha(X)$ defined in terms of the cone of effective divisors $\Lambda_{\text{eff}}(X)$ and the order of the non-trivial part of the Brauer group $|\operatorname{Br}(X)/\operatorname{Br}(F)|$. There is a similar description for arbitrary polarizations $\mathscr{L}$ (cf. [**2**]).

These conjectures have been proved flag varieties ([**7**]), toric varieties ([**2**]), and toric bundles induced from torsors ([**21**], [**5**], [**4**]). The proofs use a precise *combinatorial* description of all geometric and arithmetic invariants of the varieties: line bundles, metrizations of the line bundles etc. (for example, in terms of lattices, cones and fans). Such a description is possible because representations of reductive groups are rigid (don't admit deformations). Consequently, the corresponding varieties don't have moduli.

The only other known approach to asymptotics of rational points on algebraic varieties is the classical circle method in analytic number theory. Varieties which can be treated by this method do admit moduli. However, one of the drawbacks is that so far it works only for varieties which are complete intersections of small degree d in projective spaces $\mathbf{P}^n$ of large dimension (very roughly, $n \gg 2^d$) (mild singularities are allowed). In particular, these complete intersections have Picard group $\mathbf{Z}$. There is a promising generalization of the circle method to complete intersections in other varieties (for example, toric varieties) due to E. Peyre (cf. [**15**]), which should provide examples of asymptotics for varieties with moduli and with $\operatorname{Pic}(X)$ of higher ranks, once the necessary estimates are established. As a reference to the circle method let us mention the papers by H. Davenport, R. Heath-Brown, Ch. Hooley on smooth cubic hypersurfaces (cf. [**6**], [**10**], [**11**]), by B. Birch and by W. Schmidt on general complete intersections (cf. [**3**], [**20**]).

In this paper we prove asymptotics of rational points of bounded height on varieties which admit moduli and which at the same time are closely relatated to (non-reductive) linear algebraic groups. More precisely, we consider equivariant compactifications of the additive group $\mathbf{G}_a^n$. For $n = 2$ it can be shown that all such compactifications are obtained as blow-ups of $\mathbf{P}^2$, or Hirzebruch surfaces in points which are fixed under the action of $\mathbf{G}_a^2$. Notice that a variety (even $\mathbf{P}^2$) may admit non-isomorphic structures as an equivariant compactification of $\mathbf{G}_a^n$. A similar "minimal model program" of equivariant compactifications of $\mathbf{G}_a^n$ is a non-trivial problem already for $n = 3$ (work in progress [**9**]).

In this paper we study in detail the example of a blow-up of $\mathbf{P}^2$ in r $\mathbf{Q}$-rational points which are all contained in the line at infinity $\mathbf{P}^1$ (with the equation $x_0 = 0$). The moduli space of such surfaces X is $\mathscr{M}_{0,r}$. It is easy to see that X is a smooth projective equivariant compactification of $\mathbf{G}_a^2$ with $\mathrm{Pic}(X) = \mathbf{Z}^{r+1}$, trivial Brauer group and a simplicial cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$. Denote by $U \simeq \mathbf{G}_a^2 \subset \mathbf{P}^2$ the complement to $x_0 = 0$. Then the series

$$Z(U, K_X^{-1}, s) = \sum_{\mathbf{x} \in U(\mathbf{Q})} H_{K_X^{-1}}(\mathbf{x})^{-s}$$

is absolutely and uniformly convergent to a holomorphic function for $\mathrm{Re}(s) \gg 1$. One of the main results of this paper is the following:

Theorem 1. —

$$Z(U, K_X^{-1}, s) = \frac{h(s)}{(s-1)^{r+1}},$$

where $h(s)$ is holomorphic for $\mathrm{Re}(s) > 1 - \delta$ (for some $\delta > 0$) and

$$h(1) = \alpha(X)\tau(K_X) \neq 0.$$

Corollary 2. — *We have the following asymptotic formula:*

$$N(U, K_X^{-1}, B) = \frac{\alpha(X)\tau(K_X)}{r!} B(\log B)^r(1 + o(1)),$$

as $B \to \infty$.

In fact, we will prove asymptotics for every $\mathscr{L}$ on X such that its class is contained in the interior of $\Lambda_{\mathrm{eff}}(X)$. We will also give estimates for the growth of the function $h(s)$ in vertical strips in the neighborhood of $\mathrm{Re}(s) = 1$. This implies more precise asymptotic expansions for the counting function $N(U, \mathscr{L}, B)$.

We will address the compactifications of $\mathbf{G}_a^n$ (with $n > 2$) in subsequent papers.

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1. Geometry

1.1. Generalities. — Let G be an algebraic group and X a smooth projective variety with an action of G. We denote by $\mathrm{Pic}^G(X)$ the (abelian) group of isomorphy classes of G-linearized line bundles on X (cf. [**13**], Chap. 1, § 3, Def. 1.6). We shall say that the variety X is an equivariant compactification of G if X has an open dense subset U which is equivariantly isomorphic to G. Well known examples are given by toric varieties which are equivariant compactifications of tori (algebraic groups isomorphic to $\mathbf{G}_m^n$ over the algebraic closure of F, where $\mathbf{G}_m = \mathrm{Spec}(F[x, x^{-1}])$ is the multiplicative group scheme).

In this paper, we are interested in equivariant compactifications of $\mathbf{G}_a^n$, where $\mathbf{G}_a = \mathrm{Spec}(F[x])$ is the additive group scheme (we may call them *addic* varieties(*)).

Notice that a variety can be an equivariant compactification of a group G in many non-isomorphic ways, as the following example shows.

Example 1.2. — The projective plane $\mathbf{P}^2$ is an equivariant compactification of $\mathbf{G}_a^2$ in (essentially) two non-isomorphic ways. One of the possible actions is the standard translation action, fixing a line $\mathbf{P}^1$ at infinity. All 1-parameter subgroups are lines. The other action has exactly one fixed point. Generic 1-parameter subgroups are conics (cf. [**9**] for more details, esp. Prop. 3.2).

We quote from [**9**] the following general geometrical facts about equivariant compactifications of additive groups.

Proposition 1.3. — *Let X be a smooth projective equivariant compactification of $\mathbf{G}_a^n$ and $D = X \setminus \mathbf{G}_a^n$ the boundary.*

1. *The boundary D is a pure codimension 1 subvariety.*
2. *The Picard group $\mathrm{Pic}(X)$ is freely generated by the irreducible components $D_0, \dots, D_r$ of D.*
3. *The closed cone of effective divisors $\Lambda_{\mathit{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$ is given by*

$$\Lambda_{\mathit{eff}}(X) = \oplus_{k=0}^{r} \mathbf{R}_+[D_k].$$

(*) Scherzhafter Vorschlag von Yu. I. Manin, Weihnachten 1998

4. *There exist integers* $n_k > 1$ *such that the anticanonical class is given by*

$$-[K_X] = \sum_{k=0}^{r} n_k [D_k].$$

1.4. Blow-ups. — The basic example of an equivariant compactification of $\mathbf{G}_a^n$ is the projective space $\mathbf{P}^n$, with $\mathbf{G}_a^n$ acting on $\mathbf{P}^n = \operatorname{Proj}(F[x_0, \dots, x_n])$ by translation:

$$((t_1, \dots, t_n), (x_0 : \dots : x_n)) \mapsto (x_0 : x_1 + t_1 x_0 : \dots : x_n + t_n x_0)$$

which stabilizes the "hyperplane at infinity" given by the equation $x_0 = 0$. In this paper we consider blow-ups of the projective space $\pi \colon X \to \mathbf{P}^n$ in a subscheme $Z \subset \mathbf{P}^n$ of pure codimension $\geqslant 2$, which is contained in this hyperplane. We denote by $\mathscr{I}_Z$ the ideal sheaf of Z in $\mathbf{P}^n$ so that the blow-up is defined by the formula $X = \mathbf{Proj}(\bigoplus_j \mathscr{I}_Z^j)$. As $Z \subset Z_0$ is fixed by $\mathbf{G}_a^n$, the universal property of the blow-up implies that the action of $\mathbf{G}_a^n$ on $\mathbf{P}^n$ lifts uniquely to an action on X.

The geometry of blow-ups of arbitrary subschemes can be very complicated. We shall assume that $\mathscr{I}_Z$ is the product of a finite number of ideals $\mathscr{I}_{Z_k}$, where the Z_k are integral subschemes of the hyperplane at infinity in $\mathbf{P}^n$. The universal property of blow-ups says that X is the "minimal" scheme mapping to $\mathbf{P}^n$ on which $\mathscr{I}_Z$ becomes invertible. An easy lemma in commutative algebra implies that on X, the $\mathscr{I}_{Z_k}$ are themselves invertible. (The blow-up of a product of ideals is the universal way to make these ideals invertible; it is the same as blowing up successively Z_1, then the strict transform of Z_2, etc.) In particular, π factors as $X \to X_k \to \mathbf{P}^n$, X_k being the blow-up of Z_k in $\mathbf{P}^n$.

On X, we now have Cartier divisors D_k (which are the inverse images of the Z_k) and line bundles $\mathscr{O}_X(D_k)$ equipped with a canonical section $\mathsf{s}_{D_k} \in \Gamma(X, \mathscr{O}_X(D_k))$. Moreover, s_{D_k} and $\mathscr{O}_X(D_k)$ are pullbacks of similar objects on X_k (which we will denote by the same letters). Note also that $\mathscr{O}_X(-D_k) = \mathscr{I}_{Z_k} \cdot \mathscr{O}_X \subset \mathscr{O}_X$ and that by definition, s_{D_k} is obtained by dualizing the pull-back of the canonical inclusion $\mathscr{I}_{Z_k} \to \mathscr{O}_{\mathbf{P}^n}$.

