De Rham cohomology of varieties over fields of positive characteristic

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Abstract. These are the elaborated notes of two talks given at the Summer School in Göttingen on Higher-Dimensional Geometry over Finite Fields. We study the De Rham cohomology of smooth and proper varieties over fields of positive characteristic in case that the Hodge spectral sequence degenerates. The De Rham cohomology carries the structure of a so-called F-zip. We explain two classifications of F-zips, one stems from representation theory of algebras and the other one uses algebraic groups and their compactifications. We show how this second classification can be extended if the De Rham cohomology is endowed with a symplectic or a symmetric pairing. Throughout we illustrate the theory via the examples of (polarized) abelian varieties and (polarized) K3-surfaces.

Introduction

In this article the De Rham cohomology of certain varieties over fields of positive characteristic is studied. The text is aimed mainly at students and non-specialists having some familiarity with the usual techniques from algebraic geometry. The emphasis will be on precise definitions and examples, in particular abelian varieties and K3-surfaces. For the proofs often only a reference to the literature is given.

To put the theory for varieties over fields of positive characteristic into perspective let us first look at complex varieties: Let X be a smooth and proper scheme over the complex numbers and denote by X^{an} the associated compact complex manifold. The De Rham cohomology $H^\bullet_{\mathrm{DR}}(X/\mathbb{C})$ is a finite complex vector space. It carries several additional structures (recalled in more detail in (1.3)).

(I) As a complex vector space $H^\bullet_{\mathrm{DR}}(X/\mathbb{C})$ is isomorphic to the singular cohomology $H^\bullet(X^{\mathrm{an}}, \mathbb{C}) = H^\bullet(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ and therefore has a $\mathbb{Z}$-rational structure.

(II) The Hodge spectral sequence degenerates and $H^\bullet_{\mathrm{DR}}(X/\mathbb{C})$ is therefore endowed with the Hodge filtration whose successive quotients are the spaces $H^a(X, \Omega^b_{X/\mathbb{C}})$.

(III) The complex conjugate of the Hodge filtration is opposed to the Hodge filtration and therefore defines a canonical splitting of the Hodge filtration.

In other words, $H^\bullet_{\mathrm{DR}}(X/\mathbb{C})$ is endowed with an integral Hodge structure in the sense of [De1].

To know the $\mathbb{R}$-Hodge structure on $H^\bullet_{\mathrm{DR}}(X/\mathbb{C})$ is equivalent to the knowledge of the Hodge numbers $h^{ab}(X) = \dim_{\mathbb{C}}(H^a(X, \Omega^b_{X/\mathbb{C}}))$. This is therefore a discrete invariant. Moreover, the Hodge numbers are locally constant in families, i.e., if $X \to S$ is a smooth proper morphism of schemes of finite type over $\mathbb{C}$ the map $S(\mathbb{C}) \ni s \mapsto h^{ab}(X_s)$ is locally constant.

The integral Hodge structure is a much finer invariant. It varies in families and the study of "moduli spaces" of integral Hodge structures leads to Griffiths' period domains (see e.g. [BP]).

Consider the example of two abelian varieties X and Y over $\mathbb{C}$. Then $H^1_{\mathrm{DR}}(X/\mathbb{C})$ and $H^1_{\mathrm{DR}}(Y/\mathbb{C})$ with their real Hodge structures are isomorphic if and only if $\dim(X) = \dim(Y)$. If $H^1_{\mathrm{DR}}(X/\mathbb{C})$ and $H^1_{\mathrm{DR}}(Y/\mathbb{C})$ with their integral Hodge structures are isomorphic, X and Y are isomorphic (called to global Torelli property for abelian varieties).

Now let X be a smooth and proper scheme over an algebraically closed field k of characteristic $p > 0$. Here we do not have any analogue of the property (I) of the De Rham cohomology in the complex case (see however below). Moreover the Hodge spectral sequence does not degenerate in general, hence the analogue of property (II) does not hold. But it turns out that still many interesting varieties (e.g., abelian varieties, smooth complete intersections in the projective space, K3-surfaces, toric varieties) have property (II) (see (1.5)) and therefore we will just assume in this article that the Hodge spectral sequence degenerates. The general case has not been studied much. A notable exception is Ogus' paper [Og4].

Now consider property (III). First of all there is of course no complex conjugation and therefore no verbatim analogue of property (III). But there is another filtration on the De Rham cohomology, namely the filtration given by the second spectral sequence of hypercohomology. It plays a somewhat analogue role to the complex conjugate of the Hodge spectral sequence for varieties over $\mathbb{C}$ and therefore we call it the conjugate spectral sequence (following Katz [Ka]). The initial terms of the Hodge and the conjugate spectral sequences are linked by the Cartier isomorphism and it follows from our assumption that the conjugate spectral sequence degenerates as well (see (1.6)). Therefore we have a "conjugate" filtration on $H^\bullet_{\mathrm{DR}}(X/k)$. But it is not true in general that Hodge and conjugate filtration are opposed to each other. In fact this characterizes ordinary varieties (see (1.9)). In general we obtain the structure of an F-zip on $H^\bullet_{\mathrm{DR}}(X/k)$, a notion first defined in [MW] and recalled in (1.7). There are two ways to classify F-zips, both explained in Chapter 3. It turns out that the isomorphism class of the De Rham cohomology endowed with its F-zip structure is still a discrete invariant but it is not locally constant in families.

Again we illustrate this with the example of abelian varieties. For an abelian variety X over k of dimension g there are 2^g possible F-zip structures on $H^1_{\mathrm{DR}}(X/k)$ and in fact all of them occur (see (5.1)). In particular for an elliptic curve E there are two cases and this is the distinction whether E is ordinary or supersingular.

We conclude this introductory part with some remarks on the nonexistent analogue of property (I). To get an "integral" structure the best candidate is the crystalline cohomology $H^\bullet_{\mathrm{cris}}(X/W)$ which is W-module where $W = W(k)$ is the ring of Witt vectors of k. The relative Frobenius on X induces a semilinear map ϕ on $H^\bullet_{\mathrm{cris}}(X/W)$. If $H^\bullet_{\mathrm{cris}}(X/W)$ is a free W-module, there is a functorial isomorphism $\alpha\colon H^\bullet_{\mathrm{cris}}(X/W) \otimes_W k \xrightarrow{\sim} H^\bullet_{\mathrm{DR}}(X/k)$. In this case ϕ defines on $H^\bullet_{\mathrm{cris}}(X/W) \otimes_W k$ the structure of an F-zip such that α is an isomorphism of F-zips. Therefore we could consider $H^\bullet_{\mathrm{cris}}(X/W)$ as an "integral" version of $H^\bullet_{\mathrm{DR}}(X/k)$. Again this is a much finer invariant and it would be desirable to construct moduli spaces of such structures.

We give now a more detailed description of the content of this article. In the first chapter we recall the definition of the De Rham cohomology of an algebraic variety and of the two canonical spectral sequences converging to it. Then we consider the case of complex varieties before turning towards the actual focus of this article, the De Rham cohomology of varieties over fields of positive characteristic and its F-zip structure. In the second chapter we detail two examples: Abelian varieties (and – more generally – at level 1 truncated Barsotti-Tate groups) and K3-surfaces.

The third chapter is dedicated to the two classifications of F-zips. The first stems from the field of representation theory of algebras and is essentially due to Gelfand and Ponomarev [GP] and Crawley-Boevey [Cr]. The second uses the theory of algebraic groups and has been given in [MW]. It relates the classification problem to the study of a Frobenius-linear version of the wonderful compactification of the projective linear group. We link both classifications (3.5) and illustrate both of them by classifying at level 1 truncated Barsotti-Tate groups (3.7) and (3.8).

Often the De Rham cohomology carries additional structures like symplectic or symmetric pairings (e.g., induced by polarizations or Poincaré duality). Therefore we define in Chapter 4 symplectic and orthogonal F-zips. Now the essential ingredient for the second classification has an analogue for arbitrary reductive groups G. Specializing G to the symplectic and the orthogonal group we obtain a description of isomorphism classes of symplectic and orthogonal F-zips. In Chapter 5 this is applied to the study of the De Rham cohomology of polarized abelian varieties and of polarized K3-surfaces.

The sixth chapter deals with families of F-zips over arbitrary schemes S of characteristic p. It is shown that every F-zip over S defines a decomposition in locally closed subschemes, namely in the loci where the "isomorphism class of the F-zip is constant" (see (6.1) for two (equivalent) precise definitions). We conclude with glancing at this decomposition for the special case that S is the moduli space of principally polarized abelian varieties or the moduli space of polarized K3-surfaces.

In an appendix we recall the necessary notions and results about reductive groups which are used for the second classification.

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Notation: Fix a prime number $p > 0$. We will always denote by k a field which we assume to be perfect of characteristic p from (1.5) on. If not otherwise specified, X will be a smooth proper scheme over k which usually will satisfy a further condition (deg) from (1.5) on.

1. De Rham cohomology of varieties over fields of positive characteristic

(1.1) Hypercohomology.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and assume that $\mathcal{A}$ has sufficiently many injective objects. Let $T\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Recall that an object M in $\mathcal{A}$ is called *T-acyclic* if all derived functors $\mathrm{R}^i T(M) = 0$ for $i > 0$. An object in $\mathcal{A}$ is injective if and only if it is T-acyclic for all abelian categories $\mathcal{B}$ and for all left exact functors $T\colon \mathcal{A} \to \mathcal{B}$.

Let K be a complex of objects in $\mathcal{A}$ which is bounded below. The hypercohomology $RT(K)$ can be defined as follows. Choose a quasi-isomorphism $K \to I$ to a complex of T-acyclic objects I^j in $\mathcal{A}$ (e.g., one can choose a quasi-isomorphism of K to a complex of injective objects). Then set

$$\mathbb{R}^j T(K) := H^j(T(I)).$$

This definition is independent of the choice of $K \to I$. Clearly, each quasi-isomorphism $K \to K'$ induces an isomorphism $\mathbb{R}^j T(K) \xrightarrow{\sim} \mathbb{R}^j T(K')$. Note that if $K[a]$ ($a \in \mathbb{Z}$) is the complex $K[a]^i = K^{a+i}$ and $d_{K[a]} = (-1)^a d_K$, we have $\mathbb{R}^j T(K[a]) = \mathbb{R}^{j+a} T(M)$. If M is an object in $\mathcal{A}$, considered as a complex concentrated in degree zero, we have $\mathbb{R}^j T(M) = \mathrm{R}^j T(M)$.

Assume that K is endowed with a descending filtration of subcomplexes

$$K \supset \dots \supset \mathrm{Fil}^i(K) \supset \mathrm{Fil}^{i+1}(K) \supset \dots$$

which is biregular (i.e., the filtration $\mathrm{Fil}^\bullet K^j$ induced on each component satisfies $\mathrm{Fil}^i K^j = K^j$ for i sufficiently small and $\mathrm{Fil}^i K^j = 0$ for i sufficiently big). In that case we obtain a spectral sequence which converges to the hypercohomology of K (see e.g. [De1] 1.4)

$$E_1^{ab} = \mathbb{R}^{a+b} T(\mathrm{Fil}^a(K)/\mathrm{Fil}^{a+1}(K)) \implies \mathbb{R}^{a+b} T(K). \qquad (1.1.1)$$

On every complex there are (at least) two natural filtrations. First one can consider the so-called "naive" filtration $(\sigma^{\geq i}(K))_i$ given by

$$\sigma^{\geq i}(K)^j = \begin{cases} 0, & j < i; \\ K^j, & j \geq i. \end{cases}$$

In that case, $\mathrm{Fil}^a(K)/\mathrm{Fil}^{a+1}(K) = K^a[-a]$ is the complex consisting of the object K^a in degree a, and the spectral sequence (1.1.1) becomes

$$E_1^{ab} = \mathrm{R}^b T(K^a) \implies \mathbb{R}^{a+b}T(K), \tag{1.1.2}$$

which is sometimes also called the *first spectral sequence of hypercohomology.*

The second natural filtration is the (ascending) canonical filtration $(\tau_{\leq i}(K))_i$ given by

$$\tau_{\leq i}(K)^j = \begin{cases} K^j, & j < i; \\ \mathrm{Ker}(K^i \xrightarrow{d} K^{i+1}), & j = i; \\ 0, & j > i. \end{cases}$$

To get a descending filtration we set $\mathrm{Fil}^a K := \tau_{\leq -a}(K)$. In this case we have $\mathrm{Fil}^a(K)/\mathrm{Fil}^{a+1}(K) = H^{-a}K[a]$ and by (1.1.1) we obtain a spectral sequence $E_1^{ab} = \mathbb{R}^{2a+b}T(H^{-a}K) \implies \mathbb{R}^{a+b}T(K)$. Replacing E_r^{ab} by $E_{r+1}^{-b,a+2b}$ we get a spectral sequence

$$E_2^{ab} = \mathrm{R}^a T(H^b(K)) \implies \mathbb{R}^{a+b}T(K), \tag{1.1.3}$$

which is sometimes also called the *second spectral sequence of hypercohomology.*

(1.2) De Rham cohomology.

We apply the general remarks on hypercohomology to the de Rham complex. Let k be a field and let X be any scheme of finite type over k. Denote by $\Omega^\bullet_{X/k}$ the de Rham complex

$$0 \to \mathscr{O}_X \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \dots$$

of X over k [EGA] IV, (16.6). As X is locally of finite type, $\Omega^1_{X/k}$ is a coherent $\mathscr{O}_X$-module [EGA] IV, (16.3.9). As X is also quasi-compact, $\Omega^j_{X/k} = \bigwedge^j \Omega^1_{X/k}$ is zero for $j \gg 0$. The de Rham complex is a complex in the abelian category $\mathcal{A}$ of sheaves on X with values in k-vector spaces (it is not a complex in the category of $\mathscr{O}_X$-modules as the differentials are not $\mathscr{O}_X$-linear). Denote by $T = \Gamma$ the functor of global sections from $\mathcal{A}$ to the category of k-vector spaces. As usual we write $H^i(X, M)$ and $\mathbb{H}^i(X, K)$ instead of $\mathrm{R}^i\Gamma(M)$ and $\mathbb{R}^i\Gamma(K)$ (where M is an object in $\mathcal{A}$ and K is a complex in $\mathcal{A}$).

For each integer $i \geq 0$ the *i-th De Rham cohomology of X over k* is defined as

$$H^i_{\mathrm{DR}}(X/k) := \mathbb{H}^i(X, \Omega^\bullet_{X/k}).$$

Then (1.1.2) and (1.1.3) give spectral sequences

$${}'E_1^{ab} = H^b(X, \Omega^a_{X/k}) \implies H^{a+b}_{\mathrm{DR}}(X/k), \tag{1.2.1}$$

$${}''E_2^{ab} = H^a(X, \mathscr{H}^b(\Omega^\bullet_{X/k})) \implies H^{a+b}_{\mathrm{DR}}(X/k). \tag{1.2.2}$$

Here we denote by $\mathscr{H}^b(\Omega^\bullet_{X/k})$ the b-th cohomology sheaf of the complex $\Omega^\bullet_{X/k}$.

The first spectral sequence is usually called the *Hodge spectral sequence*. As explained above, there exists an integer $A > 0$ such that $\Omega^a_{X/k} = 0$ for $a > A$. Therefore we have

$$'E_1^{ab} = 0, \qquad \text{for } a > A,\, b > \dim(X). \tag{1.2.3}$$

By definition of a spectral sequence, the limit terms $H^{a+b}_{\mathrm{DR}}(X/k)$ are endowed with a descending filtration $'\,\mathrm{Fil}^\bullet$ such that $'\mathrm{Fil}^a / {'\mathrm{Fil}^{a+1}} = {'E^{ab}_\infty}$. As $'E^{ab}_\infty$ is a subquotient of $'E^{ab}_1$, this filtration is finite and we see that $H^n_{\mathrm{DR}}(X/k) = 0$ for all $n > A + \dim(X)$.

If X is smooth of pure dimension d over k, $\Omega^1_{X/k}$ is a locally free $\mathscr{O}_X$-module of rank d and therefore $\Omega^a_{X/k} = 0$ for $a > d$.

If X is proper over k, the initial terms in the Hodge spectral sequence are finite-dimensional vector spaces and therefore also their subquotients $'E^{ab}_\infty$ are finite-dimensional. Therefore for all $n \geq 0$ we have

$$\dim_k(H^n_{\mathrm{DR}}(X/k)) < \infty,$$

if X is a proper k-scheme.

(1.3) De Rham cohomology of complex varieties.