Let D_0 be the strict transform of Z_0 in X. We have a canonical isomorphism:

$$\pi^* \mathscr{O}_X(Z_0) \simeq \mathscr{O}_X(D_0 + \sum_{k=1}^{r} D_k). \tag{1.5}$$

Denoting by s_{D_0} the canonical section of $\mathscr{O}_X(D_0)$, the tensor product $\bigotimes_{k=0}^r \mathsf{s}_{D_k}$ equals the pull back on X of the canonical section of $\mathscr{O}(Z_0)$.

The D_k's ($0 \leqslant k \leqslant r$) form a basis of the Picard group of X. We identify the anticanonical sheaf in these coordinates:

Proposition 1.6. — *If Z is smooth, then X is a smooth variety and its anticanonical sheaf is given by*

$$\begin{aligned} K_X^{-1} &= \pi^*((n+1)Z_0) \otimes \bigotimes_{k=1}^{r-1} \mathscr{O}(-(\operatorname{codim}(Z_k)-1)D_k) \\ &= \bigotimes_{k=0}^{r-1} \mathscr{O}((2+\dim(Z_k))D_k). \end{aligned}$$

Proof. — See Hartshorne [**8**], Ex. 8.5, p. 188. □

1.7. Metrizations on blow-ups. — Let S be the spectrum of a Dedekind ring (which will be the ring of integers in F, or a localization of it, or a completion) or the spectrum of a field which is equipped with a valuation. Let X be a projective scheme over S. For a locally free sheaf $\mathscr{E}$ of finite rank on X, there are several notions of metrizations corresponding to these various cases. We recall briefly the definitions.

- If $S = \operatorname{Spec}(F)$, the spectrum of a field endowed with a valuation, a metric on a $\mathscr{E}$ is a family of norms on the fibres $\mathscr{E}_x$ for $x \in X(\bar{F})$, which vary continuously with x.
- If $\bar{F} = \mathbf{C}$, one may ask that the dependence is $\mathscr{C}^\infty$, and—independently—that the metrics are hermitian in the fibers.
- If $S = \operatorname{Spec}(F)$, where F is the fraction field of a discrete valuation ring R, any flat and projective model $(\mathscr{X}, \mathscr{E})$ over $\operatorname{Spec}(R)$ determines a metric according to which a section is of norm $\leqslant 1$ at a point iff it is integral.
- If $S = \operatorname{Spec}(F)$, F being a number field, an adelic metric on $\mathscr{E}$ is a collection of metrics for all X_v/F_v, v being the different places of F. Moreover, one assumes that there exists a model over $\operatorname{Spec}(\mathfrak{o}_F)$ which gives the same metrics except at a finite number of places. At these exceptional places the ratios of the two metrics are assumed to be bounded functions on X.

The usual definitions of metrics on subsheafs, quotients, hom's, etc. are compatible with these notions.

Let X be a quasi-projective flat scheme over S, $\mathscr{I}$ a sheaf of ideals on X and $Z = V(\mathscr{I})$. Let $\pi : Y \to X$ be the blow-up of $V(\mathscr{I})$, $Y = \mathbf{Proj}(\bigoplus_n \mathscr{I}^n)$. On Y, the inverse image of Z becomes a Cartier divisor D and the line bundle

$\mathscr{O}(D)$ is equipped with a canonical section s_D. We want to endow $\mathscr{O}_Y(D)$ with a metric and to give a formula for the norm of s_D at any point of $Y\backslash|D|$. Note that $\mathscr{O}(-D) = \mathscr{I} \cdot \mathscr{O}_Y \subset \mathscr{O}_Y$ and that s_D is the pull-back of the canonical inclusion $\mathscr{I} \to \mathscr{O}_X$.

Choose a locally free sheaf $\mathscr{E}$ of finite rank on X with a section $\sigma_Z \in \Gamma(X, \mathscr{E})$ whose scheme of zeroes is Z (existence follows from the quasi-projectivity of X). This induces a surjective homomorphism $\varphi : \mathscr{E}^\vee \to \mathscr{I}$ and a closed immersion $Y \hookrightarrow \mathbf{P}(\mathscr{E}^\vee)$ such that $\mathscr{O}(-D) = \mathscr{I} \cdot \mathscr{O}_Y = \mathscr{O}_{\mathbf{P}}(1)$ and the universal quotient map $\pi^*\mathscr{E}^\vee \to \mathscr{O}_{\mathbf{P}}(1)$ on Y is the pullback of φ. Hence, to metrize $\mathscr{O}_Y(D)$ it is sufficient to endow $\mathscr{E}^\vee$ with a metric. The quotient metric on $\mathscr{O}_{\mathbf{P}}(1)$ is defined as follows: for any local section s of $\mathscr{O}_{\mathbf{P}}(1)$, we pose

$$\|s\|(y) = \inf_t \|t\|$$

where the infimum is on the local sections t of $\mathscr{E}^\vee$ mapping to s under the canonical surjection $\pi^*\varphi : \pi^*\mathscr{E}^\vee \to \mathscr{O}_{\mathbf{P}}(1)$.

Restrict this to Y. This gives a norm on $\mathscr{O}_{\mathbf{P}}(1)|_Y = \mathscr{O}_Y(-D)$. The dual norm on $\mathscr{O}_Y(D)$ of the canonical section s_D is given by the formula

$$\|\mathsf{s}_D\| = \sup_{s\neq 0} \frac{|\langle \mathsf{s}_D, s\rangle|}{\|s\|} = \sup_t \frac{|\langle \mathsf{s}_D, \pi^*\varphi(t)\rangle|}{\|t\|}$$

the last supremum being over the non-zero local sections t of $\pi^*\mathscr{E}^\vee$. But, away from D on the blow-up, t comes from a local section of $\mathscr{E}^\vee$ and $\langle \mathsf{s}_D, \pi^*\varphi(t)\rangle$ is exactly the image of t under the surjection $\varphi : \mathscr{E}^\vee \to \mathscr{I}$. Hence, $\|\mathsf{s}_D\|$ is equal to the norm of φ, viewed as a homomorphism $\mathscr{E}^\vee \to \mathscr{O}_X$, which by the definition of the dual norm on $\mathscr{E}^\vee$ is exactly the norm of the original section $\sigma_Z \in \Gamma(X, \mathscr{E})$.

(This can be simplified if one regards the blow-up as the closure of the graph of the map $X \setminus Z \to \mathbf{P}(\mathscr{E}^\vee)$ induced by σ_Z.)

Note the precise meaning of these calculations:

- they are valid if S is any field with a valuation;
- if S is a discrete valuation ring, arithmetic intersection on the integral model gives a result which is compatible with the metrized theory on the generic fibre if the metric on $\mathscr{E}$ comes from the model;
- if S is the ring of integers of a number field, they show that we have an adelic metric in the sense of Arakelov geometry provided $\mathscr{E}$ is equipped with an adelic metric.

Hence, we have the following theorem:

Theorem 1.8. — *Let X be an algebraic variety over a field F, $\mathscr{I} \subset \mathscr{O}_X$ a sheaf of ideals on X and $\pi : Y \to X$ the blow-up of $\mathscr{Y}$. Let $\mathscr{E}$ be a locally free sheaf of finite rank on X with a section $\sigma_Z \in \Gamma(X, \mathscr{E})$ such that $V(\mathscr{I}) = \operatorname{div}(\sigma_Z)$ as schemes.*

Assume $\mathscr{E}$ is given a metric. Then the line sheaf $\mathscr{O}_Y(D)$ corresponding to the exceptional divisor D on Y has a canonical metric such that the norm of its canonical section s_D is given by the formula:

$$\|\mathsf{s}_D\|(y) = \|\sigma_Z\|(\pi(y)). \tag{1.9}$$

In particular, if $\mathscr{L}_1, \dots, \mathscr{L}_r$ are line bundles on X with sections s_i such that, as a scheme, $Z = \bigcap \operatorname{div}(\mathsf{s}_i)$, we may take $\mathscr{E} = \bigoplus \mathscr{L}_i$, $\sigma_Z = (\mathsf{s}_i)$. Assume the $\mathscr{L}_i$ to be metrized and endow $\mathscr{E}$ with the associated hermitian metric (resp. with the ℓ^∞-metric at non-archimedian places). The preceeding theorem implies that $\mathscr{O}_Y(D)$ may be metrized in such a way that

$$\|\mathsf{s}_D\|^2(y) = \sum_{i=1}^{r} \|\mathsf{s}_i\|(\pi(y)). \tag{1.10}$$

In particular, if $X = \mathbf{P}^n$, $\mathscr{L}_i = \mathscr{O}_{\mathbf{P}^n}(n_i)$, s_i corresponds to a homogeneous polynomial g_i of degree n_i and, if $\pi(y) = (x_0 : \dots : x_n)$,

$$\|\mathsf{s}_D\|^2(y) = \sum_{i=1}^{r} \frac{|g_i(x_0, \dots, x_n)|^2}{\left(\sum_{j=0}^{n} |x_i|^2\right)^{n_i}}. \tag{1.11}$$

As a last example, assume that $X = \mathbf{P}^n$ and Z is an integral divisor in Z_0. Then, the homogeneous ideal of Z is of the form $(x_0, f(x_1, \dots, x_n))$ for some homogeneous polynomial f of degree $d \geqslant 1$. If $\pi(y) = (1 : x_1 : \dots : x_n)$, then

$$\|\mathsf{s}_D\|^2(y) = \frac{1}{1 + \sum_{j=1}^{n} |x_j|^2} + \frac{|f(x_1, \dots, x_n)|^2}{\left(1 + \sum_{j=1}^{n} |x_j|^2\right)^d}. \tag{1.12}$$

All these formulas have analogues at non-archimedian places with the sum of the squares being replaced by their maximum.

1.13. Résumé. — Let F be a number field. For $1 \leqslant k \leqslant r$, choose a finite family of homogeneous polynomials $g_{k,j} \in F[x_0, \dots, x_n]$ of degree $d_{k,j}$ generating a prime ideal $\mathscr{I}_{Z_k}$ corresponding to an integral subscheme $Z_k \subset \mathbf{P}^n$. Let $\pi : X \to \mathbf{P}^n$ be the blow-up of the ideal $\mathscr{I} = \mathscr{I}_{Z_1} \cdots \mathscr{I}_{Z_k}$. On X, the inverse image of Z_k is a Cartier divisor D_k whose associated line bundle $\mathscr{O}_X(D_k)$

can be adelically metrized so that the norm of its canonical section s_{D_k} at a point $x \in X$ mapping to $(x_0 : \cdots : x_n) \in \mathbf{P}^n$ is given by

$$\|\mathsf{s}_{D_k}\|_v (x) = \max_j \frac{|g_{k,j}(x_0, \ldots, x_n)|_v}{\max(|x_0|_v, \ldots, |x_n|_v)^{d_{k,j}}}$$

at finite places v, and by

$$\|\mathsf{s}_{D_k}\|_v^2 (x) = \sum_j \frac{|g_{k,j}(x_0, \ldots, x_n)|_v^2}{(|x_0|_v^2 + \cdots + |x_n|_v^2)^{d_{k,j}}}$$

if v is an archimedean place.

We shall henceforth assume that Z_k is contained in the hyperplane at infinity $x_0 = 0$. Then one may assume that one of the $g_{k,j} = x_0$ and that the others do not depend on x_0. The universal property of the blow-up implies that $\pi : X \to \mathbf{P}^n$ is an isomorphism over $\mathbf{G}_a^n \simeq \{x_0 \neq 0\}$ and that the action of $\mathbf{G}_a^n$ on $\mathbf{P}^n$ lifts to an action on X and to an action on the line bundles $\mathscr{O}_X(D_k)$.

The following proposition can be deduced, either through explicit computations with the formulas defining $\|\mathsf{s}_{D_k}\|$, or by an abstract argument involving schemes over $\operatorname{Spec} \mathfrak{o}_F$.

Proposition 1.14. — *Assume that the polynomials $g_{k,j}$ have coefficients in $\mathfrak{o}_F$ and that they generate the homogeneous ideal $\mathscr{I}_{Z_k} \cap \mathfrak{o}_F[x_0, \ldots, x_n]$.*[†] *Then, for each place v of F, the standard maximal compact subgroup of $\mathbf{G}_a^n(F_v)$ acts isometrically on $\mathscr{O}(D_k)$.*

Let D_0 be the strict transform of the hyperplane at infinity under π. The line bundle $\mathscr{O}_X(D_0)$ is the pull-back on X of the $\mathscr{O}_{\mathbf{P}^n}(1)$ and we shall equip it with its standard metric (given by the formulas above, the family of $g_{0,j}$ being reduced to x_0). By means of equations (1.5, 1.6), we then can metrize the line bundles $\mathscr{O}_X(D_0)$ and K_X^{-1}.

2. Heights, Poisson formula

2.1. Product formula and heights. — We recall some conventions concerning absolute values in number fields.

[†] This means that the subscheme $V((g_{k,j})_j)$ of $\mathbf{P}^n_{\mathfrak{o}_F}$ is projective, surjective and flat over $\mathfrak{o}_F$.

Over $\mathbf{R}$, we set $|\cdot|_\infty$ to be the usual absolute value (such that $|2|_\infty = 2!$). If p is a prime number, the absolute value over $\mathbf{Q}_p$ is normalized by $|p|_p = 1/p$. These absolute values extend uniquely to any algebraic extension of $\mathbf{R}$ or $\mathbf{Q}_p$.

If F is a number field, we denote by $\operatorname{Val}(F)$ the set of places (equivalence classes of valuations) of F. If v is a place of F, we will denote by $|\cdot|_v$ be the unique extension of $|\cdot|_\infty$ or $|\cdot|_p$ to F_v (according to v being archimedean or not). We also set $m_v = e_v f_v$, the product of the ramification index by the local degree at v. Now, for any $x \in F$ and any valuation v of $\mathbf{Q}$,

$$\prod_{w|v} |x|_w^{m_w} = \left|N_{F/\mathbf{Q}}(x)\right|_v .$$

With these normalizations, we have the *product formula:* for any $x \in F^*$,

$$\prod_{v\in\operatorname{Val}(F)} |x|_v = \prod_{v\in\operatorname{Val}(\mathbf{Q})} \left|N_{F/\mathbf{Q}}(x)\right| = 1.$$

Let X be a projective variety over F and $\mathscr{L}$ a metrized line bundle on X. For any $x \in X(F)$, the (exponential, absolute) *height* of x with respect to the metrized line bundle $\mathscr{L}$ is defined by

$$H_{\mathscr{L}}(x) = \prod_{v\in\operatorname{Val}(F)} \|\mathsf{s}\|_v^{m_v}(x)$$

where s is any F-rational local section of $\mathscr{L}$, defined and non-zero at x. The product formula implies that the height doesn't depend on the choice of s.

2.2. Heights on blow-ups. — We keep the notations of the preceeding section. Moreover we identify $\mathbf{G}_a^n$ with its isomorphic inverse image in X under the blow-up $\pi : X \to \mathbf{P}^n$.

The metrizations above allow us to define *height functions* corresponding to complexified divisors $D(\mathbf{s}) = s_0 D_0 + \cdots + s_r D_r$. Namely, if $x = (x_1, \dots, x_n) \in \mathbf{G}_a^n(F)$, its exponential height is defined by

$$H_{D(\mathbf{s})}(x) = \prod_{k=0}^{r} \Big(\prod_{v\in\operatorname{Val}(F)} \|\mathsf{s}_{D_k}\|_v^{-m_v} (1 : x_1 : \cdots : x_n)\Big)^{s_k}.$$

This definition extends to all $x \in \mathbf{G}_a^n(\mathbf{A}_F)$ and gives a pairing

$$H : \operatorname{Pic}^G(X)_{\mathbf{C}} \times \mathbf{G}_a^n(\mathbf{A}_F) \longrightarrow \mathbf{C}^*$$

which is multiplicative as a function on $\mathrm{Pic}^G(X)$ and which is invariant under the action of the maximal compact subgroup of $\mathbf{G}_a^n(\mathbf{A}_F)$. Such a pairing had already appeared in the context of toric varieties.

The invariance of the heights is a crucial technical ingredient for the proofs of analytic properties of the height zeta functions for toric varieties and for equivariant compactifications of $\mathbf{G}_a^n$ considered in the present paper.

The "height zeta function" is the series

$$Z(s_0,\dots,s_r)=\sum_{x\in\mathbf{G}_a^n(F)} H_{D(\mathbf{s})}(x)^{-1}.$$

Its convergence in some non empty open subset of $\mathbf{C}^{r+1}$ is a consequence of the following (well known) lemma.

Lemma 2.3. — *Let V be a projective variety over a number field F and $(\mathscr{L}_i)_{1\leqslant i\leqslant d}$ a finite number of ample metrized line bundles on V. For $x\in V(F)$, define $H(\mathbf{s};x)=\prod_{i=1}^d H_{\mathscr{L}_i}(x)^{s_i}$. Then there exists an open non-empty subset Ω of $\mathbf{R}^d$ such that the series*

$$Z(\mathbf{s})=\sum_{x\in X(F)} H(\mathbf{s};x)^{-1}$$

converges absolutely and uniformly for all $\mathbf{s}\in\mathbf{C}^d$ with $\mathrm{Re}(\mathbf{s})$ contained in Ω.

Moreover, any other metrization on the $\mathscr{L}_i$ gives the same domain of convergence.

Proof. — The usual proof of Northcott's theorem establishes a polynomial bound for the number of rational points of bounded exponential height. Hence, the height zeta function of $(\mathbf{P}^n,\mathscr{O}(1))$ converges for $s\gg 0$. (There is no need to invoke Schanuel's theorem [**19**] which gives the precise *asymptotics*. Below, we give a new proof of Schanuel's theorem.)

Therefore, there are real numbers α_i such that $Z(0,\dots,s_i,\dots,0)$ converges for $\mathrm{Re}(s_i)\geqslant\alpha_i$. Now, $Z(\mathbf{s})$ converges for any $\mathbf{s}=(s_1,\dots,s_d)\in\mathbf{C}^d$ such that for each i, $\mathrm{Re}(s_i)\geqslant\alpha_i$. □

2.4. Harmonic analysis on the additive group. — We recal basic facts concerning harmonic analysis on the group of adelic points $\mathbf{G}_a^n(\mathbf{A}_F)$ (cf., for example, [**22**]). For any prime number p, we can view $\mathbf{Q}_p/\mathbf{Z}_p$ as the p-Sylow subgroup of $\mathbf{Q}/\mathbf{Z}$. This allows us to define a local character ψ_p of $\mathbf{G}_a(\mathbf{Q}_p)$ by setting

$$\psi_p\,:\,x_p\mapsto\exp(2\pi i x_p).$$

At the infinite place of $\mathbf{Q}$ we put

$$\psi_\infty\colon x_\infty \mapsto \exp(-2\pi i x_\infty),$$

(here x_∞ is viewed as an element in $\mathbf{R}/\mathbf{Z}$). The product of local characters gives a character ψ of $\mathbf{G}_a(\mathbf{A}_\mathbf{Q})$ and, by composition with the trace, a character of $\mathbf{G}_a(\mathbf{A}_F)$. For any $\mathbf{a} \in \mathbf{G}_a^n(\mathbf{A}_F)$ we obtain a character $\psi_\mathbf{a}$ of $\mathbf{G}_a^n(\mathbf{A}_F)$ by

$$\mathbf{x} \mapsto \psi \circ \operatorname{tr}_{F/\mathbf{Q}}(\langle \mathbf{a}, \mathbf{x}\rangle).$$

The choice of ψ defines a self-duality of $\mathbf{G}_a^n(\mathbf{A}_F)$ (Pontryagin duality). For $v \in \operatorname{Val}(F)$, we denote by μ_v the standard normalized local Haar measures on $\mathbf{G}_a^n(F_v)$ and by $\mu = \prod_v \mu_v$ the self-dual measure on $\mathbf{G}_a^n(\mathbf{A}_F)$. The precise normalization can be found in (cf. [**22**] or [**12**], p. 280); for $F = \mathbf{Q}$, we have $\mu_p(\mathbf{Z}_p) = 1$ and $\mu_\infty([0;1]) = 1$.