We first study the case that X is a smooth and proper scheme over the field of complex numbers $\mathbb{C}$. In that case we use the GAGA principle ([Se1] and [SGA1] Exp. XII): There is a compact complex manifold X^{an} and a morphism $i_X\colon X^{\mathrm{an}} \to X$ of locally ringed spaces such that for every complex analytic space Y and every morphism $\phi\colon Y \to X$ of locally ringed spaces there exists a unique morphism of complex analytic spaces $\psi\colon Y \to X^{\mathrm{an}}$ such that $i_X \circ \psi = \phi$. Serre's GAGA theorems tells us that $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}} := i_X^*\mathscr{F}$ is an equivalance of categories between the category of coherent $\mathscr{O}_X$-modules and the category of coherent $\mathscr{O}_{X^{\mathrm{an}}}$-modules. Moreover, i_X^* induces isomorphisms of finite-dimensional $\mathbb{C}$-vector spaces

$$H^a(X, \mathscr{F}) \xrightarrow{\sim} H^a(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}).$$

We apply this to the coherent module $\Omega^b_{X/\mathbb{C}}$. In that case $(\Omega^b_{X/\mathbb{C}})^{\mathrm{an}}$ is the sheaf $\Omega^b_{X^{\mathrm{an}}}$ of holomorphic b-forms on X^{an}. Clearly i_X^* is compatible with the naive filtration on the De Rham complex. We get a homomorphism of the Hodge spectral sequences which by the GAGA principle is an isomorphism on its initial terms (in the analytic setting the Hodge spectral sequence is sometimes also called the Hodge-Fröhlicher spectral sequence). Therefore i_X^* induces an isomorphism of filtered $\mathbb{C}$-vector spaces

$$H^n_{\mathrm{DR}}(X/k) \xrightarrow{\sim} H^n_{\mathrm{DR}}(X^{\mathrm{an}}) := \mathbb{H}^n(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}).$$

There is a rich history of the study of De Rham cohomology of komplex analytic spaces, both analytic and algebraic. We refer to [Voi], [Dm], and [Ill2] and the references given there for an extensive discussion. Here we recall just some well-known facts from complex geometry.

Degeneration of the Hodge spectral sequence.

First Fact over the complex numbers 1.1. *The Hodge spectral sequence* $'E_1^{ab} = H^b(X^{\mathrm{an}}, \Omega^a_{X^{\mathrm{an}}}) \implies H^{a+b}_{\mathrm{DR}}(X^{\mathrm{an}})$ *degenerates at* $'E_1$.

In particular the (descending) Hodge filtration $C^\bullet$ on $H^n_{\mathrm{DR}}(X^{\mathrm{an}})$ given by the Hodge spectral sequence has as graded pieces

$$C^i/C^{i+1} = H^{n-i}(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}}) = H^{n-i}(X, \Omega^i_{X/\mathbb{C}}).$$

Poincaré lemma.
The second fact is the Poincaré lemma. For this we consider the natural embedding $\mathbb{C}_{X^{\mathrm{an}}} \hookrightarrow \mathscr{O}_{X^{\mathrm{an}}}$, where $\mathbb{C}_{X^{\mathrm{an}}}$ denotes the sheaf of locally constant complex valued functions on X^{an}. The Poincaré lemma says:

Second Fact over the complex numbers 1.2. *The homomorphism* $\mathbb{C}_{X^{\mathrm{an}}} \to \Omega^\bullet_{X^{\mathrm{an}}}$ *is a quasi-isomorphism.*

Hence the initial terms of $''E_2^{ab} = H^a(X^{\mathrm{an}}, \mathscr{H}^b(\Omega^\bullet_{X^{\mathrm{an}}}))$ are zero for $b > 0$, and therefore this spectral sequence is trivial. The quasi-isomorphism $\mathbb{C}_{X^{\mathrm{an}}} \to \Omega^\bullet_{X^{\mathrm{an}}}$ induces an isomorphism

$$H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) = H^n(X^{\mathrm{an}}, \mathbb{C}_{X^{\mathrm{an}}}) \xrightarrow{\sim} H^n_{\mathrm{DR}}(X^{\mathrm{an}}) = H^n_{\mathrm{DR}}(X/\mathbb{C}),$$

where the left hand side denotes the singular cohomology and the first equality is a standard fact from topology (e.g., [Br] III, §1).

The conjugate filtration.
Note that $H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) = H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, and the complex conjugation on $\mathbb{C}$ induces an $\mathbb{R}$-linear map $\sigma \otimes z \mapsto \sigma \otimes \bar{z}$ on $H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$. If $W \subset H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$ is any complex subspace, its image $\overline{W}$ under this map is again a complex subspace. We define an ascending filtration $D_\bullet$ on $H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$ by setting

$$D_i := \overline{C^{n-i}}.$$

We call this filtration the conjugate filtation. Hodge theory now provides the following fact.

Third Fact over the complex numbers 1.3. *For all* $i \in \mathbb{Z}$ *we have*

$$D_{i-1} \oplus C^i = H^n_{\mathrm{DR}}(X^{\mathrm{an}}).$$

(1.4) Varieties over fields of positive characteristic.

Let now k be a field of positive characteristic p and let X be a k-scheme.

For any $\mathbb{F}_p$-scheme S we denote by $\mathrm{Frob}_S \colon S \to S$ the absolute Frobenius of S (i.e., Frob_S is the identity on the underlying topological spaces and sends a local section x of $\mathscr{O}_S$ to x^p). To shorten notations we denote by $\sigma \colon k \to k$ the Frobenius $a \mapsto a^p$ on k and also the Frobenius morphism $\sigma = \mathrm{Frob}_{\mathrm{Spec}(k)}$. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "F"] \arrow[drr, bend left, "\mathrm{Frob}_X"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(p)} \arrow[r, "\sigma_X"] \arrow[d, "f^{(p)}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"'] \\
& \mathrm{Spec}(k) \arrow[r, "\sigma"] & \mathrm{Spec}(k),
\end{tikzcd} \tag{1.4.1}$$

where $X^{(p)}$ is defined be the cartesian square and $F = F_{X/k}$ is the relative Frobenius of X over k, i.e., unique morphism making the above diagram commutative.

We describe this diagram locally: Assume that $X = \mathrm{Spec}(A)$ is affine. Via the choice of generators of A as a k-algebra, we can identify A with $k[\underline{X}]/(\underline{f})$ where $\underline{X} = (X_i)_{i \in I}$ is a tuple of indeterminates and $\underline{f} = (f_j)_{j \in J}$ is a tuple of polynomials in $k[\underline{X}]$. Then the diagram (1.4.1) is given by:

- $X^{(p)} = \mathrm{Spec}(A^{(p)})$ with $A^{(p)} = k[\underline{X}]/(f_j^{(p)}; j \in J)$, where for any polynomial $f = \sum_{\nu \in \mathbb{N}_0^{(I)}} a_\nu X^\nu \in k[\underline{X}]$ we set $f^{(p)} = \sum_{\nu \in \mathbb{N}_0^{(I)}} a_\nu^p X^\nu$.
- The morphism $\sigma_X^* \colon A \to A^{(p)}$ is induced by $k[\underline{X}] \to k[\underline{X}]$, $f \mapsto f^{(p)}$.
- The relative Frobenius $F^* = F_{X/k}^*$ is induced by the homomorphism of k-algebras $k[\underline{X}] \to k[\underline{X}]$ which sends an indeterminate X_i to X_i^p.

(1.5) De Rham cohomology of varieties over fields of positive characteristic.

From now on k will denote a perfect field of characteristic p, and $f \colon X \to \mathrm{Spec}(k)$ will be a smooth proper scheme over k.

We will study the analogies of the facts in (1.3).

First Fact in positive characteristic 1.4. *In general the Hodge spectral sequence* ${}'E_1^{ab} = H^b(X, \Omega^a_{X/k}) \implies H^{a+b}_{\mathrm{DR}}(X/k)$ *does* not *degenerate at* E_1.

Mumford [Mu1] has given examples of smooth projective surfaces such that the Hodge spectral sequence does not degenerate. To exclude such cases we will make from now on the following assumption.

(deg) **Assumption**: The Hodge spectral sequence degenerates at E_1.

We list some examples where this assumption holds:

(1) Any abelian variety X over k satisfies (deg) ([Od] Prop. 5.1).
(2) Any smooth proper curve C over k satisfies (deg): To see this one can either use the previous example and deduce the degeneracy of the Hodge spectral sequence for C for that of the Hodge spectral sequence of the Jacobian of C, or one can use the result of Deligne and Illusie below.
(3) Any K3-surface X over k satisfies (deg): This follows from [De2] Prop. 2.2.
(4) Every smooth complete intersection in the projective space $\mathbb{P}^n_k$ satisfies (deg) as a scheme over k (see [SGA7] Exp. IX, Thm. 1.5).
(5) Every smooth proper toric variety satisfies (deg) (see [Bl]).
(6) Let X be a smooth proper k-scheme such that $F_*(\Omega^\bullet_{X/k})$ is decomposable (i.e., isomorphic in the derived category to a complex with zero differential). Then f satisfies (deg) by results of Deligne and Illusie, see [DI], Cor. 4.1.5. Moreover, this condition is satisfied if $\dim(X) \leq p$ and f admits a smooth lifting $\tilde{f}\colon \tilde{X} \to \mathrm{Spec}(W_2(k))$, where $W_2(k)$ is the ring of Witt vectors of length 2 of k. We refer to [Ill2] for an extensive discussion of this property.

(1.6) The Cartier isomorphism.

We now come to the analogue of the Poincaré lemma: Assume that X is any k-scheme (not necessarily smooth or proper). Again the differentials of the De Rham complex $\Omega^\bullet_{X/k}$ are in general not $\mathscr{O}_X$-linear. But it follows from the local description of the relative Frobenius in (1.4) that the differentials of $F_*(\Omega^\bullet_{X/k})$ are $\mathscr{O}_{X^{(p)}}$-linear (because $d(X_i^p f) = X_i^p d(f) + pd(X_i^{p-1})f = X_i^p d(f)$).

The Cartier isomorphism describes the cohomology sheaves of the complex $F_*(\Omega^\bullet_{X/k})$. It can be defined as follows. Let x be a local section of $\mathscr{O}_X$. Then $d\sigma_X^*(x)$ is a local section of $\Omega^1_{X^{(p)}/k}$ and there exists a unique homomorphism of $\mathscr{O}_{X^{(p)}}$-modules $\gamma\colon \Omega^1_{X^{(p)}/k} \to \mathscr{H}^1(F_*(\Omega^\bullet_{X/k}))$ such that $\gamma(d\sigma_X^*(x))$ is the class of $x^{p-1}dx$.

Second Fact in positive characteristic 1.5. *For all $i \geq 0$ there exists a unique homomorphism of $\mathscr{O}_{X^{(p)}}$-modules*

$$\gamma^i\colon \Omega^i_{X^{(p)}/k} \to \mathscr{H}^i(F_*(\Omega^\bullet_{X/k}))$$

such that

$$\gamma^0(1) = 1,$$
$$\gamma^1 = \gamma,$$
$$\gamma^{i+j}(\omega \wedge \omega') = \gamma^i(\omega) \wedge \gamma^j(\omega').$$

Moreover, if X is smooth over k, γ^i is an isomorphism for all $i \geq 0$.

This family of isomorphisms is called the *(inverse) Cartier isomorphism*. It was initially defined by Cartier. The description here is due to Grothendieck and detailed in [Ka] 7. One can prove the fact that γ^i is an isomorphism along the following lines (see loc. cit. for the details): The assertion is clearly local on X. The formation of the De Rham complex and the definition of γ^i commutes with pull back via étale morphisms. Therefore we can assume that $X = \mathbb{A}^d_k$ is the d-dimensional affine space over k. The formation of the De Rham complex and the definition of γ^i also commutes with extension of scalars of the base field and hence we can assume that $k = \mathbb{F}_p$ and hence $\sigma = \mathrm{id}$. Finally using the Künneth formula we can assume that $X = \mathbb{A}^1_{\mathbb{F}_p}$. In this case it is an easy explicit calculation to check that γ^1 is an isomorphism.

Now we assume again that X is smooth and proper over k. Then $(\gamma^i)^{-1}$ induces for all $n \geq i$ a k-linear isomorphism

$$H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}/k}) \xrightarrow{\sim} H^{n-i}(X^{(p)}, \mathscr{H}^i(F_*(\Omega^\bullet_{X/k}))).$$

Using the natural isomorphisms

$$\begin{aligned} H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}/k}) &\cong \sigma^* H^{n-i}(X, \Omega^i_{X/k}), \\ H^{n-i}(X^{(p)}, \mathscr{H}^i(F_*(\Omega^\bullet_{X/k})) &\cong H^{n-i}(X^{(p)}, F_*\mathscr{H}^i(\Omega^\bullet_{X/k})) \\ &\cong H^{n-i}(X, \mathscr{H}^i(\Omega^\bullet_{X/k})), \end{aligned}$$

we obtain a k-linear isomorphism

$$\begin{aligned} \varphi_i \colon \sigma^*({}'E_1^{i,n-i}) &= \sigma^* H^{n-i}(X, \Omega^i_{X/k}) \\ &\xrightarrow{\sim} H^{n-i}(X, \mathscr{H}^i(\Omega^\bullet_{X/k})) = {}''E_2^{n-i,i}. \end{aligned} \tag{1.6.1}$$

Corollary 1.6. *If X satisfies the assumption* (deg)*, the second spectral sequence ${}''E_2^{ab} = H^a(X, \mathscr{H}^b(\Omega^\bullet_{X/k})) \implies H^{a+b}_{\mathrm{DR}}(X/k)$ degenerates at ${}''E_2$.*

In characteristic p we will call this second spectral sequence the *conjugate spectral sequence* (following Katz [Ka]). Note that this is just a name. There is no complex conjugation here.

(1.7) F-zips.

We continue to assume that k is a perfect field and that X is a smooth proper k-scheme which satisfies (deg). We fix an integer $n \geq 0$ and set $M := H^n_{\mathrm{DR}}(X/k)$. We have seen that M carries the following structure.

(a) M is a finite dimensional k-vector space.

(b) The Hodge spectral sequence (1.2.1) provides a descending filtration ${}'\mathrm{Fil}^\bullet$ on M such that ${}'\mathrm{Fil}^i / {}'\mathrm{Fil}^{i+1} = H^{n-i}(X, \Omega^i_{X/k})$. If we define $C^i := \sigma^*({}'Fil^i)$, $C^\bullet$ is a descending filtration on $\sigma^*(M)$.

(c) The conjugate spectral sequence (1.2.2) provides a second descending filtration ${}''\mathrm{Fil}^\bullet$ on M. If we define $D_i := {}''\mathrm{Fil}^{n-i}$, $D_\bullet$ is an ascending filtration on M such that $D_i/D_{i-1} = H^{n-i}(X, \mathscr{H}^i(\Omega^\bullet_{X/k}))$.

(d) The isomorphism (1.6.1) is a k-linear isomorphism

$$\varphi_i\colon C^i/C^{i+1} \xrightarrow{\sim} D_i/D_{i-1}.$$

We now give an abstract definition for such a structure which we call an F-zip. This notion was first defined in [MW]. If M is any k-vector space, a *descending filtration* on M is by definition a family $(C^i)_{i\in\mathbb{Z}}$ of subspaces of M such that $C^i \supset C^{i+1}$ for all $i \in \mathbb{Z}$ and such that $\bigcup_i C^i = M$ and $\bigcap_i C^i = 0$. An *ascending filtration* is defined analogously.

Definition 1.7. *An* F-zip *over* k *is a tuple* $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ *such that*
(a) M *is a finite dimensional* k*-vector space,*
(b) $C^\bullet$ *is a descending filtration on* $\sigma^*(M)$*,*
(c) $D_\bullet$ *is an ascending filtration on* M*,*
(d) $\varphi_i\colon C^i/C^{i+1} \xrightarrow{\sim} D_i/D_{i-1}$ *is a* k*-linear isomorphism.*
Moreover, for an F*-zip we call the function*

$$\tau = \tau_{\underline{M}}\colon i \mapsto \dim_k(C^i/C^{i+1})$$

the filtration type of $C^\bullet$ *or simply the* type of $\underline{M}$. *The elements in the support of* τ*, i.e., the* $i \in \mathbb{Z}$ *such that* $\tau(i) \neq 0$*, are called the* weights of $\underline{M}$.

The notion of a morphism of F-zips over k is defined in an obvious way and we obtain an $\mathbb{F}_p$-linear category (which is not abelian; e.g., there do not exist kernels in general). If $\underline{M}$ is an F-zip over k and k' is a field extension of k, it is clear how to define the base change $\underline{M} \otimes_k k'$ which we will usually denote by $\underline{M}_{k'}$.

The Hodge spectral sequence, the conjugate spectral sequence, and the Cartier isomorphism are functorial in X, therefore we see:

Proposition 1.8. *Fix an integer* $n \geq 0$. *Then* $X \mapsto H^n_{\mathrm{DR}}(X/k)$ *defines a contravariant functor from the category of smooth proper* k*-schemes satisfying* (deg) *to the category of* F*-zips over* k. *Moreover, if* τ *is the type of the* F*-zip* $H^n_{\mathrm{DR}}(X/k)$*, the weights on* $\underline{M}$ *are contained in* $\{0,\dots,n\}$ *and for* $0 \leq i \leq n$ *we have*

$$\tau(i) = \dim_k H^{n-i}(X, \Omega^i_{X/k}).$$

Example 1.9. For $d \in \mathbb{Z}$ the *Tate-*F*-zips* $\underline{T}(d)$ is given as follows. The underlying vector space of $\underline{T}(d)$ is just k, we have $C^i = \sigma^*(k)$ for $i \leq d$ and $C^i = 0$ for $i > d$ which implies $D_i = 0$ for $i < d$ and $D_i = k$ for $i \geq d$. Finally $\varphi_d\colon k \to k$ is given by the Frobenius σ.