For a function H on $\mathbf{G}_a^n(\mathbf{A}_F)$ we denote by $\hat{H}$ its Fourier-transform (with respect to the Haar measure μ)

$$\hat{H} : \mathbf{G}_a^n(\mathbf{A}_F) \to \mathbf{C}, \qquad \psi \mapsto \int_{\mathbf{G}_a^n(\mathbf{A}_F)} H(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}),$$

whenever the integral converges. We shall also use the notation $\mathrm{d}\mathbf{x}$ for $\mathrm{d}\mu(\mathbf{x})$.

We will use the Poisson formula in following form (cf. [**12**], p. 280).

Theorem 2.5. — *Let H be a continuous function on $\mathbf{G}_a^n(\mathbf{A}_F)$ such that both H and $\hat{H}$ are integrable and such that*

$$\sum_{\mathbf{a}\in\mathbf{G}_a^n(F)} H(\mathbf{x}+\mathbf{a})$$

is absolutely and uniformly convergent on compact subsets in $\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$. Then

$$\sum_{\mathbf{x}\in\mathbf{G}_a^n(F)} H(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbf{G}_a^n(F)} \hat{H}(\psi_\mathbf{a}).$$

For $\mathbf{s} \in \mathbf{C}^{r+1}$ and $\psi \in \mathbf{G}_a^n(\mathbf{A}_F)$, we shall denote by $\hat{H}(\mathbf{s};\psi)$ the Fourier transform of the height function $H(\mathbf{s};\cdot)^{-1}$ on $\mathbf{G}_a^n(\mathbf{A}_F)$ at the character ψ. It is the product of the local Fourier transforms of the functions $H(\mathbf{s};\cdot)^{-m_v}$ for all $v \in \operatorname{Val}(F)$.

Proposition 2.6. — *With the above notations, for all characters ψ which are non-trivial on the maximal compact subgroup of $\mathbf{G}_a^n(\mathbf{A}_F)$, we have that $\hat{H}(\mathbf{s},\psi) = 0$.*

Proof. — This follows from the invariance of the height under the maximal compact subgroups, see Prop. 1.14. □

Consequently, we have the formal identity for the height zeta function:

$$Z(\mathbf{s}) = \sum_{\mathbf{a}\in\mathbf{G}_a^n(\mathfrak{o}_F)} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}). \tag{2.7}$$

The following lemma verifies the two hypotheses of the Poisson formula 2.5 concerning H.

Lemma 2.8. — *There exists a real* α *such that for any* $\mathbf{s}\in\mathbf{C}^{r+1}$ *satisfying* $\mathrm{Re}(s_0-s_k)\geqslant\alpha$ *and* $\mathrm{Re}(s_k)\geqslant\alpha$, *and for any compact subset* K *of* $\mathbf{G}_a^n(\mathbf{A}_F)/\mathbf{G}_a^n(F)$, *the series*

$$\sum_{\mathbf{a}\in\mathbf{G}_a^n(F)} H(\mathbf{s};x+\mathbf{a})$$

converges absolutely and uniformly for $\mathbf{x}\in K$.

Proof. — If $\mathbf{s}\in\mathbf{Z}^{r+1}$, the line bundle $D(\mathbf{s})$ is ample iff all $s_k>0$ and $s_0>s_1+\cdots+s_r$. Moreover, the ample line bundles $D=(r+1)D_0+D_1+\cdots+D_r, D+D_1,\dots,D+D_r$ provide a basis of $\mathrm{Pic}(X)_{\mathbf{R}}$. Hence, Lemma 2.3 implies the existence of a real $\alpha>0$ such that the series converges absolutely when $\mathbf{x}=0$, uniformly for all $\mathbf{s}\in\mathbf{C}^{r+1}$ such that $\mathrm{Re}(s_0)>\alpha$, $\mathrm{Re}(s_0-s_k)>\alpha$.

For any $\mathbf{x}$, the function $H(\mathbf{s};\mathbf{x}+\cdot)$ is another height function for $D(\mathbf{s})$, called "twisted height" in our paper [**5**], § 2.4, esp. proposition 2.4.3. This implies the convergence for any $\mathbf{x}$. The uniformity for $\mathbf{x}\in K$ follows from the fact that the height functions can be mutually uniformly bounded. □

Now, for the proof of the meromorphic continuation of the height zeta function it will be sufficient to prove that the $\hat{H}$-series on the right-hand side of Eq. 2.7 1°) converges for some $\mathrm{Re}(s_0)>\alpha$, $\mathrm{Re}(s_0-s_k)>\alpha$ big enough, and 2°) continues meromorphically.

2.9. Integrability of local height functions. — The aim of this section is to prove a general result concerning the integrability of local height functions against a measure with singularities.

Proposition 2.10. — *Let* X *be a proper smooth variety of dimension* d *over a field* F_v *which is a finite extension of* $\mathbf{R}$ *or* $\mathbf{Q}_p$. *Fix a finite number of metrized line bundles* $\mathscr{L}_\alpha$ *on* X *together with sections* s_α. *Assume that their divisors* $\mathrm{div}(\mathsf{s}_\alpha)$ *are smooth and that their sum is a divisor* D *with normal* *crossings and let $U = X \setminus D$. Finally, let $\omega \in \Gamma(U, \Omega^d_{X/F_v})$ be a meromorphic differential form of top degree. We assume that there are integers λ_α such that the divisor of ω equals $\sum_\alpha \lambda_\alpha \operatorname{div}(s_\alpha)$. Denote by $\mathrm{d}\omega$ the associated measure on $U(F_v)$.*

Then, the integral

$$\int_{U(F_v)} \prod_\alpha \|\mathsf{s}_\alpha\|^{m_v r_\alpha}(x)\,\mathrm{d}\omega$$

converges if and only if for all α, $r_\alpha > \lambda_\alpha - 1$.

Proof. — Using a partition of unity on X for the F_v-topology, we may assume that X is a relatively compact open subset $\Omega \subset F_v^d$, with local coordinates $x_1, \dots, x_d$ and that the divisor $\sum_\alpha \operatorname{div}(s_\alpha)$ is given by the equation $x_1 \dots x_a = 0$ for some integer $0 \leqslant a \leqslant d$. The integral is then

$$I_\Omega = \int_\Omega \prod_{i=1}^a |x_i|_v^{m_v(r_{\alpha(i)} - \lambda_{\alpha(i)})} \exp\big(\sum_\alpha h_\alpha(x)\big)\,\mathrm{d}x_1 \dots \mathrm{d}x_d$$

for some functions h_α giving the metrics in our local trivialization and which are therefore continuous and bounded.

Remark that the integral of $|x|_v^{m_v s}$ over the unit ball of F_v converges if and only if $s > -1$. The Fubini theorem shows that the integral I_Ω converges if and only if for each $i \in \{1, \dots, a\}$, $r_{\alpha(i)} - \lambda_{\alpha(i)} > -1$. As any α appears in some chart, the proposition is proved. □

2.11. The local Fourier transform in the archimedean case

When $F_v = \mathbf{R}$ or $\mathbf{C}$, we want to show that the local Fourier transform of the height function as a function of $\psi_{\mathbf{a}}$ decreases rapidly when the norm of $\mathbf{a} \in F_v^n$ grows to infinity. The proof proceeds by integration by parts, which requires some estimates.

Lemma 2.12. — *Let X be a smooth projective variety over F_v and Z be a smooth closed subscheme of X. Let ∂ be a global section of $(\Omega^1_X)^\vee \otimes \mathscr{I}_Z$, i.e. a derivation on X vanishing on Z. Denote by $\pi : Y \to X$ the blow-up of $\mathscr{I}_Z$.*

1) Then the derivation $\partial|_{\pi^{-1}(X\setminus Z)}$ extends uniquely to a derivation on Y.

2) Let $\mathscr{E}$ be a vector bundle on X equipped with a smooth hermitian metric and s a global section of $\mathscr{E}$ whose divisor is Z. Then the function $\partial \log \|\mathsf{s}\|$ extends uniquely to a smooth function on Y.

Proof. — Choose local analytic coordinates on X such that Z is defined by $x_1 = \cdots = x_a = 0$. Then, Y may be embedded in $\mathbf{P}^{a-1} \times \mathbf{A}^d$ with coordinates $((t_1 : \cdots : t_a), (x_1, \ldots, x_d))$ and is given there by the equations $t_i x_j = t_j x_i$ for $i \in \{1, \ldots, a\}$. We consider the chart $t_a \neq 0$. Then, local coordinates on Y are $t_1, t_2, \ldots, t_{a-1}, x_a, x_{a+1}, \ldots, x_d$ and $\pi : Y \to X$ is given by $x_i = t_i x_a$ if $i < a$.

On X, the derivation ∂ has the form

$$\partial = \sum_{i=1}^{d} h_i \frac{\partial}{\partial x_i},$$

for some functions $h_i \in (x_1, \ldots, x_a)$. Now, we have to verify that if $i < a$, ∂t_i is a regular function on Y. But

$$\partial t_i = \partial(x_a/x_i) = h_a(x)\frac{1}{x_i} - h_i(x)\frac{x_a}{x_i^2} = \frac{1}{x_a}h_a(x)t_i - \frac{1}{x_a}h_i(x)t_i \in \mathscr{O}_Y$$

since

$$h_j(x) \in (t_1 x_a, \ldots, t_{a-1}x_a, x_a) = (x_a).$$

For the statement concerning norms, we may fix the coordinates so that $\|\mathsf{s}\|^2(x) = \sum_{i=1}^a |x_i|^2$. Then,

$$\begin{aligned}\partial \log \|\mathsf{s}\| &= \frac{1}{|x_1|^2 + \cdots + |x_a|^2}\Big(\sum_{i=1}^{a} 2x_i h_i(x)\Big) \\ &= \frac{1}{|t_1|^2 + \cdots + |t_{a-1}|^2 + 1}\Big(\sum_{i=1}^{a} 2t_i \frac{h_i(x)}{x_a}\Big)\end{aligned}$$

is regular on Y. □

Proposition 2.13. — *For any compact subset $K \subset \mathbf{R}^{r+1}$ where $H(\mathbf{s};\cdot)^{-m}$ is integrable, and for any integer $d \geqslant 1$, there exists a constant $c(d,K)$ such that for any $\mathbf{a} \in \mathbf{C}^n$ and any $\mathbf{s} \in \mathbf{C}^{r+1}$ with* $\mathrm{Re}(\mathbf{s}) \in K$*,*

$$\left|\hat{H}(\mathbf{s};\psi_{\mathbf{a}})\right| \leqslant c(d,K)\left(\frac{1 + \|\Im(\mathbf{s})\|}{1 + \|\mathbf{a}\|}\right)^d.$$

Proof. — The 2 preceeding lemmas imply that $\partial^\alpha(\log\|\mathsf{s}_D\|)(x)$ are bounded on $\mathbf{G}_a^n(F)$. Moreover, $\|\mathsf{s}_D\|$ tends to 0 at infinity. We thus may integrate by parts d-times. □

3. Projective space

This section is included to illustrate our approach in the simplest example. We give yet another proof of asymptotics for the number of rational points of bounded height on the standard projective space $\mathbf{P}^n$ over a number field F with the standard metrization of the line bundle $\mathscr{O}(1)$ given by the model $\mathbf{P}^n_{\mathfrak{o}_F}$ at the finite places and by the L^2-norms at the archimedian places.

To keep this section as self-contained as possible, we reprove the estimates needed without referring to the general estimates of the preceeding section. For simplicity, we assume moreover that $F=\mathbf{Q}$.

We will denote by $\mathbf{A}$ the ring of adeles $\mathbf{A}_{\mathbf{Q}}$, by p a prime number. We have the normalized valuations $|\cdot|_p$ with $|p|_p = p^{-1}$ and the usual absolute value $|\cdot|_\infty$. If $\mathbf{a}\in\mathbf{G}_a^n(\mathbf{A})$, we denote by $\psi_{\mathbf{a}}$ the corresponding character by the identification of $\mathbf{G}_a^n(\mathbf{A})$ with its Pontryagin dual.

We are interested in the height zeta function

$$Z(s) = \sum_{\mathbf{x}\in\mathbf{G}_a^n(\mathbf{Q})} H(\mathbf{x})^{-s} \tag{3.1}$$

where $H(\mathbf{x}) = H_\infty(\mathbf{x})\prod_p H_p(\mathbf{x})$ with

$$H_v(\mathbf{x}) := \|\mathbf{x}\|_v = \begin{cases} (1+\sum_{j=1}^n |x_j|_v^2)^{1/2} & \text{if } v|\infty \\ \max(1,\max_j |x_j|_v) & \text{if } v \text{ is finite.}\end{cases}$$

The series (3.1) converges absolutely and uniformly to a holomorphic function for $\operatorname{Re}(s)\gg 0$. For all s such that the both sides converge, we have the Poisson-formula identity (cf. 2.5)

$$Z(s) = \sum_{\psi_{\mathbf{a}}} \hat{H}(s;\psi_{\mathbf{a}}), \tag{3.2}$$

absolutely. This identity is the starting point for a meromorphic continuation of $Z(s)$. We now compute (resp. estimate) the local Fourier transforms.

Lemma 3.3. — *Let p be a prime number. For all s with $\operatorname{Re}(s)>n$, $H_p(s;\cdot)$ is integrable on $\mathbf{Q}_p^n$ and its Fourier transform at the trivial character ψ_0 is given by*

$$\hat{H}_p(s;\psi_0) = \frac{1-p^{-s}}{1-p^{-(s-n)}}. \tag{3.4}$$

Proof. — We decompose the domain of integration $\mathbf{Q}_p^n$ into subdomains

$$U(\alpha) = \{\mathbf{x} = (x_1, ..., x_n) \, ; \, \|\mathbf{x}\|_p = p^\alpha\},$$

for $\alpha \geqslant 1$ and

$$U(0) = \{\mathbf{x} = (x_1, ..., x_n) \, ; \, \|\mathbf{x}\|_p \leqslant 1\}$$

Then

$$\begin{aligned}\hat{H}_p(s;\psi_0) &= \int_{U(0)} H(\mathbf{x})^{-s} \, \mathrm{d}\mathbf{x} + \sum_{\alpha \geqslant 1} \int_{U(\alpha)} H(\mathbf{x})^{-s} \, \mathrm{d}\mathbf{x}, \\ &= 1 + \sum_{\alpha \geqslant 1} p^{-\alpha s} \cdot \mathrm{vol}(U(\alpha)).\end{aligned}$$

One has $\mathrm{vol}\, U(0) = 1$ and for $\alpha \geqslant 1$,

$$\mathrm{vol}(U(\alpha)) = p^{\alpha n} \, \mathrm{vol}(\mathbf{Z}_p^n \setminus (p\mathbf{Z}_p)^n) = p^{\alpha n}(1 - p^{-n}).$$

For all s with $\mathrm{Re}(s) > n$, the geometric series converges absolutely and we obtain

$$\begin{aligned}\hat{H}_p(s;\psi_0) &= 1 + (1 - \frac{1}{p^n}) \sum_{\alpha \geqslant 1} p^{-\alpha(s-n)}, \\ &= 1 + (1 - \frac{1}{p^n}) \cdot \frac{1}{p^{s-n}} \cdot \frac{1}{1 - p^{-(s-n)}}.\end{aligned}$$

Simplifying, we obtain (3.4). □

For all $\mathbf{a} = (a_1, ..., a_n) \in \mathbf{Z}^n$ let $S(\mathbf{a})$ be the set of all primes which divide all a_j.

Lemma 3.5. — *For all* $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$*, all* s *with* $\mathrm{Re}(s) > n$ *and all* $p \notin S(\mathbf{a})$ *we have*

$$\hat{H}_p(s;\psi_{\mathbf{a}}) = 1 - p^{-s}. \tag{3.6}$$

Proof. — As above, we have

$$\hat{H}_p(s,\psi_{\mathbf{a}}) = 1 + \sum_{\alpha \geqslant 1} p^{-\alpha s} \int_{U(\alpha)} \psi_{\mathbf{a}}(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

We first integrate over the set $V(\alpha)$ of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| \leqslant p^\alpha$.

$$\int_{V(\alpha)} \psi_{\mathbf{a}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = p^{\alpha n} \int_{\mathbf{Z}_p^n} \psi_{\mathbf{a}/p^\alpha}(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

If $\alpha \geqslant 1$, as p doesn't divide all the a_j, this is the integral of a non trivial character on a compact group, hence 0. For $\alpha = 0$, we get 1. Therefore, as $V(0) = U(0)$ and $U(\alpha) = V(\alpha) \setminus V(\alpha - 1)$ for $\alpha \geqslant 1$,

$$\int_{U(\alpha)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \begin{cases} 0 & \text{for } \alpha \geqslant 2 \\ -1 & \text{for } \alpha = 1. \end{cases}$$

This implies the lemma. □

Lemma 3.7. — *For all $\varepsilon > 0$ there exist constants c and $\delta > 0$ such that for all s with $\operatorname{Re}(s) > n + \varepsilon$ and all $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$ we have the estimate*

$$\left| \prod_{p \in S(\mathbf{a})} \hat{H}_p(s; \psi_{\mathbf{a}}) \right| \leqslant c \cdot (1 + \|\mathbf{a}\|)^{\delta}. \tag{3.8}$$

Proof. — In the integral, we replace $\psi_{\mathbf{a}}$ by 1, s by $\operatorname{Re}(s)$ and use the computation in (3.3). For $\operatorname{Re}(s) \geqslant n + \varepsilon$, we obtain

$$\left| \hat{H}_p(s; \psi_{\mathbf{a}}) \right| \leqslant \frac{1}{1 - p^{-\varepsilon}}.$$

If a is a positive integer, we have an inequality

$$\prod_{p|a} \frac{1}{p^{\varepsilon}} \ll \ln(1 + a)$$

which can be deduced e.g. from the Prime Number Theorem. This gives us equation (3.8). □

We now turn to the estimations of the local Fourier transform for the place at infinity. For the trivial character, we can—as we could in the non-archimedian case—explicitely compute the relevant integral:

Lemma 3.9. — *For all s with $\operatorname{Re}(s) > n$, $H_\infty(s; \cdot)$ is integrable on $\mathbf{R}^n$ and its Fourier transform at the trivial character ψ_0 is given by*

$$\hat{H}_\infty(s; \psi_0) = \pi^{n/2} \frac{\Gamma((s-n)/2)}{\Gamma(s/2)}.$$

Lemma 3.10. — *For all $\delta > 0$ and all compacts K in the domain $\operatorname{Re}(s) > n$ there exists a constant $c(\delta, K)$ such that for all $\mathbf{a} \in \mathbf{Z}^n$ and all $s \in K$ we have*

$$\left| \hat{H}_\infty(s; \psi_{\mathbf{a}}) \right| \leqslant c(\delta, K)(1 + \|\mathbf{a}|)^{-\delta}$$

Proof. — By a unitary change of variables, we may assume $\mathbf{a} = (\|\mathbf{a}\|, 0, \dots, 0)$. Thus,

$$\begin{aligned}\hat{H}_\infty(s;\psi_{\mathbf{a}}) &= \int_{\mathbf{R}^n} (1+\|x\|^2)^{-s/2} \exp(-2\pi i \|\mathbf{a}\| x_1)\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^{n-1}} (1+|x_1|^2 + \|\mathbf{x}'\|)^{-s/2} \exp(-2\pi i \|a\| x_1)\, \mathrm{d}x_1\, \mathrm{d}\mathbf{x}' \\ &= \int_{\mathbf{R}} (1+|x_1|^2)^{-(s-n+1)/2} \exp(-2\pi i \|a\| x_1)\, dx_1 \int_{\mathbf{R}^{n-1}} \frac{\mathrm{d}\mathbf{y}}{(1+\|\mathbf{y}\|^2)^{s/2}}\end{aligned}$$

For any $k > 0$, the kth derivative of $t \mapsto (1+t^2)^{-s}$ is of the form $P_k(t)(1+t^2)^{-s-k}$ with P_k a polynomial of degree k whose coefficients are polynomials in s. Thus we can integrate by parts and get for any k an expression

$$\int_{\mathbf{R}} (1+t^2)^{-(s-n+1)/2} \exp(-2\pi i \|a\| t)\, dt = \frac{1}{(\pi i \|a\|)^k} \int_{\mathbf{R}} \frac{P_k(t)}{(1+t^2)^{k-\frac{s-n+1}{2}}}\, dt$$

which imply the lemma. □

Remark 3.11. — It follows from the arguments above that the Fourier transform has polynomial growth in vertical strips.