If X is a geometrically connected smooth proper k-scheme of dimension d, we have isomorphisms

$$H^0_{\mathrm{DR}}(X/k) \cong \underline{T}(0), \qquad H^{2d}_{\mathrm{DR}}(X/k) \cong \underline{T}(2d).$$

For the projective space of dimension d we have

$$H^i_{\mathrm{DR}}(\mathbb{P}^d_k/k) = \begin{cases} \underline{T}(i), & \text{if } 0 \leq i \leq 2d \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

(1.8) The category of F-zips.

For the proof of the following proposition we refer to [Wd3].

Proposition 1.10. *The category of F-zips over k has the following properties.*
(1) *There exist finite direct sums, and every F-zip $\underline{M}$ is a direct sum of a finite family of indecomposable F-zips (i.e., F-zips which are not isomorphic to a direct sum of nonzero F-zips). This finite family is uniquely determined by $\underline{M}$ up to order.*
(2) *There exists a natural $\otimes$-structure on this category and with this structure the category of F-zips is a rigid $\mathbb{F}_p$-linear $\otimes$-category.*
(3) *The Tate-F-zips are projective and injective in the following sense: For every F-zip $\underline{M}$ and for every surjective morphism $\underline{M} \to \underline{T}(d)$ (resp. for every injective morphism $\underline{T}(d) \to \underline{M}$) there exists a section (resp. a retraction).*

For every F-zip $\underline{M}$ and every $d \in \mathbb{Z}$ we set $\underline{M}(d) := \underline{M} \otimes \underline{T}(d)$. This is the F-zip obtained from $\underline{M}$ by shifting the indices of the filtrations $C^\bullet$ and $D_\bullet$ be d.

It follows by an easy descend argument from $\bar{k}$ to k that every F-zip $\underline{M}$ over k admits a unique (up to order) decomposition

$$\underline{M} = \underline{M}_{\mathrm{mw}} \oplus \bigoplus_{j=1}^{t} \underline{M}_{(d_j)}, \tag{1.8.1}$$

where $\underline{M}_{\mathrm{mw}}$ is an F-zip such that every indecomposable summand of $\underline{M}_{\mathrm{mw}}$ has mixed weights, i.e., more than a single weight, and where $(\underline{M}_{(d_j)})_{\bar{k}}$ is isomorphic to a direct sum of Tate-F-zips of weight d_j for pairwise different integers d_j.

(1.9) Ordinary varieties.

We now come to the analogue of the third fact 1.3. Let X be as above and fix $n \geq 0$. Consider the F-zip $(M := H^n_{\mathrm{DR}}(X/k), C^\bullet, D_\bullet, \varphi_\bullet)$ defined above.

Third Fact in positive characteristic 1.11. *In general, $\sigma^*(D_{i-1})$ and C^i are not complementary subspaces of $\sigma^*(M)$.*

This leads us to the following definition.

Definition 1.12. *An F-zip $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ is called* ordinary *if we have $\sigma^*(D_{i-1}) \oplus C^i = \sigma^*(M)$ for all $i \in \mathbb{Z}$.*

A smooth proper k-scheme satisfying (deg) *is called* ordinary, *if $H^n_{\mathrm{DR}}(X/k)$ with its natural F-zips structure is ordinary for all $n \geq 0$.*

In [IR] (4.12), Illusie and Raynaud define the notion of ordinariness for a smooth proper scheme X over k using the De Rham-Witt complex. It follows from loc. cit. (4.13) that the above definition is equivalent to the definition given by Illusie and Raynaud.

(1.10) Chern class of a line bundle.

Let X be any k-scheme. Let $\mathrm{dlog}\colon \mathscr{O}_X^\times \to \Omega^1_{X/k}$ the logarithmic derivation, given on local sections by $x \mapsto dx/x$. We consider dlog as a morphism

$$\mathrm{dlog}\colon \mathscr{O}_X^\times \to \sigma^{\geq 1}\Omega^\bullet_{X/k}[1]$$

of complexes of abelian sheaves on X. Taking the first hypercohomology we obtain a map

$$c_1'\colon \operatorname{Pic}(X) = H^1(X, \mathscr{O}_X^\times) \to \mathbb{H}^2(X, \sigma^{\geq 1}\Omega^\bullet_{X/k}).$$

The exact sequence

$$0 \longrightarrow \sigma^{\geq 1}\Omega^\bullet_{X/k} \longrightarrow \Omega^\bullet_{X/k} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

provides a long exact sequence

$$\cdots \to \mathbb{H}^2(X, \sigma^{\geq 1}\Omega^\bullet_{X/k}) \xrightarrow{\alpha} H^2_{\mathrm{DR}}(X/k) \xrightarrow{\beta} H^2(X, \mathscr{O}_X) \to \dots \tag{1.10.1}$$

and we denote by $c_1\colon \operatorname{Pic}(X) \to H^2_{\mathrm{DR}}(X/k)$ the composition of c_1' with α. This is the Chern class map.

Now assume that X is a smooth proper k-scheme satisfying (deg) and endow $H^2_{\mathrm{DR}}(X/k)$ with its F-zip structure. As the Hodge spectral sequence degenerates, the long exact sequence (1.10.1) decomposes in short exact sequences and the map $\sigma^*(\beta)$ is nothing but the map $\sigma^*(H^2_{\mathrm{DR}}(X/k)) = C^0 \to C^0/C^1 = \sigma^*(H^2(X, \mathscr{O}_X))$. Therefore we see that for any line bundle $\mathscr{L}$ on X we have $\sigma^*(c_1(\mathscr{L})) \in C^1$.

(1.11) Smooth proper families and F-zips over schemes.

There is a relative version of the notion of an F-zip and an F-zip structure on the De Rham cohomology. Here we give only a brief outline and refer to [MW] for the details.

Let S be an arbitrary $\mathbb{F}_p$-scheme and let $f\colon X \to S$ be a smooth and proper morphism of schemes. We set $H^n_{\mathrm{DR}}(X/S) = \mathbb{R}^n f_*(\Omega^\bullet_{X/S})$. The general formalism explained in (1.1) (now applied to the functor f_*) provides two spectral sequences

$$'E_1^{ab} = R^b f_*(\Omega^a_{X/S}) \implies H^{a+b}_{\mathrm{DR}}(X/S),$$
$$''E_2^{ab} = R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S})) \implies H^{a+b}_{\mathrm{DR}}(X/S),$$

called the *Hodge spectral sequence* and the *conjugate spectral sequence*. Again we have a Cartier isomorphism which induces an isomorphism of $\mathscr{O}_S$-modules

$$\mathrm{Frob}_S^*({}'E_1^{ba}) \xrightarrow{\sim} {}''E_2^{ab}.$$

We say that f satisfies the condition (deg) if the following two conditions hold.
(a) The Hodge spectral sequence degenerates at E_1.
(b) The $\mathscr{O}_S$-modules $R^b f_*(\Omega^a_{X/S})$ are locally free of finite rank for all $a, b \geq 0$.
If f satisfies (deg), the formation of the Hodge spectral sequence (and in particular the formation of $R^b f_*(\Omega^a_{X/S})$ and of $H^n_{\mathrm{DR}}(X/S)$) commutes with base change $S' \to S$. It follows that the conjugate spectral sequence degenerates at E_2 and that its formation commutes with base change as well.

Examples for morphisms $f\colon X \to S$ satisfying (deg) are again abelian schemes, smooth proper (relative) curves, $K3$-surfaces, smooth complete intersections in $\mathbb{P}(\mathscr{E})$ for some vector bundle $\mathscr{E}$ on S.

Similar as in (1.7) we make the following definitions. If M is a locally free $\mathscr{O}_S$-module of finite rank, a descending filtration on M is a family $(C^i)_{i\in\mathbb{Z}}$ of $\mathscr{O}_S$-submodules of M which are locally direct summands such that $C^i \supset C^{i+1}$ for all $i \in \mathbb{Z}$ and such that $\bigcup_i C^i = M$ and $\bigcap_i C^i = 0$. An *ascending filtration* is defined analogously.

Definition 1.13. *An* F-zip over S *is a tuple* $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ *such that*
(a) *M is a locally free $\mathscr{O}_S$-module of finite rank,*
(b) *$C^\bullet$ is a descending filtration on* $\mathrm{Frob}_S^*(M)$*,*
(c) *$D_\bullet$ is an ascending filtration on M,*
(d) *$\varphi_i\colon C^i/C^{i+1} \xrightarrow{\sim} D_i/D_{i-1}$ is an $\mathscr{O}_S$-linear isomorphism.*
Again we can define the type of $\underline{M}$ *as* $\mathbb{Z} \ni i \mapsto \mathrm{rk}_{\mathscr{O}_S}(C^i/C^{i+1})$*, which is now a function with values in the set of locally constanst functions on S with values in $\mathbb{N}_0$.*

Note that this definition differs slightly from the definition given in [MW] although they are equivalent if S is the spectrum of a perfect ring.

The same construction of an F-zip structure on the De Rham cohomology for $S = \mathrm{Spec}(k)$ can be done for an arbitrary $\mathbb{F}_p$-scheme S and we obtain for each $n \geq 0$ a functor $(f\colon X \to S) \mapsto H^n_{\mathrm{DR}}(X/S)$ from the category of smooth proper S-schemes satisfying (deg) to the category of F-zips over S. This functor commutes with arbitrary base change $g\colon S' \to S$ (in the obvious sense).

2. Examples I

We continue to assume that k is a perfect field of characteristic $p > 0$.

(2.1) Abelian Varieties.

Let X be an abelian variety over k of dimension $g \geq 1$. Consider $H^1_{\mathrm{DR}}(X/k)$ with its F-zip structure. The $\otimes$-structure on the category of F-zips over k allows to form the F-zip $\bigwedge^n(H^1_{\mathrm{DR}}(X/k))$ for $n \geq 0$ (see [Wd3] for details) and the cup product defines an isomorphism of F-zips

$$\bigwedge^n (H^1_{\mathrm{DR}}(X/k) \xrightarrow{\sim} H^n_{\mathrm{DR}}(X/k).$$

In particular we see that the F-zip $H^1_{\mathrm{DR}}(X/k)$ already determines all F-zips $H^n_{\mathrm{DR}}(X/k)$ for $n \geq 0$.

Now let $H^1_{\mathrm{DR}}(X/k) = (M, C^\bullet, D_\bullet, \varphi_\bullet)$. Then M is a k-vector space of dimension $2g$ and the filtrations are given by

$$C^0 = \sigma^* M \supset C^1 = \sigma^* H^0(X, \Omega^1_{X/k}) \supset C^2 = 0,$$
$$D_{-1} = 0 \subset D_0 = H^1(X, \mathscr{H}^0(\Omega^\bullet_{X/k})) \subset D_1 = M$$

with $\dim(C^1) = \dim(D_0) = g$. The Cartier isomorphism induces two nontrivial isomorphisms

$$\varphi_0 \colon \sigma^*(M)/C^1 \xrightarrow{\sim} D_0,$$
$$\varphi_1 \colon C^1 \xrightarrow{\sim} M/D_0.$$

Moreover, the following assertions are equivalent.

(1) X is ordinary (i.e., the F-zips $H^n_{\mathrm{DR}}(X/k)$ are ordinary in the sense of Definition 1.12 for all $n \geq 0$).
(2) The F-zip $H^1_{\mathrm{DR}}(X/k)$ is ordinary (i.e., $\sigma^*(D_0) \oplus C^1 = \sigma^*(M)$).
(3) $X[p](\bar{k}) \cong (\mathbb{Z}/p\mathbb{Z})^g$.
(4) The Newton polygon of X has only slopes 0 and 1.

The last two (equivalent) conditions are often used to define the notion of ordinariness for an abelian variety. Here $X[p]$ denotes the kernel of the multiplication with p on X. In fact, the next three sections show that $X[p]$ determines the F-zip $H^1_{\mathrm{DR}}(X/k)$ (and vice versa).

(2.2) At level 1 truncated Barsotti-Tate groups.

Let X be an abelian variety over k. Then the kernel $G := X[p]$ of the multiplication with p is a so-called at level 1 truncated Barsotti-Tate group of height $2g$ and dimension g. We explain now what this means.

For any group scheme G over k, the relative Frobenius $F_{G/k} \colon G \to G^{(p)}$ defined in (1.4) is a homomorphism of group schemes. Moreover, if G is commutative, by [SGA3] Exp. VII_A, 4.3 there exists a natural homomorphism $V_{G/k} \colon G^{(p)} \to G$ of group schemes over k, called the *Verschiebung of* G, such that

$$V_{G/k} \circ F_{G/k} = p\,\mathrm{id}_G\,. \tag{2.2.1}$$

If $F_{G/k}$ is an epimorphism (i.e., if G is smooth over k), (2.2.1) determines $V_{G/k}$ uniquely.

An *at level 1 truncated Barsotti-Tate group over k* (or, shorter, a BT_1) is a finite commutative group scheme G over k which is annihilated by the multiplication with p such that the complex

$$G \xrightarrow{F_{G/k}} G^{(p)} \xrightarrow{V_{G/k}} G$$

of abelian fppf-sheaves is exact. The underlying scheme of G is of the form $\mathrm{Spec}(A)$ for a finite-dimensional k-algebra and $\dim_k(A)$ is a power of p and we define the *height h of G* by $p^h = \dim_k(A)$. As G is in general not smooth, the Lie algebra $\mathrm{Lie}(G)$ can be nonzero even though the underlying scheme of G is of dimension zero. We call $\dim(\mathrm{Lie}(G))$ the *dimension of G* and denote it by $\dim(G)$.

At level 1 truncated Barsotti-Tate groups can be described via Dieudonné theory. We call a *Dieudonné space over k* a triple (M, F, V), where M is a finite-dimensional k-vector space and $F\colon \sigma^*(M) \to M$ and $V\colon M \to \sigma^*(M)$ are k-linear maps such that $F \circ V = 0$, $V \circ F = 0$ and $\mathrm{Ker}(F) = \mathrm{Im}(F)$ (note that these conditions imply that $\mathrm{Ker}(V) = \mathrm{Im}(F)$). Then we have the following theorem (see e.g. [Dem] or [BBM] 3).

Theorem 2.1. *Crystalline Dieudonné theory provides a contravariant functor $\mathbb{D}$ from the category of BT_1 over k to the category of Dieudonné spaces over k, and this functor is an equivalence of categories. Moreover, if $\mathbb{D}(G) = (M, F, V)$, we have* $\mathrm{height}(G) = \dim_k(M)$ *and* $\dim(G) = \dim_k(M/\mathrm{Im}(F))$.

(2.3) Dieudonné spaces and F-Zips.

We can consider every Dieudonné space as a special case of an F-zip as follows. If (M, F, V) is a Dieudonné space, we set

$$C^i = \begin{cases} \sigma^*(M), & i \leq 0; \\ \mathrm{Ker}(F), & i = 1; \\ 0, & i \geq 2; \end{cases} \qquad\qquad D_i = \begin{cases} 0, & i \leq -1; \\ \mathrm{Ker}(V), & i = 0; \\ M, & i \geq 1. \end{cases}$$

Finally we define $\varphi_0\colon \sigma^*(M)/\mathrm{Ker}(F) \to \mathrm{Ker}(V) = \mathrm{Im}(F)$ as the isomorphism induced by F and $\varphi_1\colon \mathrm{Im}(V) = \mathrm{Ker}(F) \to M/\mathrm{Ker}(V)$ as the inverse of the isomorphism induced by V. Then $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ is an F-zip.

This construction is clearly functorial in (M, F, V) and thus we obtain a functor from the category of Dieudonné spaces over k to the category of F-zips over k. Moreover, it is easy to see that this induces an equivalence of the category of Dieudonné spaces with the category of F-zips $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ such that $C^0 = \sigma^*(M)$ and $C^2 = 0$, i.e., the support of the type τ of $\underline{M}$ is contained in $\{0, 1\}$. We will call such F-zips *Dieudonné-zips*. From Theorem 2.1 we obtain.

Corollary 2.2. *Crystalline Dieudonné theory together with the construction above gives an equivalence $G \mapsto \underline{M}(G)$ between the category of BT_1 G over k and the category of Dieudonné zips $\underline{M}$ over k (considered as a full subcategory of the category of all F-zips over k). Via this equivalence we have* $\mathrm{height}(G) = \dim_k(M)$ *and* $\dim(G) = \tau(1)$*, where τ is the type of $\underline{M}(G)$.*

(2.4) Truncated Barsotti-Tate groups and de Rham cohomology of abelian varieties.

We now have constructed two functors from the category of abelian varieties over k to the category of F-zips over k. The first is the functor $X \mapsto H^1_{\mathrm{DR}}(X/k)$ constructed in (2.1). The second is the composition of the functors $X \mapsto X[p]$ and $G \mapsto \mathbb{D}(G)$ where we consider every Dieudonné space as an F-zip as explained in (2.3). We have the following result by [Od] Corollary 5.11.

Theorem 2.3. *The two functors $X \mapsto H^1_{\mathrm{DR}}(X/k)$ and $X \to \mathbb{D}(X[p])$ are isomorphic.*

Hence we see that $H^1_{\mathrm{DR}}(X/k)$ together with its F-zip structure and the p-torsion $X[p]$ determine each other. This is in effect only a special case ($n = 1$) of the more general result that for all $n \geq 1$ the p^n-torsion $X[p^n]$ and the crystalline cohomology $H^1_{\mathrm{cris}}(X/W_n(k))$ (which is a free $W_n(k)$-module M together with $W_n(k)$-linear maps $F\colon \sigma^*(M) \to M$ and $V\colon M \to \sigma^*(M)$ such that $F \circ V = p\,\mathrm{id}_M$ and $V \circ F = p\,\mathrm{id}_{\sigma^*(M)}$) determine each other (see loc. cit.).

(2.5) Decomposition of truncated Barsotti-Tate groups.

Let us first recall some facts on finite commutative group schemes (see e.g. [Dem]). To simplify the exposition, let us assume that k is algebraically closed.

Every finite commutative group scheme G over k admits a unique decomposition $G = G_p \times G^p$, where G_p is a group scheme killed by a power of p and where G^p is a finite étale group scheme of order prime to p.

Every such G_p admits itself a unique decomposition $G_p = G_{\text{ét}} \times G_{\mathrm{mult}} \times G_{\mathrm{bi}}$, where

- $G_{\text{ét}}$ is étale, i.e., $G_{\text{ét}}$ is isomorphic to a product of constant group schemes associated to groups of the form $\mathbb{Z}/p^l\mathbb{Z}$.
- G_{mult} is of multiplicative type, i.e., G_{mult} is isomorphic to a product of group schemes of roots of unities of the form μ_{p^l}.
- G_{bi} is bi-infinitesimal, i.e., G and its Cartier dual $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ are local as schemes and therefore the spectra of local Artinian k-algebras. Equivalently, there exist no nontrival homomorphisms of group schemes $\mathbb{Z}/p\mathbb{Z} \to G_{\mathrm{bi}}$ and $\mu_p \to G_{\mathrm{bi}}$.

In particular, every BT_1 admits such a decomposition $G = G_{\text{ét}} \times G_{\mathrm{mult}} \times G_{\mathrm{bi}}$.

Let $\underline{M}(G)$ be the Dieudonné-zip over k corresponding to G via the equivalence in Corollary 2.2. Then (1.8.1) has the form $\underline{M} = \underline{T}(0)^{\oplus f} \oplus \underline{T}(1)^{\oplus e} \oplus \underline{M}_{\mathrm{mw}}$ and

$$\underline{M}(G_{\text{ét}}) = \underline{T}(0)^{\oplus f}, \quad \underline{M}(G_{\mathrm{mult}}) = \underline{T}(1)^{\oplus e}, \quad \underline{M}(G_{\mathrm{bi}}) = \underline{M}_{\mathrm{mw}}.$$

(2.6) K3-surfaces.

We now study the De Rham cohomology of K3-surfaces. For this let k be a perfect field of characteristic $p > 0$. By definition, a K3-surface over k is a smooth proper connected k-scheme X of dimension 2 such that $H^1(X, \mathscr{O}_X) = 0$ and such that its canonical bundle $K_X = \Omega^2_{X/k}$ is trivial (i.e., isomorphic to $\mathscr{O}_X$). Note that every K3-surface is automatically projective.

Then it is well-known (see e.g. [De2]) that the Hodge spectral sequence degenerates at E_1 and that the Hodge numbers $(\dim_k(H^b(X, \Omega^a_{X/k})))_{0 \le a,b \le 2}$ are given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{2.6.1}$$

Hence we see that

$$\dim_k(H^n_{\mathrm{DR}}(X/k)) = \begin{cases} 1, & n = 0, 4; \\ 22, & n = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The F-zip structure on $M := H^2_{\mathrm{DR}}(X/k)$ is given by two filtrations $C^\bullet$ and $D_\bullet$ with

- $C^i = \sigma^*(M)$ for $i \le 0$, $\dim_k(C^1) = 21$, $\dim_k(C^2) = 1$, and $C^i = 0$ for $i \ge 3$,
- $D_i = 0$ for $i \le -1$, $\dim_k(D_0) = 1$, $\dim_k(D_1) = 21$, and $D_i = M$ for $i \ge 2$,
- two isomorphisms $\varphi_0 \colon \sigma^*(M)/C^1 \xrightarrow{\sim} D_0$ and $\varphi_2 \colon C^2 \xrightarrow{\sim} M/D_1$ of one-dimensional spaces, and an isomorphism $\varphi_1 \colon C^1/C^2 \xrightarrow{\sim} D_1/D_0$ of spaces of dimension 20.

Moreover, the following assertions are equivalent.

(1) X is ordinary (i.e., the F-zips $H^n_{\mathrm{DR}}(X/k)$ are ordinary in the sense of Definition 1.12 for all $n \ge 0$).
(2) The F-zip $H^2_{\mathrm{DR}}(X/k)$ is ordinary.
(3) The Newton polygon of X has precisely the slopes 0, 1, and 2.

3. Classification of F-zips

In this section we will give two different classifications of F-zips over algebraically closed fields. The first classification (3.1) stems from representation theory of algebras and is a variation of a result of Crawley-Boevey [Cr] which goes back in this special case to work of Gelfand and Ponomarew [GP]. For the special case of Dieudonné zips this classification has obtained indepedently by Oort [Oo].

The second classification is shown by defining the variety X of rigidified F-zips and relate it to the wonderful compactification of the projective linear group by de Concini and Procesi ((3.2) and (3.3)). This classification was obtained in [MW] by relating X to a third variety and it was observed in [Wd2] how to relate this work to the wonderful compactification (see also below in (3.3)).

Notation: Let k be an algebraically closed field of characteristic p.

(3.1) Classification via representation theory.

Recall that by Proposition 1.10 every F-zip over k is a direct sum of indecomposable F-zips and that this decomposition is unique up to order. Argueing as in [Cr] it is possible to give the following description of isomorphism classes of indecomposable F-zips (see [Wd3]).

A *word* is by definition a formal sequences $\mathfrak{x} = x_1x_2\dots x_n$ with $n \geq 0$ where $x_j \in \mathbb{Z}$. A *rotation* of $\mathfrak{x}$ is a word of the form $x_{j+1}\dots x_nx_1\dots x_j$ for some j. The *product* $\mathfrak{x}\mathfrak{y}$ of two words is defined by placing them next to each other. We set

$$\mathscr{B} = \{\, \mathfrak{x} \text{ word} \mid \text{all nontrivial rotations of } \mathfrak{x} \text{ are different from } \mathfrak{x}\,\}/\sim$$

with $\mathfrak{x} \sim \mathfrak{y}$ if $\mathfrak{y}$ is a rotation of $\mathfrak{x}$. The elements of $\mathscr{B}$ are called *bands.*

To each representative $\mathfrak{x} = x_1\dots x_n$ of a band we associate an F-zip $\underline{M}(\mathfrak{x}) = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ over $\mathbb{F}_p$ as follows. Let M_0 be an $\mathbb{F}_p$-vector spaces with basis $(e_1,\dots,e_n)$. We set $x_0 := x_n$ and $x_{n+1} := x_1$ and define

$$\begin{aligned}
C^i &:= \bigoplus_{\substack{1\leq l\leq n,\\ x_l\geq i}} \mathbb{F}_pe_l,\\
D_i &:= \bigoplus_{\substack{1\leq l\leq n,\\ x_{l-1}\leq i}} \mathbb{F}_pe_l,\\
\varphi_i\colon C^i/C^{i+1} &= \bigoplus_{\substack{1\leq l\leq n,\\ x_l=i}} \mathbb{F}_pe_l\\
&\xrightarrow{e_l\mapsto e_{l+1}} D_i/D_{i-1} = \bigoplus_{\substack{1\leq l\leq n,\\ x_{l-1}=i}} \mathbb{F}_pe_l = \bigoplus_{\substack{1\leq l\leq n,\\ x_l=i}} \mathbb{F}_pe_{l+1}.
\end{aligned} \tag{3.1.1}$$

Theorem 3.1. *The above construction* $[\mathfrak{x}] \mapsto \underline{M}(\mathfrak{x})_k$ *induces a bijection*

$$\mathscr{B} \leftrightarrow \left\{\begin{matrix}\textit{isomorphism classes of}\\ \textit{indecomposable } F\textit{-zips over } k\end{matrix}\right\}.$$

For example, the band corresponding to a Tate-F-zip $\underline{T}(d)$ is the word consisting of the single letter $\mathbf{d}$.

(3.2) The scheme of F-zips.

For the second classification we define the variety of rigidified F-zips. Here "rigidified" means that we fix the underlying module of the F-zip. Again let k be algebraically closed.

We first fix a filtration type, i.e., a function

$$\tau\colon \mathbb{Z} \to \mathbb{N}_0 \tag{3.2.1}$$

with finite support $i_1 > \cdots > i_r$. We set for $j = 1, \dots, r$

$$\begin{aligned} n_j &:= \tau(i_j), \\ n &:= \sum_{j=1}^{r} n_j. \end{aligned} \tag{3.2.2}$$

We will now define an $\mathbb{F}_p$-scheme which parametrizes F-zips of type τ with fixed underlying module. Therefore let M_0 be a fixed $\mathbb{F}_p$-vector space of dimension n. Let X_τ be the $\mathbb{F}_p$-scheme whose S-valued points (for S an $\mathbb{F}_p$-scheme) are given by

$$X_\tau(S) = \{\, \underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \mid \underline{M} \text{ F-zip of type } \tau \text{ over } S \\ \text{with } M = M_0 \otimes_{\mathbb{F}_p} \mathscr{O}_S \,\}.$$

Let $G = \mathrm{GL}(M_0)$, which we consider as an algebraic group over $\mathbb{F}_p$. Then X_τ has a natural action of $G \times G$ defined as follows. If $\underline{M} = (M_0 \otimes_{\mathbb{F}_p} \mathscr{O}_S, C^\bullet, D_\bullet, \varphi_\bullet)$ is an S-valued of X_τ and $(h_1, h_2) \in G(S) \times G(S)$, we set

$${}^{(h_1,h_2)}\underline{M} = (M_0 \otimes_{\mathbb{F}_p} \mathscr{O}_S, (F(h_2)(C^i))_i, (h_1(D_i))_i, (h_1 \varphi_i F(h_2)^{-1})_i), \tag{3.2.3}$$

where $h_1 \varphi_i F(h_2)^{-1}$ denotes the composition

$$\begin{tikzcd} F(h_2)(C^i)/F(h_2)(C^{i+1}) \arrow[d, "F(h_2)^{-1}"'] & h_1(D_i)/h_1(D_{i-1}) \\ C^i/C^{i+1} \arrow[r, "\varphi_i"] & D_i/D_{i-1}. \arrow[u, "h_1"'] \end{tikzcd}$$

The $G \times G$-variety has the following properties ([MW] Lemma 5.1 and Lemma 4.2, [Wd2] (3.3)).

Proposition 3.2. *X_τ is a smooth connected $\mathbb{F}_p$-scheme of dimension equal to $\dim(G) = n^2$. The $G \times G$-action on X_τ is transitive.*

Embedding $G \hookrightarrow G \times G$ diagonally, the $G \times G$-action restricts to a G-action on X_τ. Every F-zip over k of type τ is isomorphic to an F-zip whose underlying k-vector space is equal to $M_0 \otimes_{\mathbb{F}_p} k$ and two such F-zips are isomorphic if and only if they are in the same $G(k)$-orbit. Hence:

Proposition 3.3. *Associating to each k-valued point of X_τ the isomorphism class of the corresponding F-zip defines a bijection*

$$\big\{ G(k)\text{-orbits on } X_\tau(k) \big\} \leftrightarrow \left\{ \begin{matrix} \text{isomorphism classes of F-zips} \\ \text{over k of type } \tau \end{matrix} \right\}.$$

To study the $G(k)$-orbits on $X_\tau(k)$ we relate X_τ to a part of the wonderful compactification $\bar{H}$ of $H := \mathrm{PGL}(M_0)$. We refer to the appendix for the necessary facts about reductive groups and the definition and some properties of $\bar{H}$.

(3.3) Classification via the wonderful compactification.

Recall that we fixed a filtration type τ (3.2.1) which gives a flag type $(n_1, \dots, n_r)$ (3.2.2) in the sense of Example 7.2. Further we fixed an n-dimensional $\mathbb{F}_p$-vector space M_0, where $n = n_1 + \cdots + n_r$. Set $G = \mathrm{GL}(M_0)$ and $H = \mathrm{PGL}(M_0)$, considered as algebraic groups over $\mathbb{F}_p$.

To classify F-zips over the algebraically closed field k we proceed in two steps.

First step.
We construct a morphism

$$\gamma\colon X_\tau \longrightarrow \overline{H}_{(n_1,\dots,n_r)},$$

where $\overline{H}_{(n_1,\dots,n_r)}$ is the $H \times H$-orbit in $\overline{H}$ corresponding to $(n_1, \dots, n_r)$ (Example 7.9). This morphism will be equivariant with respect to the canonical projection $G \times G \to H \times H$. Here the action of $G \times G$ on X_τ is the one defined in (3.2.3), and the action of $H \times H$ on $\overline{H}_{(n_1,\dots,n_r)}$ is the composition of the homomorphism

$$F_2\colon H \times H \to H \times H, \quad (h_1, h_2) \mapsto (h_1, F(h_2)) \tag{3.3.1}$$

and the action defined in (7.7.2). In particular it will induce a map

$$[\gamma]\colon \{G(k)\text{-orbits of } X_\tau(k)\} \to \{H_F(k)\text{-orbits of } \overline{H}_{(n_1,\dots,n_r)}(k)\},$$

where H_F is defined in (7.7.5). We will show that $[\gamma]$ is in fact a bijection.

Second step.
We invoke [MW] and [Wd2] to classify the H_F-orbits of $\overline{H^1_J}$ in the general setting of (7.7).

Making the first step.
We will define γ only on k-valued points although is is not difficult – albeit notationally a bit cumbersome – to define γ also on S-valued points where S is an arbitrary $\mathbb{F}_p$-scheme. For this we refer to [Wd3].

Set $M := M_0 \otimes_{\mathbb{F}_p} k$ and let $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ be a k-valued point of X_τ. Note that φ_i is an isomorphism of a zero-dimensional vector space except for $i \in \{i_1, \dots, i_r\}$.

First we define a "projective version" PX_τ of X_τ. Set $T = \mathbb{G}^r_{m,\mathbb{F}_p}$. Then T acts on X_τ by

$$(C^\bullet, D_\bullet, (\varphi_{i_1}, \dots, \varphi_{i_r})) \cdot (t_i)_{1 \le i \le r} = (C^\bullet, D_\bullet, (t_1^{-1}\varphi_{i_1}, \dots, t_r^{-1}\varphi_{i_r})).$$

This is a free action. Embed $\mathbb{G}_{m,\mathbb{F}_p}$ diagonally into T. We set $PX_\tau = X_\tau/\mathbb{G}_{m,\mathbb{F}_p}$ and $\bar{X}_\tau = X_\tau/T$. We can define an action of $H \times H$ on PX_τ and $\bar{X}_\tau$ by the same recipe as the action of $G \times G$ on X_τ (3.2.3). The canonical morphisms

$$X_\tau \xrightarrow{\pi} PX_\tau \xrightarrow{\psi} \bar{X}_\tau$$

are clearly equivariant with respect to $G \times G \to H \times H$. It is easy to check that $\psi \circ \pi$ induces a bijection

$$\{G(k)\text{-orbits of } X_\tau(k)\} \xrightarrow{\sim} \{H(k)\text{-orbits of } \bar{X}_\tau(k)\}.$$

For a k-valued point $\underline{M} \in PX_\tau(k)$ denote by $H_{\underline{M}} \subset H \times H$ the stabilizer of $\underline{M}$.

Lemma 3.4. *$H_{\underline{M}}$ is a smooth algebraic group with* $\dim(H_{\underline{M}}) = \dim(H)$.