Theorem 3.12. — *The series*

$$Z(s) = \sum_{\psi_{\mathbf{a}}} \hat{H}(s;\psi_{\mathbf{a}})$$

converges absolutely and uniformly to a holomorphic function for s with $\operatorname{Re}(s) > n+1$. *The function $Z(s)$ admits a meromorphic continuation to the domain* $\operatorname{Re}(s) > n$ *with exactly one simple pole at $s = n+1$. The residue at this pole equals*

$$\lim_{s\to n+1} (s-(n+1))\hat{H}(s;\psi_0) = \lim_{s\to n+1} (s-(n+1)) \int_{\mathbf{G}_a^n(\mathbf{A}_{\mathbf{Q}})} H(s;\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Proof. — Choose a real number $\delta > n$. From lemmas 4.5, 4.7 and 4.10, it follows that for any compact $K \in]n;+\infty[$, any $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$, the product of the local Fourier transforms at the character $\psi_{\mathbf{a}}$ converges to a holomorphic function of s which satisfies the inequality

$$\left|\hat{H}(s;\psi_{\mathbf{a}})\right| \leqslant c(K)(1+|\Im(s)|)^\delta (1+\|a\|)^{-\delta}, \qquad \operatorname{Re}(s) \in K.$$

Hence, the sum over all non-trivial ψ converges absolutely and locally uniformly to a holomorphic function in the domain $\operatorname{Re}(s) > n$.

At the trivial character, we have, if $\mathrm{Re}(s) > n+1$,

$$\hat{H}(s;\psi_0) = \frac{\zeta(s-n)\Gamma((s-n)/2)}{\zeta(s)\Gamma(s/2)}.$$

This has a simple pole at $s = n$ and extends meromorphically to the domain $\mathrm{Re}(s) > n$, with no other pole there. □

The identification of the residue and Peyre's Tamagawa constant in [**14**] is straightforward, granted the meromorphic continuation of $\hat{H}(s;\psi_0)$.

4. Blow-ups of $\mathbf{P}^2$

4.1. Preliminaries. — We continue to work over $\mathbf{Q}$ and we keep the notations of previous sections.

Let us consider the projective plane $\mathbf{P}^2$ with coordinates (x_0, x_1, x_2) and its Zariski open subset $U \subset \mathbf{P}^2$ given by $x_0 \neq 0$. Denote by X the blow-up of $\mathbf{P}^2$ in r distinct points $Z_1, \dots, Z_r$ which are contained in the line at infinity $Z_0 \subset \mathbf{P}^2$ which is given by $x = 0$.

For all $k \in \{1, \dots, r\}$, there is a linear form $\ell_k \in \mathbf{Z}[x_1, x_2]$ with primitive coefficients such that $Z_k = \mathscr{V}(x_0, \ell_k)$. For $k = 1, \dots, r$, we denote by D_k the inverse image of Z_k in X and by D_0 the strict transform of the line Z_0. The variety X is smooth; the anticanonical class is given by

$$-[K_X] = 3[D_0] + 2\sum_{k=1}^{r-1}[D_k].$$

In the sequel, we shall identify a point $\mathbf{x} \in \mathbf{G}_a^2$ with the point with homogeneous coordinates $(1 : \mathbf{x})$ in $\mathbf{P}^2$ or with its pre-image in the blow-up. It follows from the general theory of height functions on blow-ups that for all $k \in \{1, \dots, r\}$, a local height function for the divisor D_k at such a point $\mathbf{x}$ is given by

$$H_{k,p}(\mathbf{x}) = \frac{\max(1, \|\mathbf{x}\|_p)}{\max(1, |\ell_k(\mathbf{x})|_p)}$$

at a finite place p, and by an analogous formula where $\max(1,\cdot)$ is replaced by $\sqrt{1+\cdot^2}$ at the infinite place. For D_0, we have

$$H_{0,p}(\mathbf{x}) = \max(1, \|\mathbf{x}\|_p)\prod_{k=1}^{r} H_{k,p}^{-1}(\mathbf{x})$$

(with the same convention if $v=\infty$). The global height is given by

$$H_k(\mathbf{x}) = H_{k,\infty}(\mathbf{x}) \cdot \prod_p H_{k,p}(\mathbf{x})$$

and for $\mathbf{s} = (s_0, \dots, s_r) \in \mathbf{C}^r$, we define

$$H(\mathbf{s};\mathbf{x}) := \prod_{k=0}^{r} H_k(\mathbf{x})^{s_k}$$

the global height corresponding to the complexified line bundle $D(\mathbf{s})$.

From 2.5, we see that the height zeta function for X has the following formal "Fourier expansion":

$$\sum_{\mathbf{x}\in\mathbf{Q}^2} H(\mathbf{s};\mathbf{x})^{-1} = \sum_{\mathbf{a}\in\mathbf{Z}^2} \hat{H}(\mathbf{s};\psi_{\mathbf{a}})$$

We have the decomposition

$$\hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = \hat{H}_\infty(\mathbf{s};\psi_{\mathbf{a},\infty}) \cdot \prod_p \hat{H}_p(\mathbf{s};\psi_{\mathbf{a},p}).$$

As in the case of $\mathbf{P}^n$, we compute the local Fourier transforms for almost all places and estimate them at the remaining bad places.

Let S be the set of primes of bad reduction of the schematic closure of $\bigcup_k Z_k$ in $\mathbf{P}^2_{\mathbf{Z}}$. A prime p belongs to S if there exist two linear forms ℓ_k and ℓ_j such that p divides $\det(\ell_k,\ell_j)$.

4.2. Decomposition of the domain. — Fix a prime $p \notin S$ which we may omit from the notations for norms, etc. Define subsets of $\mathbf{Q}_p^2$ as follows:

- $U(0) = \mathbf{Z}_p^2$;
- if $1 \leqslant \beta \leqslant \alpha$ and $k \in \{1,\dots,r\}$, $U_k(\alpha,\beta)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^2$ such that $\|\mathbf{x}\| = p^\alpha$ and $|\ell_k(\mathbf{x})| = p^{\alpha-\beta}$;
- if $\alpha \geqslant 1$ and $k \in \{1,\dots,r\}$, $U_k(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^2$ such that $\|\mathbf{x}\| = p^\alpha$ and $|\ell_k(\mathbf{x})| \leqslant 1$;
- if $\alpha \geqslant 1$, $U(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^2$ such that $\|\mathbf{x}\| = p^\alpha$ and all $|\ell_j(\mathbf{x})| = p^\alpha$.

As $p \notin S$, these sets furnish a partition of $\mathbf{Q}_p^2$. This decomposition is well adjusted to our local heights since they are constant on each subset:

- on $U(0)$, all H_k's are 1;
- on $U_k(\alpha,\beta)$, $H_k = p^\beta$, the other H_j with $j \geqslant 1$ are 1 and $H_0 = p^{\alpha-\beta}$;
- on $U_k(\alpha)$, $H_k = p^\alpha$, the other H_j are 1;

– on $U(\alpha)$, $H_0 = p^\alpha$ and all other are 1.

In other words,

$$H(\mathbf{s};\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in U(0); \\ p^{\alpha s_0} & \text{if } \mathbf{x} \in U(\alpha),\ \alpha \geqslant 1; \\ p^{\alpha s_0 + \beta(s_k - s_0)} & \text{if } \mathbf{x} \in U_k(\alpha,\beta),\ 1 \leqslant \beta < \alpha; \\ p^{\alpha s_k} & \text{if } \mathbf{x} \in U_k(\alpha,\beta),\ \beta \geqslant \alpha. \end{cases} \tag{4.3}$$

The table on p. 29 summarizes this information.

4.4. Some integrals of characters. — We identify $\mathbf{a} \in \mathbf{Z}^2$ with the linear form it defines on $\mathbf{G}_a^2$ as well as with the character $\psi_{\mathbf{a}}$ of $\mathbf{G}_a^2(\mathbf{A_Q})$ it determines.

We will say that a character is *generic* if $\mathbf{a}$ is not proportional to any ℓ_k. A non trivial character is special if it is proportional to some (necessarily unique) ℓ_k.

If $\mathbf{a}$ is generic, $S(\mathbf{a})$ is the set of primes p such that p divides some determinant $\det(\ell_k, \mathbf{a})$.

If $\mathbf{a}$ is special for ℓ_k, $S(\mathbf{a})$ is the set of primes such that p divides some determinant $\det(\ell_j, \mathbf{a})$ for $j \neq k$.

Note that $S \subset S(\mathbf{a})$ for any non-trivial $\mathbf{a}$ and that if $p|\mathbf{a}$, then $p \in S(\mathbf{a})$.

We now compute the integral of $\psi_{\mathbf{a}}$ over the subsets defined in the previous subsection, at least in the cases $\mathbf{a} = 0$, $\mathbf{a}$ special and $\mathbf{a}$ generic.