Proof. The smoothness of $H_{\underline{M}}$ is easy to check with the infinitesimal lifting criterion. From the transitivity of the $G \times G$-action on X_τ (Proposition 3.2) it follows that the $H \times H$-action on PX_τ (and hence on $\bar{X}_\tau$) is transitive as well. As $\dim(PX_\tau) = \dim(X_\tau) - 1 = \dim(G) - 1 = \dim(H)$, we see that $\dim(H_{\underline{M}}) = \dim(H)$. $\square$

We define an F_2-equivariant morphism (F_2 was defined in (3.3.1))

$$\rho\colon PX_\tau \to \mathcal{V}_H, \qquad \underline{M} \mapsto \mathrm{Lie}(H_{\underline{M}}),$$

where $\mathcal{V}$ is defined in (7.7). Note that for $\underline{M}_1, \underline{M}_2 \in PX_\tau(k)$ which have the same image in $\bar{X}_\tau$, the stabilizers $H_{\underline{M}_1}$ and $H_{\underline{M}_2}$ are equal. Therefore ρ factorizes over $PX_\tau \twoheadrightarrow \bar{X}_\tau$. Now it follows from [Lu] 12.3 (specialized to the case PGL_n) that ρ induces an F_2-equivariant isomorphism

$$\iota\colon \bar{X}_\tau \xrightarrow{\sim} \overline{H}_{(n_1,\dots,n_r)}.$$

This concludes the first step.

Making the second step.
In this second step we will describe the H-orbits of $\overline{H^1_J}$ in the general setting of (7.7) and for an arbitrary subset $J \subset I$ where (W, I) is the Weyl system of H.

To describe this classification we have to introduce the following notation. Denote by W_J the subgroup of W generated by J. In any left coset $W_J w$ ($w \in W$) there exists a (necessarily unique) element ${}^J w$ such that $\ell({}^J w) \leq \ell(w')$ for all $w' \in W_J w$. We denote by ${}^J W \subset W$ the set of these representatives of minimal length of $W_J \backslash W$. We further set

$$d_J := \dim(H) - \max\{\, \ell(w) \mid w \in {}^J W \,\} = \dim(H) - \dim(\mathrm{Par}_J),$$

where Par_J is the scheme of parabolic subgroups of H of type J (7.4) and (7.5) (the equality being a general fact from the theory of root systems of a reductive group).

Theorem 3.5. *There exists a natural bijection*

$$\begin{aligned} {}^J W &\leftrightarrow \{H_F(k)\text{-orbits of } \overline{H^1_J}\}, \\ w &\mapsto {}^w\overline{H^1_J} \end{aligned} \tag{3.3.2}$$

such that

$$\dim({}^w\overline{H^1_J}) = \ell(w) + d_J. \tag{3.3.3}$$

Proof. Note that via the commutative diagram (7.7.4) it suffices to describe the H_F-orbits of $\overline{H}_J$. Now by [Wd2], $\overline{H}_J$ is $H \times H$-equivariantly isomorphic to another $H \times H$-scheme $Z_J = Z_{H,J}$ over $\mathbb{F}_p$. By [MW] Theorem 4.11 there is a natural bijection between the H-orbits of Z_J and ${}^J W$. □

We will not explain in which sense (3.3.2) is natural but refer to [MW] for this. Instead we will make the Theorem 3.5 explicit in the case $H = H^1 = \mathrm{PGL}_n$ in the next section.

(3.4) Classification of F-zips.

By Example 7.4, for $H = \mathrm{PGL}_n$ we can identify

$$W = S_n, \qquad I = \{\, \tau_i \mid i = 1, \dots, n-1 \,\},$$

and $J \subset I$ corresponds to a flag type, i.e., to a tuple $(n_1, \dots, n_r)$ with $\sum n_j = n$. By the definition of this correspondence in (7.4.2) and (7.5.2) we have

$$W_J = S_{n_1} \times S_{n_2} \times \dots S_{n_r},$$

which is considered as a subgroup of $W = S_n$ in the obvious way.

For $j = 0, \dots, r$ we set

$$m_j := n_1 + \dots + n_j.$$

From the description of the length of a permutation in (7.5.2) it follows easily that

$$\begin{aligned} {}^J W = \big\{\, w \in S_n \ \mid w^{-1}(m_{j-1}+1) < \dots < w^{-1}(m_j), \\ \text{for all } j = 1, \dots, r \big\}. \end{aligned} \tag{3.4.1}$$

Let w_0 be the longest elements in S_n, i.e., $w_0(i) = n+1-i$ for all $1 \le i \le n$. Let $w_{0,J}$ be the longest element in W_J and set $w_0^J := w_0 w_{0,J}$, i.e.,

$$w_0^J(i) = n - m_j - m_{j-1} + i, \qquad \text{for } m_{j-1} < i \le m_j.$$

To shorten notations we set

$$w^\circ := ww_0^J \tag{3.4.2}$$

for $w \in S_n$.

We will now associate to each $w \in S_n$ an F-zip ${}^w\underline{M} = (M_0, C^\bullet, D_\bullet, \varphi_\bullet)$ over $\mathbb{F}_p$ (where M_0 is our fixed n-dimensional $\mathbb{F}_p$-vector space), such that

$$ {}^J W \ni w \mapsto ({}^w\underline{M})_k \in X_\tau(k)$$

defines the bijection

$$ {}^J W \leftrightarrow \{H_F(k)\text{-orbits of } \overline{H}_J\} \leftrightarrow \{\mathrm{GL}_n(k)\text{-orbits of } X_\tau(k)\}$$

constructed in the two steps of (3.3).

Note that the Frobenius on $\mathbb{F}_p$ is the identity and therefore $\sigma^*(M_0) = M_0$. As $C^\bullet$ will be a filtration of type τ (3.2.1), we have $C^i = C^{i+1}$ and $D_i = D_{i-1}$ for $i \notin \{i_1, \dots, i_r\}$. Therefore it suffices to define C^{i_j}, D_{i_j}, and φ_{i_j} for $j = 1, \dots, r$. Recall the convention $i_1 > \cdots > i_r$.

We choose a basis $(e_1, \dots, e_n)$ of M_0. Then ${}^w\underline{M}$ is defined by

$$\begin{aligned} C^{i_j} &:= \bigoplus_{l=1}^{m_j} \mathbb{F}_p e_l, \\ D_{i_j} &:= \bigoplus_{l=m_{j-1}+1}^{n} \mathbb{F}_p e_{w^\circ(l)}, \\ \varphi_{i_j} \colon C^{i_j}/C^{i_j+1} &= \bigoplus_{l=m_{j-1}+1}^{m_j} \mathbb{F}_p e_l \\ &\xrightarrow{e_l \mapsto e_{w^\circ(l)}} D_{i_j}/D_{i_j-1} = \bigoplus_{l=m_{j-1}+1}^{m_j} \mathbb{F}_p e_{w^\circ(l)}. \end{aligned} \tag{3.4.3}$$

Note that ${}^w\underline{M}$ also depends on τ. Altogether we obtain:

Theorem 3.6. *Let τ be a filtration type and let $(n_1, \dots, n_r)$ be the associated flag type. Associating to $w \in S_n$ the F-zip ${}^w\underline{M}$ induces a bijection*

$$ {}^J W \leftrightarrow \left\{ \begin{matrix} \textit{isomorphism classes of } F\textit{-zips} \\ \textit{of type } \tau \textit{ over } k \end{matrix} \right\},$$

where ${}^J W$ is described in (3.4.1).

Of course, ${}^J W$ depends only on τ and we will often write ${}^\tau W$ instead of ${}^J W$.

Description of ordinary F-zips.
It follows from this explicit description that we have the following criterion for an F-zip to be ordinary in the sense of Definition 1.12.

Corollary 3.7. *Let $\underline{M}$ be an F-zip. The following assertions are equivalent.*
(1) *$\underline{M}$ is ordinary.*
(2) *$\underline{M}$ is a direct sum of Tate-F-zips (Example 1.9).*
(3) *The permutation corresponding to $\underline{M}$ via the bijection in Theorem 3.6 is the element of maximal length in ${}^JW$, i.e., the element $w_{0,J}w_0$.*

Decomposition in Tate-F-zips and mixed weight F-zips.
Let $\underline{M}$ be an F-zip over k with set of weights $\{i_1 > \cdots > i_r\}$ and flag type $(n_1, \dots, n_r)$. As k is algebraically closed now, the decomposition (1.8.1) of an F-zip $\underline{M}$ over k has the form

$$\underline{M} = \underline{M}_{\mathrm{mw}} \oplus \bigoplus_{j=1}^{r} \underline{T}(i_j)^{\oplus t_j}$$

with $0 \leq t_j \leq n_j$.

We use the notations of (3.4.1). Let $w \in {}^JW$ be the permutation corresponding to the isomorphism class of $\underline{M}$. It follows from the explicit description in Theorem 3.6 that

$$t_j = \#\bigl\{\, i \in \{m_{j-1}+1, \dots, m_j\} \mid w^\circ(i) = i \bigr\}.$$

(3.5) Connecting the two classifications.

We now relate the representation theoretic classification of F-zips via bands in (3.1) with the group theoretic classification of (3.3). For this we again fix a type τ as in (3.2.1) and associate to each $w \in W = S_n$ a collection $\mathcal{X}$ of words $\mathfrak{x}_1, \dots, \mathfrak{x}_c$ such that for $w \in {}^\tau W$ we have

$${}^w(\underline{M}) \cong \underline{M}(\mathcal{X}) := \bigoplus_{a=1}^{c} \underline{M}(\mathfrak{x}_a),$$

where $\underline{M}(\mathfrak{x}_a)$ is the F-zip defined in (3.1). We first define a map

$$\iota\colon \{1, \dots, n\} \to \{i_1, \dots, i_r\} \subset \mathbb{Z},$$
$$\iota(\nu) = i_j, \qquad \text{for} \quad \sum_{h=1}^{j-1} n_h < \nu \leq \sum_{h=1}^{j} n_h.$$

Now write w° as a product of cycles

$$[\nu_{11}, \nu_{12}, \dots, \nu_{1d_1}] \dots [\nu_{c1}, \nu_{c2}, \dots, \nu_{cd_c}]$$

and then associate words $\mathfrak{x}_1, \dots, \mathfrak{x}_c$ of integers by

$$\mathfrak{x}_a = \iota(\nu_{a1})\iota(\nu_{a2}) \cdots \iota(\nu_{ad_a}).$$

From this comparison we also see:

Proposition 3.8. *Let $\underline{M}$ be the F-zip of type τ corresponding to a permutation $w \in {}^{\tau}W$ via Theorem 3.6. Then the number of indecomposable summands of $\underline{M}$ is the same as the number of cycles of w°. In particular, $\underline{M}$ is indecomposable if and only if w° consists of one cycle only.*

(3.6) A simple example.

We fix an integer $n \geq 1$ and define a filtration type by

$$\tau\colon \mathbb{Z} \to \mathbb{N}_0, \qquad \tau(i) = \begin{cases} 1, & i = 0; \\ n-1, & i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

In particular F-zips of type τ will be given by Dieudonné zips (2.3). We have $i_1 = 1 > i_2 = 0$, $J = (n-1, 1)$ and ${}^{\tau}W = {}^{J}W$ consists of the permutations w_i for $i = 1, \dots, n$ where

$$w_i := \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & n \\ 1 & \dots & i-1 & n & i & \dots & n-1 \end{pmatrix}.$$

We have $w_1^{\circ} = \mathrm{id}$ and for $i \geq 2$:

$$w_i^{\circ} = \begin{pmatrix} 1 & \dots & i-2 & i-1 & i & \dots & n-1 & n \\ 2 & \dots & i-1 & n & i & \dots & n-1 & 1 \end{pmatrix}.$$

The corresponding F-zip ${}^{w_i}\underline{M}$ is given by $C^1 = \bigoplus_{l=1}^{n-1} \mathbb{F}_p e_l$, $D_0 = \mathbb{F}_p e_{w_i^{\circ}(n)}$ (i.e., $D_0 = \mathbb{F}_p e_1$ for $i > 1$ and $D_0 = \mathbb{F}_p e_n$ for $i = 1$, and $\varphi_0\colon e_n \mapsto e_{n-1}$, $\varphi_1\colon e_l \mapsto e_{w_i^{\circ}(l)}$ for $l = 1, \dots, n-1$.

Writing w_i° as product of cycles we get

$$w_i^{\circ} = [1\ 2\ \dots\ i-1\ n][i][i+1]\dots[n-1].$$

The corresponding collection of words is then given by

$$\mathcal{X} : \mathfrak{x}_1 = \mathbf{1}^{i-1}\mathbf{0},\ \mathfrak{x}_2 = \mathbf{1},\ \mathfrak{x}_3 = \mathbf{1},\ \dots,\ \mathfrak{x}_{n-i+1} = \mathbf{1}.$$

Example of elliptic curves.
The special case $n = 2$ classifies the F-zip structure on $\underline{M}(E) = H^1_{\mathrm{DR}}(E/k)$ where E is an elliptic curve over the algebraically closed field k (2.1). We see that up to isomorphism there are only two F-zips of this type, isomorphic to ${}^{w}\underline{M}_k$ for $w \in S_2$ or to $\underline{M}(\mathcal{X})_k$ where $\mathcal{X} = \mathbf{1}, \mathbf{0}$ or $\mathcal{X} = \mathbf{10}$. Then E is supersingular if and only if $\underline{M}(E) \cong {}^{w}\underline{M}_k$ for $w = \mathrm{id}$ if and only if $\underline{M}(E) \cong \underline{M}(\mathcal{X})_k$ for $\mathcal{X} = \mathbf{10}$. Otherwise, E is ordinary.

(3.7) Classification of at level 1 truncated Barsotti-Tate groups.

Fix integers $0 \leq d \leq h$. To classify BT_1 of height h and dimension d over an algebraically clsed field k it suffices by Corollary 2.2 to parametrize the isomorphism classes of Dieudonné zips $\underline{M}$ of type τ, where $\tau(0) = h - d$ and $\tau(1) = d$. In this case we have (with the notations of (3.4))

$$^{\tau}W = \{\, w \in S_h \mid w^{-1}(1) < \cdots < w^{-1}(d), w^{-1}(d+1) < \cdots < w^{-1}(h) \,\}.$$

Therefore the two classifications give the following theorem.

Theorem 3.9. *The constructions in* (3.4.3) *and in* (3.1.1) *give bijections*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of at level 1 truncated} \\ \textit{Barsotti-Tate groups over } k \textit{ of height } h \textit{ and dimension } d \end{array} \right\}$$

$$\leftrightarrow {}^{\tau}W$$

$$\leftrightarrow \left\{ \begin{array}{c} \textit{unordered tuples of bands } \mathfrak{x}_1, \dots, \mathfrak{x}_t \textit{ in the letters } 0 \textit{ and } 1 \\ \textit{where } 0 \textit{ appears } h - d \textit{ times and } 1 \textit{ appears } d \textit{ times} \end{array} \right\}$$

(3.8) The p-rank and a-number.

Let k be algebraically closed. For every abelian variety X over k often two numerical invariants are considered, the p-rank and the a-number. Both depend only on the p-torsion $X[p]$ which is a BT_1 whose classification was obtained in (3.7). We will explain how to read off these invariants from the two classifications.

Let G be a BT_1 of height h and dimension d and let $\underline{M}$ be the corresponding Dieudonné zip (Corollary 2.2).

The p-rank.
As $G(k)$ is a finite abelian group killed by p, we have $G(k) \cong (\mathbb{Z}/p\mathbb{Z})^f$. This integer $f \geq 0$ is called the *p-rank of* G. By (2.5) the p-rank is simply the multiplicity of $\underline{T}(0)$ in $\underline{M}$. For the classifications this means the following.

If $\mathfrak{x}_1, \dots, \mathfrak{x}_r$ are the bands corresponding to the indecompsable summands of the F-zip $\underline{M}$ via the classification (3.1), the p-rank of f equals the number of bands $\mathfrak{x}_i$ which consist of the single letter $\mathbf{0}$.

If $w \in {}^{\tau}W$ is the permutation corresponding to the isomorphism class of $\underline{M}$ via Theorem 3.9, the p-rank of X is the number of $d + 1 \leq i \leq h$ such that $w^{\circ}(i) = i$.

The a-number.
Let α_p the finite commutative group scheme over k such that

$$\alpha_p(R) = \{\, a \in R \mid a^p = 0 \,\} \subset (R, +)$$

for all k-algebras R. This is a group scheme killed by p which is not a BT_1 (Frobenius and Verschiebung are both zero on α_p).

Consider the abelian group $\mathrm{Hom}(\alpha_p, G)$ of homomorphisms of group schemes over k. The obvious k-vector space structure on $\alpha_p(R)$ makes $\mathrm{Hom}(\alpha_p, G)$ into a k-vector space. Its dimension is called the *a-number of G* and denoted by $a(G)$.

Let (M, F, V) be the Dieudonné space associated to G via the equivalence of categories in Theorem 2.1 and $\underline{M}$ the corresponding Dieudonné zip. Then

$$a(\underline{M}) := a(G) = \dim_k(\mathrm{Ker}(F) \cap \sigma^*(\mathrm{Ker}(V))) = \dim_k(C^1 \cap \sigma^*(D_0)).$$

In terms of bands or permutations this number can be described as follows.

If $G = G_1 \times G_2$, we have $a(G) = a(G_1) + a(G_2)$ and hence it suffices to describe the a-number of an indecomposable Dieudonné zip $\underline{M}$ whose isomorphism class is therefore given by a single band $\mathfrak{x} = x_0 x_1 \dots x_{n-1}$ where $x_i \in \{\mathbf{0}, \mathbf{1}\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Then the description of the F-zip corresponding to $\mathfrak{x}$ in (3.1.1) shows

$$a(\underline{M}) = \{\, i \in \mathbb{Z}/n\mathbb{Z} \mid x_i = \mathbf{0},\ x_{i+1} = \mathbf{1} \,\}.$$

We now assume that $d \leq h - d$ (otherwise we replace G by its Cartier dual, i.e., $\underline{M}$ by $\underline{M}^\vee(1)$, see Example 4.2 below). If $w \in {}^{\tau}W$ be the corresponding permutation (3.7), it follows immediately from the explicit description in (3.4.3) that

$$a(\underline{M}) = \#\{\, 1 \leq i \leq d \mid w^\circ(i) > d \,\}.$$

Via both descriptions (and also via the definition) we see that the a-number of an abelian variety X is zero if and only if the p-rank of X is equal to $\dim(X)$, i.e., if and only if X is ordinary.

4. F-zips with additional structures

Often the De Rham cohomology of an algebraic variety comes equipped with additional structures, e.g. pairings induced by Poincaré duality or by a polarization. Here we just consider two examples.

Notation: Let k be a perfect field of characteristic p.

(4.1) Symplectic and orthogonal F-Zips.

Let $d \in \mathbb{Z}$ and let S be an $\mathbb{F}_p$-scheme.

Definition 4.1. *A* d-symplectic F-zip over S *is an F-zip $\underline{M}$ over S together with a perfect pairing $\psi\colon \bigwedge^2(\underline{M}) \to \underline{T}(d)$, where $\underline{T}(d)$ is the Tate-zip of weight d ($\underline{T}(d)$ was in Example 1.9 only defined over a field, but the definition of $\underline{T}(d)$ over an arbitrary $\mathbb{F}_p$-scheme should be clear).*

Similarly, a d-orthogonal F-zip over S *is an F-zip $\underline{M}$ over S together with a perfect pairing $\psi\colon \mathrm{Sym}^2(\underline{M}) \to \underline{T}(d)$. When we speak of orthogonal F-zips, we will always assume that $p \neq 2$.*

For a more explicit definition of symplectic and orthogonal F-zips see also [MW] 6.1.

For a vector space M we denote by $M^\vee$ its dual. Note first that the existence of a perfect pairing $\psi\colon \underline{M} \otimes \underline{M} \to \underline{T}(d)$ implies in particular the existence of an isomorphism $M \xrightarrow{\sim} M^\vee$ which induces a k-linear isomorphism

$$C^i \xrightarrow{\sim} (C^{d+1-i})^\perp \subset \sigma^*(M)^\vee = \sigma^*(M^\vee).$$

This implies

$$\tau(i) = \tau(d-i) \tag{4.1.1}$$

for the type τ of $\underline{M}$.

We now fix a function $\tau\colon \mathbb{Z} \to \mathbb{N}_0$ with finite support $i_1 > \cdots > i_r$ and which satisfies (4.1.1). Again we set $n_j := \tau(i_j)$ and $n := n_1 + \cdots + n_r$. Then (4.1.1) implies

$$i_j + i_{r+1-j} = d, \tag{4.1.2}$$

$$n_j = n_{r+1-j} \tag{4.1.3}$$

for all $j = 1, \dots, r$.

Example 4.2. Let G be a BT_1 over a perfect field k and let $\underline{M}$ be the corresponding Dieudonné zip (2.3). Then the Cartier dual $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ of G is again a BT_1 and the corresponding Dieudonné zip is $\underline{M}^\vee(1)$.

Let k be algebraically closed and let $\underline{M}$ be an indecomposable F-zip over k with associated band $\mathfrak{x} = x_1 x_2 \dots x_n$ (via (3.1)). Then $\underline{M}^\vee$ is given by the band $\mathfrak{x}^\vee := (-x_1)(-x_2)\dots(-x_n)$ and $\underline{M}(d)$ (for $d \in \mathbb{Z}$) corresponds to the band $\mathfrak{x}(d) := (x_1+d)(x_2+d)\dots(x_n+d)$.

Fix $d \in \mathbb{Z}$. We will now classify d-symplectic and d-orthogonal F-zips over k, where k is algebraically closed. For d-orthogonal F-zips we first have to define a further invariant, namely its discriminant.

(4.2) Classification of symplectic F-zips.

We fix an n-dimensional $\mathbb{F}_p$-vector space $M_0 \neq 0$ together with symplectic (i.e. alternating and non-degenerate) pairing $\psi_0\colon \bigwedge^2(M_0) \to \mathbb{F}_p$. This implies that n is even, say $n = 2g$. Set $G = \mathrm{Sp}(M_0, \psi_0)$, considered as an algebraic group over $\mathbb{F}_p$.

As in (3.2) we define an $\mathbb{F}_p$-scheme $X_\tau^{d\text{-symp}}$ whose S-valued points are those d-symplectic F-zips $((M, C^\bullet, D_\bullet, \varphi_\bullet), \psi)$ of type τ over S such that $(M, \psi) = (M_0, \psi_0) \otimes_{\mathbb{F}_p} \mathscr{O}_S$. The same definition as (3.2.3) defines a $G \times G$-action on $X_\tau^{d\text{-symp}}$ such that, if we embed $G \hookrightarrow G \times G$ diagonally, the $G(k)$-orbits of $X_\tau^{d\text{-symp}}(k)$ correspond to isomorphism classes of d-symplectic F-zips of type τ over k.

The center of G is μ_2, the group of second roots of unity, acting by multiplication of (M_0, ψ_0). Set $H := \mathrm{PSp}(M_0, \psi_0) = G/\mu_2$. The $H \times H$-orbits of the wonderful compactification $\overline{H}$ of H are given by conjugacy classes $\mathcal{C}_H$ of parabolic subgroups of H (7.8). As for PGL_n, $\mathcal{C}_H$ is in bijection to flag types $(n_1, \dots, n_r)$ such that $n_1 + \cdots + n_r = 2g$ and such that (4.1.3) holds. In particular, our fixed type τ gives such a conjugacy class $(n_1, \dots, n_r)$ of parabolic subgroups. Let (W, I) be the Weyl group of H together with its set of simple reflections and let J be the subset of I which corresponds to our fixed conjugacy class $(n_1, \dots, n_r)$ via (7.5.1). Let $\overline{H}_J$ be the $H \times H$-orbit corresponding to J.

The same arguments as in (3.3) show that there exists a morphism

$$X_\tau^{d\text{-symp}} \to \overline{H}_J,$$

which induces a bijection from the set of $G(k)$-orbits of $X_\tau^{d\text{-symp}}(k)$ to the set of $H_F(k)$-orbits of $\overline{H}_J(k)$.

Now we can use the general Theorem 3.5 and obtain:

Theorem 4.3. *There exists a natural bijection*

$$\left\{ \begin{matrix} \textit{isomorphism classes of d-symplectic} \\ \textit{F-zips over } k \textit{ of type } \tau \end{matrix} \right\} \leftrightarrow {}^J W$$

We will make ${}^J W$ more explicit in the symplectic case. The Weyl group W and its set of simple reflections I can again be described by the relative position of flags as in the case of PGL_n. In addition there is a symmetry condition imposed by the symplectic from. More precisely we can identify

$$\begin{aligned} W &= \{\, w \in S_{2g} \mid w(i) + w(2g+1-i) = 2g+1 \text{ for all } i \,\}, \\ I &= \{\, s_i := \tau_i \tau_{2g-i} \mid i = 1, \dots, g-1 \,\} \cup \{ s_g := \tau_g \}. \end{aligned} \tag{4.2.1}$$

We set $m_j := n_1 + \cdots + n_j$. The flag type $(n_1, \dots, n_r)$ corresponds via (7.5.1) to the subset

$$J = \{\, s_i \mid i \notin \{m_1, m_2, \dots, m_r\} \,\}$$

of I. Then ${}^J W$ consists of those elements in W such that

$$w^{-1}(m_{i-1}+1) < \cdots < w^{-1}(m_i), \qquad \text{for } i = 1, \dots, r.$$

(4.3) The discriminant of an orthogonal F-zip.

Fix $d \in \mathbb{Z}$ and let $(\underline{M}, \psi)$ be a d-orthogonal F-zip over an $\mathbb{F}_p$-scheme S. We will define the *discriminant of* $(\underline{M}, \psi)$, denoted by $\mathrm{disc}(\underline{M}, \psi)$, in two steps.

Assume first that M is locally free of rank one. Let n be the weight of $\underline{M}$. For every étale morphism $\iota\colon U \to S$ consider the set

$$B(U) := \{\, e \in \Gamma(U, \iota^*(M)) \mid e \text{ generates } M,\ \varphi_n(\sigma^*(e)) = e \,\}.$$

The constant étale sheaf $\underline{\mathbb{F}}_p^\times$ acts by multiplication on B and this makes B into an $\underline{\mathbb{F}}_p^\times$-torsor. For any $e \in B(U)$ we have $\psi(e \otimes e) \in \underline{\mathbb{F}}_p^\times$ and its image in $\underline{\mathbb{F}}_p^\times/(\underline{\mathbb{F}}_p^\times)^2$ is independent of e. This defines a global section $\mathrm{disc}(\underline{M}, \psi)$ of the constant étale sheaf $\underline{\mathbb{F}}_p^\times/(\underline{\mathbb{F}}_p^\times)^2$.

Now let $(\underline{M}, \psi)$ be arbitrary. Then the maximal exterior power $\det(\underline{M})$ of $\underline{M}$ inherits a non-degenerate symmetric bilinear form $\det(\psi)$ by

$$\det(\psi)(m_1 \wedge \cdots \wedge m_h, m_1' \wedge \cdots \wedge m_h') = \sum_{\pi \in S_h} \mathrm{sgn}(\pi) \prod_{i=1}^{h} \psi(m_i, m'_{\pi(i)}).$$

But now the underlying $\mathscr{O}_S$-module of $\det(\underline{M})$ is locally free of rank one and we set $\mathrm{disc}(\underline{M}, \psi) = \mathrm{disc}(\det(\underline{M}), \det(\psi))$.

If $(\underline{M}_1, \psi_1)$ and $(\underline{M}_2, \psi_2)$ are two d-orthogonal F-zips over S, we can build their orthogonal sum $(\underline{M}, \psi)$ in the obvious way. Then clearly we have $\det(\underline{M}, \psi) = \det(\underline{M}_1, \psi_1) \otimes \det(\underline{M}_2, \psi_2)$ and we see that

$$\mathrm{disc}(\underline{M}, \psi) = \mathrm{disc}(\underline{M}_1, \psi_1) \cdot \mathrm{disc}(\underline{M}_2, \psi_2). \tag{4.3.1}$$

(4.4) Classification of orthogonal F-zips.

To deal with the orhogonal case, recall that we assume that $p \neq 2$. Again let k be algebraically closed.

For an arbitrary field K, by a *symmetric space over* K we mean a finite-dimensional K-vector space M together with a non-degenerate symmetric pairing $\psi\colon \mathrm{Sym}^2(M) \to K$. The *dimension of* (M, ψ) is by definition the dimension of the underlying vector space M and $\mathrm{disc}(\psi) \in K^\times/(K^\times)^2$ denotes the discriminant of (M, ψ).

We now fix an integer $n \geq 0$ and $\delta \in \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$. Any two symmetric spaces over $\mathbb{F}_p$ of dimension n and discriminant δ are isomorphic (see e.g. [Se2] chap. IV, 1.7). Let (M_0, ψ_0) such a symmetric space. Set $G = \mathrm{SO}(M_0, \psi_0)$ considered as an algebraic group over $\mathbb{F}_p$. Of course, G depends up to isomorphism only on n and δ.

To simplify the exposition we will from now on assume that $n = 2m + 1$ is odd. In this case the isomorphism class of G depends only on n and not on δ (any algebraic group over $\mathbb{F}_p$ which is isomorphic to G after base change to an algebraically closed field (i.e., any $\mathbb{F}_p$-form of G), is already isomorphic to G: as $\mathbb{F}_p$ has cohomological dimension 1, G has no nontrivial inner forms, and as the Dynkin diagram of G (which is of type (B_m)) has no automorphisms, G has no outer automorphisms and therefore every form of G is inner). Moreover G is adjoint in this case (Example 7.5). Both statements are not true if n is even.

Again we define an $\mathbb{F}_p$-scheme $X_{\tau,\delta}^{d\text{-orth}}$ whose S-valued points are those $d-$orthogonal F-zips $((M, C^\bullet, D_\bullet, \varphi_\bullet), \psi)$ of type τ over S whose discriminant in the sense of (4.3) is δ and such that $(M, \psi) = (M_0, \psi_0) \otimes_{\mathbb{F}_p} \mathscr{O}_S$. The same definition as (3.2.3) defines a $G \times G$-action on $X_{\tau,\delta}^{d\text{-orth}}$ such that, if we embed $G \hookrightarrow G \times G$ diagonally, the $G(k)$-orbits of $X_{\tau,\delta}^{d\text{-orth}}(k)$ correspond to isomorphism classes of d-orthogonal F-zips of type τ and discriminant δ over k.

As in the symplectic case, the $G \times G$-orbits of the wonderful compactification $\overline{G}$ of G correspond to flag types $(n_1, \dots, n_r)$ such that $n_1 + \dots + n_r = 2m+1$ and such that (4.1.3) holds. (Note that the analogue statement would not hold if n were even!) In particular, our fixed type τ gives such a conjugacy class $(n_1, \dots, n_r)$ of parabolic subgroups. Let (W, I) be the Weyl group of G together with its set of simple reflections and let J be the subset of I which corresponds to our fixed conjugacy class $(n_1, \dots, n_r)$ via (7.5.1). Let $\overline{G}_J$ be the $G \times G$-orbit corresponding to J.

As in the linear and the symplectic case we obtain by the general classification of G_F-orbits on $\overline{G}_J$ (Theorem 3.5):

Theorem 4.4. *There exists a natural bijection*

$$\left\{ \begin{matrix} \textit{isomorphism classes of d-orthogonal} \\ F\textit{-zips over } k \textit{ of type } \tau \end{matrix} \right\} \leftrightarrow {}^J W$$

The Weyl group W, its set of simple reflections I, and ${}^J W$ have the same description as in the symplectic case (4.2) (replacing g by m).

5. Examples II

As examples we will study abelian varieties endowed with a polarization of degree prime to p and polarized K3-surfaces.

In this section we assume that k is a perfect field of characteristic p.

(5.1) Prime-to-p-polarized abelian varieties.

We now apply the general results of (4.2) to describe the isomorphism classes of the first De-Rham cohomology of abelian varieties endowed with a polarization of degree prime to p.

Let X be an abelian variety of dimension $g \geq 1$ over a perfect field k. We denote by $X^\vee$ the dual abelian variety and by $\xi\colon X \to X^\vee$ a polarization of degree d [Mu2]. We assume that d is prime to p.

Via the canonical perfect pairing $H^1_{\mathrm{DR}}(X/k) \otimes H^1_{\mathrm{DR}}(X^\vee/k) \to k$ (see e.g. [BBM] 5.1), we can identify $H^1_{\mathrm{DR}}(X^\vee/k)$ with $H^1_{\mathrm{DR}}(X/k)^\vee$ and ξ induces a k-linear map $\beta^{-1}\colon H^1_{\mathrm{DR}}(X/k)^\vee \to H^1_{\mathrm{DR}}(X/k)$ which is an isomorphism because d is prime to p. We denote by $\psi\colon H^1_{\mathrm{DR}}(X/k) \otimes H^1_{\mathrm{DR}}(X/k) \to k$ the pairing associated to $\beta := (\beta^{-1})^{-1}$. As ξ is a polarization, we have $\xi^\vee = \xi$ and this implies $\beta^\vee = -\beta$ [BBM] 5.2.13. Therefore ψ is skew-symmetric and hence alternating if $p > 2$. For $p = 2$, ψ is also alternating, but one has to work harder to see this (e.g. [dJ] 2).

By (2.1), the type τ of the F-zip $H^1_{\mathrm{DR}}(X/k)$ is given by $\tau(i) = g$ for $i = 0, 1$ and $\tau(i) = 0$ otherwise. The filtration $C^\bullet$ and $D_\bullet$ are given by $C^1 = \sigma^* H^0(X, \Omega^1_{X/k})$ and $D_0 = H^1(X, \mathscr{H}^0(\Omega^\bullet_{X/k}))$.