Remark that for any $\mathbf{a}$, $\int_{U(0)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$.

***Lemma 4.5* (Trivial character).** — *Let p be a prime not in S. Then,*

$$\operatorname{vol} U_k(\alpha,\beta) = p^{2\alpha-\beta}\frac{(p-1)^2}{p^2}; \tag{4.5a}$$

$$\operatorname{vol} U_k(\alpha) = p^{\alpha}\frac{p-1}{p}; \tag{4.5b}$$

$$\operatorname{vol} U(\alpha) = p^{2\alpha}\frac{(p-1)(p+1-r)}{p^2}. \tag{4.5c}$$

Lemma 4.6 **(Generic character)**. — *Let* $\mathbf{a}$ *be a generic character and* $p \notin S(\mathbf{a})$. *Then,*

$$\int_{U_k(\alpha,\beta)} \psi_{\mathbf{a}} = 0 \tag{4.6a}$$

$$\int_{U_k(\alpha)} \psi_{\mathbf{a}} = \begin{cases} -1 & \text{if } \alpha = 1 \\ 0 & \text{else;} \end{cases} \tag{4.6b}$$

$$\int_{U(\alpha)} \psi_{\mathbf{a}} = \begin{cases} -1 + r & \text{if } \alpha = 1 \\ 0 & \text{else;} \end{cases} \tag{4.6c}$$

Lemma 4.7 **(Special character)**. — *Let* $\mathbf{a}$ *a character which is special for* ℓ_k. *If* $p \notin S(\mathbf{a})$ *and* $j \neq k$, *one has*

$$\int_{U_j(\alpha,\beta)} \psi_{\mathbf{a}} = 0 \tag{4.7a}$$

$$\int_{U_j(\alpha)} \psi_{\mathbf{a}} = \begin{cases} -1 & \text{if } \alpha = 1 \\ 0 & \text{else;} \end{cases} \tag{4.7b}$$

$$\int_{U_k(\alpha,\beta)} \psi_{\mathbf{a}} = \begin{cases} -p^{\alpha}\frac{p-1}{p} & \text{if } \beta = \alpha - 1 \\ 0 & \text{else;} \end{cases} \tag{4.7c}$$

$$\int_{U_k(\alpha)} \psi_{\mathbf{a}} = p^{\alpha}\frac{p-1}{p} \tag{4.7d}$$

$$\int_{U(\alpha)} \psi_{\mathbf{a}} = \begin{cases} -(p+1-r) & \text{if } \alpha = 1 \\ 0 & \text{else;} \end{cases} \tag{4.7e}$$

Proof. — We prove the three lemmas simultaneously. By a unitary change of variables, we may assume that $\ell_k(\mathbf{x}) = x_1$. Then one has

$$U_k(\alpha,\beta) = p^{\beta-\alpha}\mathbf{Z}_p^* \times p^{-\alpha}\mathbf{Z}_p^*$$

and

$$U_k(\alpha) = \mathbf{Z}_p \times p^{-\alpha}\mathbf{Z}_p^*,$$

hence their volumes as in formulas (4.5a) and (4.5b).

If p doesn't divide $\det(\ell_k, \mathbf{a})$, we may change variables and even assume that $\mathbf{a} = (0, 1)$. Then,

$$\int_{U_k(\alpha,\beta)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = p^{2\alpha-\beta}\frac{p-1}{p}\int_{\mathbf{Z}_p^*} \exp(2\pi i u/p^{\alpha})\,du$$

and the last integral has already been calculated when we studied the case of $\mathbf{P}^n$: one finds 0 if $\alpha \geqslant 2$ and $-1/p$ if $\alpha = 1$. But $\alpha > \beta \geqslant 1$, so $\alpha \neq 1$. This proves formulas (4.6a) and (4.7a).

Similarly,

$$\int_{U_k(\alpha)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = p^{\alpha}\int_{\mathbf{Z}_p^*} \exp(2\pi i u/p^{\alpha})\,du$$

is -1 for $\alpha = 1$ and 0 else. Formulas (4.6b) and (4.7b) are therefore proved.

We now treat the case of a character $\mathbf{a}$ which is special for ℓ_k. A unitary change of variables allows to assume $\ell_k(\mathbf{x}) = x_1$ and $\mathbf{a} = (1, 0)$. Then,

$$\int_{U_k(\alpha,\beta)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = p^{2\alpha-\beta}\frac{p-1}{p}\int_{\mathbf{Z}_p^*} \exp(2\pi i x/p^{\alpha-\beta})$$

is 0 if $\alpha - \beta \neq 1$ and is equal to

$$p^{2\alpha-\alpha+1}\frac{p-1}{p}\frac{(-1)}{p} = -p^{\alpha}\frac{p-1}{p}$$

if $\alpha = \beta - 1$, as stated in (4.7c). Equation (4.7d) follows from

$$\int_{U_k(\alpha)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbf{Z}_p\times p^{-\alpha}\mathbf{Z}_p^*} \exp(2\pi i x_1)\,\mathrm{d}\mathbf{x} = p^{\alpha}\frac{p-1}{p}.$$

To compute the volume of $U(\alpha)$, it is useful to remark that $U(\alpha)$ is $p^{-\alpha}$ times the complementary subset in $\mathbf{Z}_p^2$ of $1 + (p-1)r$ disjoint balls of radius $1/p$. Therefore,

$$\operatorname{vol} U(\alpha) = p^{2\alpha}\big(1 - \frac{1+(p-1)r}{p^2}\big) = p^{2\alpha}\frac{(p-1)(p+1-r)}{p^2},$$

as in formula (4.5c).

If $p \nmid \mathbf{a}$, remark that the integral of $\psi_{\mathbf{a}}$ over $p^{-\alpha}(\mathbf{Z}_p^2 \setminus p\mathbf{Z}_p^2)$ is -1 for $\alpha = 1$ and 0 for $\alpha \geqslant 2$. We now need to substract the integrals over the disjoint subsets $U_k(\alpha, \beta)$ and $U_k(\alpha)$.

For a generic character, one gets 0 if $\alpha \geqslant 2$ and $-1+r$ if $\alpha = 1$; this establishes formula (4.6c). Finally, if $\mathbf{a}$ is special for ℓ_k, one has

$$\int_{U(1)} \psi_{\mathbf{a}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = -1 + (r-1) - (p-1) = -(p+1-r)$$

and $\int_{U(\alpha)} \psi_{\mathbf{a}} = 0$ for $\alpha \geqslant 2$, as claimed in (4.7e). □

4.8. The local Fourier transform at ψ_0. — We still assume $p \notin S$ and compute the local Fourier transform at the trivial character ψ_0. By the general result 2.10 on Fourier transforms of height functions, $H(\mathbf{s};\cdot)^{-1}$ is integrable on $\mathbf{G}_a^n(\mathbf{Q}_p)$ as soon as $\mathrm{Re}(s_0) > 2$ and all $\mathrm{Re}(s_k) > 1$. We then have:

$$\hat{H}(\mathbf{s};\psi_0) = \int_{U(0)} + \left(\sum_{k=1}^{r} \sum_{1\leqslant\beta<\alpha} \int_{U_k(\alpha,\beta)} + \sum_{\alpha=1}^{\infty} \int_{U_k(\alpha)} \right) + \sum_{\alpha=1}^{\infty} \int_{U(\alpha)}$$

and we compute each sum separately. The integral over $U(0)$ is 1. Now, for a fixed k, the integral over all $U_k(\alpha,\beta)$ is

$$\begin{aligned}
\sum_{1\leqslant\beta<\alpha} \int_{U_k(\alpha,\beta)} &= \frac{(p-1)^2}{p^2} \sum_{1\leqslant\beta<\alpha} p^{-\alpha s_0} p^{-\beta(s_k-s_0)} p^{2\alpha} p^{-\beta} \\
&= \frac{(p-1)^2}{p^2} \sum_{\beta=1}^{\infty} p^{-\beta(s_k-s_0+1)} \sum_{\alpha=\beta+1}^{\infty} p^{-\alpha(s_0-2)} \\
&= \frac{(p-1)^2}{p^2} \sum_{\beta=1}^{\infty} p^{-\beta(s_k-s_0+1)} p^{-\beta(s_0-2)} \frac{1}{p^{s_0-2}-1} \\
&= \frac{(p-1)^2}{p^2} \frac{1}{p^{s_0-2}-1} \sum_{\beta=1}^{\infty} p^{-\beta(s_k-1)} \\
&= \frac{(p-1)^2}{p^2} \frac{1}{p^{s_0-2}-1} \frac{1}{p^{s_k-1}-1}
\end{aligned}$$

The sum over all $U_k(\alpha)$ (k fixed) equals

$$\sum_{\alpha=1}^{\infty} \int_{U_k(\alpha)} = \frac{p-1}{p} \sum_{\alpha=1}^{\infty} p^{-\alpha s_k} p^{\alpha} = \frac{p-1}{p} \sum_{\alpha=1}^{\infty} p^{-\alpha(s_k-1)} = \frac{p-1}{p} \frac{1}{p^{s_k-1}-1}$$

Finally, the sum over all $U(\alpha)$ is

$$\sum_{\alpha=1}^{\infty}\int_{U(\alpha)} = \frac{(p-1)(p+1-r)}{p^2}\sum_{\alpha=1}^{\infty} p^{-\alpha s_0}p^{2\alpha} = \frac{(p-1)(p+1-r)}{p^2}\sum_{\alpha=1}^{\infty} p^{-\alpha(s_0-2)}$$
$$= \frac{(p-1)(p+1-r)}{p^2}\frac{1}{p^{s_0-2}-1} = \frac{(p-1)(p+1-r)}{p^2}\frac{1}{p^{s_0-2}-1}$$