Proposition 5.1. *$H^1_{\mathrm{DR}}(X/k)$ together with its F-zip structure and the alternating pairing ψ is a 1-symplectic F-zip of type τ.*

Proof. It remains to show that the k-linear isomorphism β is an isomorphisms of F-zips $H^1_{\mathrm{DR}}(X/k) \xrightarrow{\sim} H^1_{\mathrm{DR}}(X/k)^\vee(1)$. This is well-known in the language of Dieudonné spaces (2.3). Therefore define

$$\begin{aligned}
F &\colon \sigma^*(H^1_{\mathrm{DR}}(X/k)) \twoheadrightarrow \sigma^*(H^1_{\mathrm{DR}}(X/k))/C^1 \xrightarrow{\varphi_0} D_0 \hookrightarrow H^1_{\mathrm{DR}}(X/k), \\
V &\colon H^1_{\mathrm{DR}}(X/k) \twoheadrightarrow H^1_{\mathrm{DR}}(X/k)/D_0 \xrightarrow{\varphi_1^{-1}} C^1 \hookrightarrow \sigma^*(H^1_{\mathrm{DR}}(X/k)).
\end{aligned}$$

Going through the definition of the dual and the Tate twist of an F-zip, the assertion of the lemma is now equivalent to the equality

$$\psi(F(m_1), m_2) = \sigma^*(\psi)(m_1, V(m_2)) \tag{5.1.1}$$

for all $m_1 \in \sigma^*(H^1_{\mathrm{DR}}(X/k))$ and $m_2 \in H^1_{\mathrm{DR}}(X/k)$. Note that (5.1.1) implies (but is not equivalent to) that $H^0(X, \Omega^1_{X/k})$ and $H^1(X, \mathscr{H}^0(\Omega^\bullet_{X/k}))$ are Lagrangian subspaces of $H^1_{\mathrm{DR}}(X/k)$. □

Altogether we have associated to every abelian variety endowed with a polarization of degree prime to p a 1-symplectic F-zip $(H^1_{\mathrm{DR}}(X/k), \psi)$ of type τ. This construction is (contravariantly) functorial in (X, ξ). Conversely for fixed d prime to p, every 1-symplectic F-zip of type τ is isomorphic to $(H^1_{\mathrm{DR}}(X/k), \psi)$ for some abelian variety endowed with a polarization of degree d. For $p > 2$ this is a special case ($F = \mathbb{Q}$) of [Wd1] (7.2).

The flag type associated to τ is given by (g, g) and the subset J of simple reflections in the Weyl group is $\{s_1, \dots, s_{g-1}\}$ (with the notations of (4.2)).

Now assume that k is algebraically closed. By Theorem 4.3 isomorphism classes of 1-symplectic F-zips over k of type τ are given by ${}^JW$, where

$${}^JW = \{\, w \in W \mid w^{-1}(1) < \dots < w^{-1}(g) \,\}. \tag{5.1.2}$$

Note that every $w \in {}^JW$ automatically satisfies $w^{-1}(g+1) < \dots < w^{-1}(2g)$.

(5.2) Connection with Oort's classification.

Oort has given a different classification of symplectic Dieudonné spaces (i.e., Dieudonné spaces together with a symplectic pairing satisfying (5.1.1)) in [Oo]. We now explain the connection between the two classifications.

We can describe the set ${}^JW$ in (5.1.2) also as follows. For $w \in {}^JW$ we define a map

$$\begin{aligned}
&\varphi_w \colon \{0, \dots, g\} \to \{1, \dots, g\}, \\
&\qquad \varphi_w(i) = \#\{\, a \in \{1, \dots, g\} \mid w(a) > i \,\}.
\end{aligned} \tag{5.2.1}$$

We obtain a map $\varphi_w \colon \{1, \dots, g\} \to \mathbb{N}_0$ such that $\varphi_w(j) \leq \varphi_w(j+1) \leq \varphi_w(j) + 1$, i.e., an elementary sequence in the language of [Oo]. Then the isomorphism class of the 1-symplectic F-zip of type τ corresponding to $w \in {}^J W$ is given by the isomorphism class of the symplectic Dieudonné space corresponding to φ_w by the construction in [Oo] (9.1).

(5.3) Polarized $K3$-surfaces.

We now recall first the notion of a polarization on a K3-surface and some facts about K3-surfaces in general. Then we show that any K3-surface with polarization of degree prime to p gives rise to a 2-orthogonal F-zip of type τ, where

$$\tau(i) = \begin{cases} 1, & i = 0, 2; \\ 19, & i = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.1}$$

These are classified by the general classification of orthogonal F-zips in (4.4) which we make more explicit in this case.

Some facts about K3-surfaces.
Let X be a K3-surface over the perfect field k (2.6). A *polarization of* X is an ample line bundle $\mathscr{L}$. Its self-intersection $(\mathscr{L}, \mathscr{L})$ is called its *degree*. Hirzebruch-Riemann-Roch tells us

$$\chi(\mathscr{L}) = \chi(\mathscr{O}_X) + \frac{1}{2}\big((\mathscr{L}, \mathscr{L}) - (\mathscr{L}, K_X)\big).$$

By (2.6.1) we have $\chi(\mathscr{O}_X) = 2$ and by the definition of a K3-surface, K_X is trivial. We obtain $\chi(\mathscr{L}) = 2 + (\mathscr{L}, \mathscr{L})/2$ and in particular the degree is an even number.

The Picard functor $\mathrm{Pic}_{X/k}$ is representable by a group scheme locally of finite type and its Lie algebra is isomorphic to $H^1(X, \mathscr{O}_X)$ by [BLR] chapter 8. As $H^1(X, \mathscr{O}_X) = 0$ by (2.6.1), $\mathrm{Pic}_{X/k}$ is étale and its identity component $\mathrm{Pic}^0_{X/k}$ is trivial. Therefore $\mathrm{Pic}(X)$ is equal to the Néron-Severi group $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and this is a finitely generated free abelian group.

Next we recall some facts about cup-product and chern classes for K3-surfaces. As a reference we use [De2], where crystalline versions of these constructions are explained. But the crystalline cohomology $H^i_{\mathrm{cris}}(X/W(k))$ is a free $W(k)$-module, where $W(k)$ denotes the ring of Witt vectors of k by loc. cit. Prop. 1.1. Therefore we have

$$H^i_{\mathrm{cris}}(X/W(k)) \otimes_{W(k)} k = H^i_{\mathrm{DR}}(X/k)$$

for all $i \geq 0$ (see e.g. [Ill1] 1.3(b)) and all our constructions are just "(mod p)-versions" of the crystalline theory explained in [De2].

Composing the cup-product with the trace map we get a non-degenerate symmetric pairing

$$\psi\colon H^2_{\mathrm{DR}}(X/k) \otimes H^2_{\mathrm{DR}}(X/k) \xrightarrow{\cup} H^4_{\mathrm{DR}}(X/k) \xrightarrow{\mathrm{tr}} k. \tag{5.3.2}$$

For all $\mathscr{L} \in \mathrm{Pic}(X)$ we have

$$(\mathscr{L}, \mathscr{L}) = \psi(c_1(\mathscr{L}), c_1(\mathscr{L})), \tag{5.3.3}$$

where $c_1 \colon \mathrm{Pic}(X) = \mathrm{NS}(X) \to H^2_{\mathrm{DR}}(X/k)$ is the Chern class map (1.10). Here we consider the integer on the left hand side as an element of k.

Orthogonal F-zips associated to polarized K3-surfaces.
Let X be a K3-surface over k. We have seen in (2.6) that $H^2_{\mathrm{DR}}(X/k)$ carries the structure of an F-zip of type $(1, 20, 1) \in \mathbb{N}_0^{\{0,1,2\}} \subset \mathbb{N}_0^{\mathbb{Z}}$. The following lemma follows from work of Ogus (namely that in the language of [Og2] §1 the second crystalline cohomology is a K3-crystal).

Lemma 5.2. *The pairing ψ (5.3.2) defines on $H^2_{\mathrm{DR}}(X/k)$ the structure of a 2-orthogonal F-zip of type $\tilde{\tau}$.*

From now on we fix an integer $d \geq 1$. Let $\mathscr{L}$ be a polarization on X of degree $2d$. We assume that p does not divide $2d$. Then (5.3.3) implies that the subspace $\langle c_1(\mathscr{L}) \rangle$ of $H^2_{\mathrm{DR}}(X/k)$ generated by $c_1(\mathscr{L})$ is nonzero and that the restriction of ψ to $\langle c_1(\mathscr{L}) \rangle$ is non-degenerate. We denote by $H_{\mathrm{prim}}(X, \mathscr{L})$ the orthogonal complement of $\langle c_1(\mathscr{L}) \rangle$. It follows from [Og1] that $\langle c_1(\mathscr{L}) \rangle$ carries an induced F-zip structure and that there exists an isomorphism $\langle c_1(\mathscr{L}) \rangle \xrightarrow{\sim} \underline{T}(1)$ such that the restriction of ψ to $\langle c_1(\mathscr{L}) \rangle$ corresponds via this isomorphism to the natural pairing $\underline{T}(1) \otimes \underline{T}(1) \to \underline{T}(2)$. Therefore also the orthogonal complement $H_{\mathrm{prim}}(X, \mathscr{L})$ carries the structure of a 2-orthogonal F-zip over k.

Altogether we associated to each polarized K3-surface $(X, \mathscr{L})$ of degree $2d$ a 2-orthogonal F-zip $(H_{\mathrm{prim}}(X, \mathscr{L}), \psi)$ whose type is the function τ defined in (5.3.1). We will call such an F-zip a K3-zip.

Lemma 5.3. *Denote by δ the image of $-2d$ in $\mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$. Then the discriminant of the K3-zip $(H_{\mathrm{prim}}(X, \mathscr{L}), \psi)$ (in the sense of (4.3)) is δ.*

Proof. By [Og2] §1 (1.4) we have $\mathrm{disc}(H^2_{\mathrm{DR}}(X/k)) = -1$ and by the definition of $2d$, we have $\mathrm{disc}(\langle c_1(\mathscr{L}) \rangle) = 2d$. Therefore the lemma follows from (4.3.1). □

Now let k be algebraically closed. By the general classification Theorem 4.4, isomorphism classes of K3-zips over k with discriminant δ are in bijection with ${}^J W$, where

$$\begin{aligned} W &= \{\, w \in S_{20} \mid w(i) + w(21 - i) = 21 \text{ for all } i \,\}, \\ {}^J W &= \{\, w \in W \mid w^{-1}(2) < \cdots < w^{-1}(19) \,\}. \end{aligned} \tag{5.3.4}$$

Therefore we have a bijection

$${}^J W \leftrightarrow \{1, \dots, 20\}, \quad w \mapsto w^{-1}(1). \tag{5.3.5}$$

Note that the element in ${}^J W$ corresponding to $r \in \{1, \dots, 20\}$ has length $r - 1$.

Description of K3-zips.
We will now give an explicit description of the K3-zip over the algebraically closed field k corresponding to w_r. Therefore fix an $r \in \{1, \dots, 20\}$. Let M_0 be a 21-dimensional $\mathbb{F}_p$-vector space and choose a basis $e_1, \dots, e_{21}$ of M_0. Let ψ_0 be the symmetric bilinear form on M associated to the quadratic from

$$\sum x_i e_i \mapsto x_1 x_{21} + \cdots + x_{10} x_{12} + x_{11}^2.$$

Note that we can identify W with

$$W' := \{\, w \in S_{21} \mid w(i) + w(22-i) = 22 \text{ for all } i \,\}$$

by sending $w \in W$ to $w' \in W'$ defined by $w'(i) := w(i)$ for $i \leq 10$, $w'(11) := 11$, and $w'(i) = w(i-1)$ for $i \geq 12$. Under this bijection ${}^J W$ is send to

$${}^J W' := \{\, w \in W' \mid w^{-1}(2) < \cdots < w^{-1}(20) \,\}.$$

As in (5.3.5) we have a bijection

$$\{1, \dots, 10, 12, \dots, 21\} \leftrightarrow {}^J W', \qquad r \mapsto w_r. \tag{5.3.6}$$

We denote the K3-zip corresponding to w_r by ${}^r\underline{M}$. Note that we have $\ell(w_r) = r-1$.

The underlying symmetric space of ${}^r\underline{M}$ is $(M_0, \psi_0)_k$ and the underlying F-zip is $({}^{w_r}\underline{M})_k$, where ${}^{w_r}\underline{M}$ is the F-zip of type τ constructed in Example 3.4.3. We distinguish three cases.

$r = 21$: In this case w_r is the element of maximal length,

$${}^1\underline{M} \cong \underline{T}(0) \oplus \underline{T}(1)^{\oplus 19} \oplus \underline{T}(2),$$

and ψ induces a perfect duality between $\underline{T}(0)$ and $\underline{T}(2)$. This is the case where $\underline{M}$ is ordinary (Corollary 3.7).

$12 \leq r \leq 20$: In this case

$${}^r\underline{M} \cong \underline{M}_1 \oplus \underline{T}(1)^{\oplus 2r-23} \oplus \underline{M}_2,$$

where $\underline{M}_1 \cong {}^{w_n}\underline{M}$ is the indecomposable F-zip defined in Example 3.6 for $n = 22 - r$, and $\underline{M}_2$ is in perfect duality via ψ to $\underline{M}_1$ and hence $\underline{M}_2 \cong \underline{M}_1^\vee(2)$. In other words, via the classification (3.1), $\underline{M}_1$ is given by the word $\mathbf{1}^{(21-r)}\mathbf{0}$, and $\underline{M}_2$ is given by the word $\mathbf{1}^{(21-r)}\mathbf{2}$, where $\mathbf{i}^{(d)}$ means that $\mathbf{i}$ is repeated d-times.

$1 \leq r \leq 10$: Then

$${}^r\underline{M} \cong \underline{N} \oplus \underline{T}(1)^{\oplus 21-2r},$$

where $\underline{N}$ is an indecomposable F-zip such that ψ induces on $\underline{N}$ the structure of a 2-othogonal F-zip. The band corresponding to $\underline{N}$ via the classification in (3.1) is given by $\mathbf{21}^{(r-1)}\mathbf{01}^{(r-1)}$.

6. Families of F-zips

(6.1) F-zip stratification.

Let τ be a filtration type (3.2.1) and let $\underline{M}$ be an F-zip over an $\mathbb{F}_p$-scheme S. For $w \in {}^\tau W$ we define a locally closed subschemes S_w of S which is the "locus in S where $\underline{M}$ is of isomorphism class w". We will do this in two equivalent ways.

Definition of S_w via the stack of F-zips.
The first way is via the moduli stack of F-zips of type τ. For this we define the algebraic stack (in the sense of [LM])

$$\mathscr{X}_\tau := [G\backslash X_\tau],$$

where X_τ is the scheme of F-zips $\underline{M}$ of type τ such that $M = M_0 \otimes_{\mathbb{F}_p} \mathscr{O}_S$ for fixed $\mathbb{F}_p$-vector space M_0 (defined in (3.2)) and where $G = \mathrm{GL}(M_0)$. The action of G on X_τ is given via (3.2.3), where $G \hookrightarrow G \times G$ is diagonally embedded. Then for any $\mathbb{F}_p$-scheme S the category of 1-morphisms $S \to \mathscr{X}_\tau$ is equivalent to the category of F-zips over S of type τ where as morphisms we take only isomorphisms of F-zips. By Proposition 3.2 this is a smooth algebraic stack of dimension 0.

Let k be an algebraically closed extension of $\mathbb{F}_p$. The G_k-orbits on $(X_\tau)_k$ are parametrized by ${}^\tau W$ (3.3) and are already defined over $\mathbb{F}_p$, because the standard F-zips of (3.4.3) are defined over $\mathbb{F}_p$. For $w \in {}^\tau W$ let ${}^w X_\tau \subset X_\tau$ this orbit. Then ${}^w\mathscr{X}_\tau := [G\backslash {}^w X_\tau]$ is an algebraic substack of $\mathscr{X}_\tau$ which is smooth of dimension $\ell(w) - \dim(\mathrm{Par}_J)$ by (3.3.3).

Now an F-zip $\underline{M}$ over S of type τ defines a 1-morphism $\gamma_{\underline{M}}\colon S \to \mathscr{X}_\tau$ and we define $S_w \subset S$ as the inverse image of ${}^w\mathscr{X}_\tau$ under $\gamma_{\underline{M}}$.

Definition of S_w via isotrivial F-zips.
We now translate the definition of S_w given above in a language which avoids algebraic stacks. For this we use the explicitly defined "standard" F-zips ${}^w\underline{M}$ for $w \in {}^\tau W$ over $\mathbb{F}_p$. We say that an F-zip $\underline{M}$ over an $\mathbb{F}_p$-scheme T is *isotrivial of isomorphism class* w if for any $t \in T$ there exists an open affine neighborhood U of t and a faithfully flat morphism $\alpha\colon V \to U$ of finite presentation such that $\alpha^*(\underline{M}_{|U}) \cong {}^w\underline{M} \otimes_{\mathbb{F}_p} \mathscr{O}_V$. Note that in this definition we could have replaced ${}^w\underline{M}$ by any other F-zip over $\mathbb{F}_p$ which becomes isomorphic to ${}^w\underline{M}$ over some field extension (or, equivalently, over some finite field extensions) of $\mathbb{F}_p$.