Putting all this together, we have

$$\begin{aligned}\hat{H}(\mathbf{s};\psi_0) &= 1+\frac{p-1}{p^2}\frac{1}{p^{s_0-2}-1}\sum_{k=1}^{r}\frac{1}{p^{s_k-1}-1}\big((p-1)+p(p^{s_0-2}-1)\big)\\ &\quad + \frac{(p-1)(p+1-r)}{p^2}\frac{1}{p^{s_0-2}-1}\\ &= 1+\frac{p-1}{p^2}\frac{p^{s_0-1}-1}{p^{s_0-2}-1}\sum_{k=1}^{r}\frac{1}{p^{s_k-1}-1}+\frac{(p-1)(p+1-r)}{p^2}\frac{1}{p^{s_0-2}-1}\\ &= 1+\frac{p-1}{p^2}\frac{1}{p^{s_0-2}-1}\left((p+1-r)+\sum_{k=1}^{r}\frac{p^{s_0-1}-1}{p^{s_k-1}-1}\right)\\ &= 1+\frac{p^2-1}{p^{s_0}-p^2}+\frac{p-1}{p^{s_0}-p^2}\sum_{k=1}^{r}\frac{p^{s_k-1}-p^{s_0-1}}{p^{s_k-1}-1} \qquad (4.9)\end{aligned}$$

We remark that if $(s_0,s_1,\dots,s_r)=(3,2,\dots,2)$, corresponding to the anti-canonical class K_X^{-1}, this gives

$$\hat{H}(K_X^{-1},\psi_0) = 1+\frac{p^2-1}{p^3-p^2}+r\frac{p-1}{p^3-p^2}\frac{p^2-p}{p-1} = 1+\frac{r+1}{p}+\frac{1}{p^2}=\frac{1}{p^2}\#X(\mathbf{F}_p),$$

the expected local density at p.

4.10. The local Fourier transform at a generic character. — Let $\mathbf{a}$ be a generic character and $p\notin S(\mathbf{a})$. In that case, the summation is easier. The integrals over $U_k(\alpha,\beta)$ are 0, as are the integrals over $U_k(\alpha)$ or $U(\alpha)$ if $\alpha\geqslant 2$. Therefore

$$\hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = 1-\sum_{k=1}^{r}p^{-s_k}+(r-1)p^{-s_0}. \qquad (4.11)$$

For K_X^{-s}, this specializes to

$$\hat{H}(K_X^{-s},\psi_{\mathbf{a}}) = 1-rp^{-2s}+(r-1)p^{-3s}.$$

4.12. The local Fourier transform at a special character. — If $\mathbf{a}$ is special for ℓ_k and $p \notin S(\mathbf{a})$, it behaves as if it were generic for the other ℓ_j. Besides $U(0)$, $U(1)$ and $U_j(1)$ for $j \neq k$, remain the integrals over $U_k(\alpha, \alpha - 1)$ for $\alpha \geqslant 2$ and the one over $U(\alpha)$ for $\alpha \geqslant 2$.

$$\begin{aligned}
\hat{H}(\mathbf{s}; \psi_{\mathbf{a}}) &= 1 - \sum_{j \neq k} p^{-s_j} + (r - p - 1) p^{-s_0} \\
&\quad + \sum_{\alpha=1}^{\infty} p^{\alpha} \frac{p-1}{p} p^{-\alpha s_k} - \sum_{\alpha \geqslant 2} p^{\alpha} \frac{p-1}{p} p^{-\alpha s_0 - (\alpha - 1)(s_k - s_0)} \\
&= 1 - \sum_{j \neq k} p^{-s_j} + (r - p - 1) p^{-s_0} \\
&\quad + \frac{p-1}{p} \frac{1}{p^{s_k - 1} - 1} - \frac{p-1}{p} p^{s_k - s_0} \frac{p^{-s_k - 1}}{p^{s_k - 1} - 1} \\
&= 1 - \sum_{j \neq k} p^{-s_j} + (r - p - 1) p^{-s_0} + (p-1)(1 - p^{1 - s_0}) \frac{1}{p^{s_k} - p} \qquad (4.13)
\end{aligned}$$

For K_X^{-s}, one has

$$\hat{H}(K_X^{-s}, \psi_{\mathbf{a}}) = 1 - (r-1) p^{-2s} + (r - p - 1) p^{-3s} + (p-1) \frac{1 - p^{1-3s}}{p^{2s} - 1}.$$

4.14. Bad reduction, general estimates. — If $p \in S(\mathbf{a})$, then the previous analysis doesn't say anything about the behaviour of $\hat{H}(\mathbf{s}; \psi_{\mathbf{a}})$. However, for any compact contained in the domain of integrability of the height function, there is a uniform estimate

$$\left| \hat{H}_p(\mathbf{s}; \psi_{\mathbf{a}}) \right| \leqslant C$$

where the constant C doesn't depend on $p \in S$. For $p \in S(\mathbf{a}) \setminus S$, we replace $\psi_{\mathbf{a}}$ by 1 and insert the estimates for the trivial character.

It follows that

$$\prod_{p \in S(\mathbf{a})} \left| \hat{H}_p(\mathbf{s}; \psi_{\mathbf{a}}) \right| \leqslant C'(1 + \|a\|)^{\kappa}$$

for some constant C' and some $\kappa > 0$.

4.15. Meromorphic continuation. — We split the sum over all characters in $r+2$ parts: the trivial character is treated separately; then the generic characters; then the characters which are special for ℓ_k, k varying from 1 to r:

$$\sum_{\mathbf{a}} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = \hat{H}(\mathbf{s};\psi_0) + \sum_{\mathbf{a}\text{ generic}} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}) + \sum_{k=1}^{r} \sum_{\mathbf{a}\text{ special for }\ell_k} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}).$$

Let Z_0, Z_{gen} and Z_k (for $1 \leqslant k \leqslant r$) be the functions defined by the corresponding series.

Each global Fourier transform at a generic character defines a holomorphic function of $\mathbf{s}$ in the domain $\mathrm{Re}(s_0) > 2$ and $\mathrm{Re}(s_k) > 1$ for all $1 \leqslant k \leqslant r$. Moreover, the estimate at infinity 2.13 ensures that the sum over all generic characters converges locally uniformly. Therefore, Z_{gen} extends to a holomorphic function in that domain.

For the characters which are special for ℓ_k, the product of the local Fourier transform defines a holomorphic function of $\mathbf{s}$ in the domain $\mathrm{Re}(s_0) > 2$, $\mathrm{Re}(s_j) > 1$ if $j \neq k$ and $\mathrm{Re}(s_k) > 2$. It extends to a meromorphic function in the domain $\mathrm{Re}(s_0) > 2$, $\mathrm{Re}(s_j) > 1$ and $\mathrm{Re}(s_k) > 1$ with a simple pole along the hypersurface $s_k = 2$.

And finally, for the trivial character, we have absolute convergence for $\mathrm{Re}(s_0) > 3$ and $\mathrm{Re}(s_k) > 2$ for $1 \leqslant k \leqslant r$, and meromorphic continuation to the domain $\mathrm{Re}(s_0) > 2$, $\mathrm{Re}(s_k) > 1$, with principal part $1/(s_0-3)(s_1-2)\dots(s_r-2)$.

The estimates of Prop. 2.13 as well as standard estimates for the growth of the Riemann zeta function in vertical strips imply that (away from poles) $Z(\mathbf{s})$ has polynomial growth in vertical strips.

Therefore, we have proven the following theorem:

Theorem 4.16. — *The height zeta function $Z(\mathbf{s})$ converges in the domain $\mathrm{Re}(s_0) > 3$, $\mathrm{Re}(s_k) > 2$. Moreover, there exists a holomorphic function g in the domain $\mathrm{Re}(s_0) > 2$, $\mathrm{Re}(s_k) > 1$ such that*

$$Z(\mathbf{s}) = g(\mathbf{s}) \frac{1}{(s_0-3)(s_1-2)\dots(s_r-2)}$$

and $g(K_X^{-1}) \neq 0$. Moreover, g has polynomial growth in vertical strips.

Corollary 4.17. — *There exist a polynomial P_X of degree r and an $\alpha > 0$ such that the number of rational points in U of anticanonical height $\leqslant B$ satisfies*

$$N(U, -K_X, B) = BP_X(\log B) + O(B^{1-\alpha}).$$

Moreover, if $\tau(K_X)$ denotes the Tamagawa number, the leading coefficient of P_X is equal to

$$\frac{1}{r!}\frac{\tau(K_X)}{3\cdot 2^r},$$

as predicted by Peyre's refinement of Manin's conjecture.

TABLE 1

	$U(0)$	$U_k(\alpha,\beta)$	$U_k(\alpha)$	$U(\alpha)$
volume	1	$p^{2\alpha-\beta}\frac{(p-1)^2}{p^2}$	$p^{\alpha}\frac{p-1}{p}$	$p^{2\alpha}\frac{(p-1)(p+1-r)}{p^2}$
H_0	1	$p^{\alpha-\beta}$	1	p^{α}
H_j $(j\neq k)$	1	1	1	1
H_k	1	p^{β}	p^{α}	1
$H(\mathbf{s};\cdot)$	1	$p^{\alpha s_0+\beta(s_k-s_0)}$	$p^{\alpha s_k}$	$p^{\alpha s_0}$
Integrals of a generic character $\psi_{\mathbf{a}}$				
$\alpha=1$	1		-1	$-1+r$
$\alpha\geqslant 2$	1	0	0	0
Integrals of a character $\psi_{\mathbf{a}}$ special for ℓ_k				
any α	1		$p^{\alpha}\frac{p-1}{p}$	
$\alpha=1$				$-(p+1-r)$
$\alpha\geqslant 2$			0	
$\alpha=\beta-1$		$-p^{\alpha}\frac{p-1}{p}$		

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Antoine Chambert-Loir, Institut de mathématiques de Jussieu, Boite 247, 4, place Jussieu, F-75252 Paris Cedex 05 • *E-mail :* `chambert@math.jussieu.fr`

Yuri Tschinkel, Dept. of Mathematics, U.I.C., Chicago, (IL) 60607-7045, U.S.A. *E-mail :* `yuri@math.uic.edu`