Now let $\underline{M}$ be an arbitary F-zip of type τ over S. Then S_w is the (necessarily unique) subscheme of S which satisfies the following universal property. A morphism $\alpha\colon T \to S$ factors through S_w if and only if $\alpha^*(\underline{M})$ is isotrivial of isomorphism class w.

Set-theoretically S is the disjoint union of the subschemes S_w. But even set-theoretically it is not true in general that the closure of a subscheme S_w is the union of other subschemes of the form $S_{w'}$.

Clearly there is a variant for this for d-symplectic or d-orthogonal F-zips over a scheme S, where again we obtain subschemes S_w which are now indexed by $w \in {}^J W$, where ${}^J W$ is described in (4.2) and in (4.4), respectively.

(6.2) Example: The Ekedahl-Oort stratification for the moduli space of principally polarized abelian varieties.

Fix $g \geq 1$. Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian schemes of dimension g in characteristic p, i.e., for each $\mathbb{F}_p$-scheme S the category of 1-morphisms $S \to \mathcal{A}_g$ is the category of principally polarized g-dimensional abelian schemes over S, where the morphisms are isomorphisms between abelian schemes preserving the principal polarization. This is a Deligne-Mumford stack by [FC] chapter 1. As we did not define the notion of an F-zip over an algebraic stack (which is not difficult), we consider the following variant of $\mathcal{A}_g$. Fix an integer $N \geq 3$ prime to p and consider the moduli stack $\mathcal{A}_{g,N}$ of principally polarized abelian schemes of dimension g in characteristic p together with a level-N-structure (loc. cit.). This is in fact a smooth quasi-projective scheme over $\mathbb{F}_p$ of dimension $g(g+1)/2$.

Let (X, ξ, η) be the universal principally polarized abelian scheme with level-N-structure over $\mathcal{A}_{g,N}$. As explained in (1.11), $H^1_{\mathrm{DR}}(X/\mathcal{A}_{g,N})$ carries the structure of an F-zip. Again it is not difficult to see that the principal polarization induces on $H^1_{\mathrm{DR}}(X/\mathcal{A}_{g,N})$ the structure of a 1-symplectic F-zip over $\mathcal{A}_{g,N}$. By (6.1) we obtain locally closed subschemes $(\mathcal{A}_{g,N})_w$ for $w \in {}^JW$, where ${}^JW$ is described in (5.1). The collection of these subschemes is called the *Ekedahl-Oort stratification of* $\mathcal{A}_{g,N}$. They have been defined as reduced subschemes in [Oo].

We recall briefly some properties of the Ekedahl-Oort stratification from [Oo], [Wd1], [MW], and [Wd2].

(1) For all $w \in {}^JW$ the Ekedahl-Oort stratum $(\mathcal{A}_{g,N})_w$ is quasi-affine and smooth of dimension $\ell(w)$. In particular, they are all nonempty.

(2) The closure of $(\mathcal{A}_{g,N})_w$ is the union of those Ekedahl-Oort strata $(\mathcal{A}_{g,N})_{w'}$ such that $w' \preceq w$ where $\preceq$ can be described explicitly. The partial order is a refinement of the Bruhat order on ${}^JW$ (it is strictly finer if and only if $g \geq 5$).

(3) There is a unique closed Ekedahl-Oort stratum (namely $(\mathcal{A}_{g,N})_{\mathrm{id}}$) and a unique open (and therefore dense) stratum (namely $(\mathcal{A}_{g,N})_{w_{0,J}w_0}$) which equals the ordinary locus in $\mathcal{A}_{g,N}$. For $w \neq \mathrm{id}$ the closure of $(\mathcal{A}_{g,N})_w$ is connected.

(6.3) Example: The moduli space of polarized K3-surfaces.

Fix an integer $d \geq 1$ and assume that p does not divide $2d$. We now consider the moduli stack $\mathcal{F}_{2d}$ of K3-surfaces together with a polarization of degree $2d$. This is a smooth Deligne-Mumford stack by work of Rizow [Ri] (see also [Ol]). Again the universal polarized K3-surface defines a morphism from $\mathcal{F}_{2d}$ into the algebraic stack classifying 2-orthogonal F-zips of the type τ defined in (5.3.1).

As above we obtain locally closed substacks $(\mathcal{F}_{2d})_s$ which are indexed by $s \in \{1, \dots, 20\}$ via the bijection (5.3.5). This is a refinement of the height stratification (this is essentially shown in [Og3]; we omit the details).

7. Appendix: On reductive groups and the wonderful compactification

Notation: In the appendix we denote by κ an arbitrary field and by k an algebraically closed extension of κ. If X is a scheme over κ, we set $X_k := X \otimes_\kappa k$. Moreover, if G is any group, $g \in G$, and H is a subgroup of G, we set ${}^gH := gHg^{-1}$.

We first recall some notions and facts about reductive groups ((7.1) – (7.6)). We refer to [Sp] or [SGA3] for the proofs.

(7.1) Examples of reductive groups.

Let G be a connected reductive group over κ. Here we will need only the following examples of reductive groups.

(A) For $n \geq 1$ let V be an n-dimensional κ-vector space. Then $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, and $\mathrm{PGL}(V)$, considered as algebraic groups over κ, are reductive groups.
(B) Let $V = \kappa^{2m+1}$ and q be the quadratic form $q(x_0, x_1, \dots, x_{2m}) = x_0^2 + x_1x_{2m} + \dots x_m x_{m+1}$. Then $\mathrm{SO}(V, q)$ is a reductive group.
(C) For $g \geq 1$, the algebraic group Sp_{2g} is a reductive group over κ.
(D) Let $V = \kappa^{2m}$ and q be the quadratic form $q(x_1, \dots, x_{2m}) = x_1x_{2m} + \dots x_m x_{m+1}$. Then $\mathrm{SO}(V, q)$ is a reductive group.

(7.2) Maximal tori.

Denote by $\mathbb{G}_m$ the multiplicative group over κ, i.e., $\mathbb{G}_m(R) = R^\times$ for every κ-algebra R. Recall that a *torus over* κ is an algebraic group T over κ such that $T_k \cong (\mathbb{G}_m)_k^n$ for some integer $n \geq 1$. A torus is *split* if $T \cong \mathbb{G}_m^n$. If G is a an algebraic group, an algebraic subgroup T of G is called a *maximal torus of G* if it is a torus such that T_k is maximal in the set of subtori of G_k. Maximal tori always exist.

Example 7.1. If $G = \mathrm{GL}_n$, the subgroup D of diagonal matrices in GL_n is a maximal torus which is also split. More generally, an algebraic subgroup T of G is a maximal torus if and only if T_k is conjugate in G_k to D_k. If κ is not separably closed, there exist also non-split maximal tori. In particular such tori are not conjugate to D over κ.

For every maximal torus T in SL_n (resp. in PGL_n) there exists a unique maximal torus T' in GL_n such that $T = T' \cap \mathrm{SL}_n$ (resp. such that T is the image of T' under the canonical homomorphism $\mathrm{GL}_n \to \mathrm{PGL}_n$).

(7.3) Parabolic subgroups and Borel subgroups.

An algebraic subgroup P of the reductive group G over κ is called a *parabolic subgroup* if the quotient G/P is a proper κ-scheme. Such subgroups are automatically connected and their own normalizers. A parabolic subgroup B of G is called *Borel subgroup* if B_k is a minimal parabolic subgroup of G_k.

Example 7.2. For $G = \mathrm{GL}(V)$, $G = \mathrm{PGL}(V)$, or $G = \mathrm{SL}(V)$ (notations of Example 7.1(A)), a subgroup P of G is a parabolic subgroup if and only if $P = P_{\mathcal{F}}$ is the stabilizer of a flag $\mathcal{F}$ of subspaces

$$\mathcal{F}\colon \quad 0 = V_0 \subset V_{n_1} \subset V_{n_1+n_2} \subset \cdots \subset V_{n_1+\cdots+n_r} = V_n = V$$

of V where $\dim(V_i) = i$ and $(n_1, \dots, n_r)$ is a tuple of integers $n_i \geq 1$ such that $n_1 + \cdots + n_r = n$. We call this tuple the *flag type of* $\mathcal{F}$.

Note that $\mathcal{F} \mapsto P_{\mathcal{F}}$ defines a bijection between flags in V and parabolic subgroup of G. Two parabolic subgroups of G are conjugate if and only if the correspondoing flags have the same flag type. A parabolic subgroup $P_{\mathcal{F}}$ is a Borel subgroup if and only if $\mathcal{F}$ is a complete flag, i.e., if its flag type is equal to $(1, \dots, 1)$.

If G is one of the reductive groups defined in Example 7.1(B) – (D), we can consider the natural embedding $G \hookrightarrow \mathrm{GL}_n$ for $n = 2m+1$, $n = 2g$, $n = 2m$ respectively. Then every parabolic subgroup P and any Borel subgroup B of G is of the form $P = P' \cap G$ and $B = B' \cap G$ where P' is a parabolic subgroup and B' is a Borel subgroup of GL_n.

(7.4) Scheme of parabolic subgroups.

To simplify the exposition we will assume from now on that G is *split*, i.e., that G has a split maximal torus (7.2). This implies that G admits a Borel subgroup. G is split if κ is algebraically closed by definition. Moreover, all the examples given in (A) – (D) are split over arbitrary fields κ.

We denote by $\mathcal{C}_G$ the set of conjugacy classes of parabolic subgroups of G. We fix a Borel subgroup B of G and a conjugacy class $J \in \mathcal{C}_G$. Then there exists a unique parabolic subgroup $P_C \in J$ of G which contains B. In particular, any two Borel subgroups of G are conjugate. We define a partial order on $\mathcal{C}_G$ by setting

$$J_1 \leq J_2 \quad \text{if and only if} \quad P_{J_1} \subset P_{J_2}. \tag{7.4.1}$$

As a parabolic subgroup is its own normalizer, the map $G \ni g \mapsto {}^g P_J$ defines a bijection $G/P_J \xrightarrow{\sim} C$. This allows to define the scheme Par_J of parabolic subgroups in J. It is a smooth and proper scheme over κ. In particular we get the scheme of Borel subgroups of G which is denoted by Bor.

Example 7.3. Let G be $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, or $\mathrm{PGL}(V)$. Two parabolic subgroups $P_{\mathcal{F}_1}$ and $P_{\mathcal{F}_2}$ of G are conjugated if and only if the types of $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal. By associating to a flag type $(n_1, \dots, n_r)$ the set

$$J_{(n_1,\dots,n_r)} := \{1, \dots, n-1\} \setminus \{n_1, n_1+n_2, \dots, n_1+\cdots+n_{r-1}\}, \tag{7.4.2}$$

we identify $\mathcal{C}_G$ with the set of subsets of $I := \{1, \dots, n-1\}$ (as a partially ordered set). The conjugacy class of a Borel subgroup corresponds to the empty set, and the conjugacy class of G (consisting only of G itself) corresponds to I.

(7.5) Weyl groups.

The scheme of pairs of Borel subgroups $\mathrm{Bor} \times \mathrm{Bor}$ carries a G-action by

$$g \cdot (B_1, B_2) := ({}^gB_1, {}^gB_2).$$

We denote by W the set of $G(k)$-orbits on $\mathrm{Bor}(k) \times \mathrm{Bor}(k)$. For $w \in W$ we denote the corresponding orbit by $\mathcal{O}(w)$. Moreover we set

$$\begin{aligned} \ell(w) &:= \dim(\mathcal{O}(w)) - \dim(\mathrm{Bor}), \qquad \text{for } w \in W; \\ I &:= \{\, w \in W \mid \ell(w) = 1 \,\}. \end{aligned}$$

We define a group structure on W as follows. Let $w_1, w_2 \in W$. We can choose Borel subgroups B_1, B_2, and B_3 of G_k containig a common maximal torus of G_k such that $(B_1, B_2) \in \mathcal{O}(w_1)$ and $(B_2, B_3) \in \mathcal{O}(w_2)$. Then we define $w := w_1w_2$ as the orbit $\mathcal{O}(w)$ such that $(B_1, B_3) \in \mathcal{O}(w)$. This defines a group structure on W. The identity of W is given by the orbit of Borel subgroups (B_1, B_2) such that $B_1 = B_2$ and the inverse on W is induced by $(B_1, B_2) \mapsto (B_2, B_1)$. Then (W, I) is a Coxeter system ([BouLie] chap. IV, §1) with length function ℓ. In particular, I generates W and $\ell(w)$ is the minimal number of elements $s_1, \dots, s_d \in I$ such that $w = s_1s_2 \dots s_d$.

For $i \in I$ we say that a parabolic subgroup P of G has *type* $\{i\}$ if for any two Borel subgroups $B_1, B_2 \subset P_k$ with $B_1 \neq B_2$ we have $(B_1, B_2) \in \mathcal{O}(i)$. If P is any parabolic subgroup of G, we define the *type of* P as the subset J of I which consists of those $i \in I$ such that P contains a parabolic subgroup of type $\{i\}$. Two parabolic subgroups of G have the same type if and only if they are conjugate. Therefore we obtain a bijection

$$\alpha\colon \mathcal{C}_G \leftrightarrow 2^I. \tag{7.5.1}$$

This bijection is an isomorphism of partially ordered sets for the order on $\mathcal{C}_G$ defined in (7.4.1) and the inclusion order on the set of subsets of I. If J is a subset of I we also write Par_J instead of $\mathrm{Par}_{\alpha^{-1}(J)}$ for the scheme of parabolic subgroups of type J.

Example 7.4. Let G be $\mathrm{GL}(V)$, $\mathrm{PGL}(V)$, or $\mathrm{SL}(V)$. By Example 7.2 each Borel subgroup of G_k is the stabilizer of a unique flag

$$\mathcal{F}\colon \quad 0 = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V \otimes_\kappa k$$

with $\dim(V_i) = i$. Consider two such flags $\mathcal{F}$ and $\mathcal{F}'$ corresponding to Borel subgroups B and B'. Set $\mathrm{gr}^i_{\mathcal{F}} = V^i/V^{i-1}$. This is a one-dimensional k-vector space and therefore there exists a unique $\pi(i) = \pi_{B,B'}(i) \in \{1, \dots, n\}$ such that the graded piece $\mathrm{gr}^{\pi(i)}_{\mathcal{F}'} \mathrm{gr}^i_{\mathcal{F}}$ of the flag induced by $\mathcal{F}'$ on $\mathrm{gr}^i_{\mathcal{F}}$ is nonzero. The map $\mathrm{Bor}(k) \times \mathrm{Bor}(k) \ni (B, B') \mapsto \pi^{-1}_{B,B'}$ induces an isomorphism of groups

$$\iota\colon W \xrightarrow{\sim} S_n.$$

We use ι to identify W with S_n. Via this identification we have for $w \in S_n$

$$\begin{aligned} \ell(w) &= \#\{\, (i,j) \mid i,j \in \{1,\dots,n\}, i < j, w(i) > w(j) \,\}; \\ I &= \{\tau_1, \tau_2, \dots, \tau_{n-1}\}, \end{aligned} \tag{7.5.2}$$

where $\tau_i \in S_n$ denotes transposition of i and $i+1$.

In (7.4.2) we already identified conjugacy classes of parabolic subgroups of G with subsets of $\{1,\dots,n-1\}$. Sending $i \in \{1,\dots,n-1\}$ to $\tau_i \in I$ defines the bijection α in (7.5.1).

(7.6) Adjoint groups.

Recall that a connected linear algebraic group H over $\mathbb{F}_p$ is called *adjoint* if the scheme-theoretical center is trivial. Such an algebraic group is automatically reductive.

Example 7.5. Examples for adjoint groups are PGL_n for $n \geq 1$ and $\mathrm{SO}(V,q)$ where (V,q) is the odd-dimensional quadratic space defined in (7.1) (B). In the latter case $\hat{H} = \mathrm{O}(V,q)$ is a linear algebraic group whose connected component of the identity is $\mathrm{SO}(V,q)$. If $p \neq 2$, $\mathrm{O}(V,q)$ has two connected components.

On the other hand, SL_n (for $n > 1$), GL_n (for $n \geq 1$), Sp_{2g} (for $g \geq 1$), and $\mathrm{SO}(V,q)$, where (V,q) is the even-dimensional quadratic space defined in (7.1) (D), are examples of algebraic groups which are not adjoint.

(7.7) The wonderful compactification.

We will now define the wonderful compactification of an adjoint group.

The algebraic group.
Let $\hat{H}$ be a linear algebraic group over κ and denote by H the connected component of the identity. We assume that H is an adjoint group over κ. We further fix a connected component H^1 of $\hat{H}$. To simplify the exposition we assume that H is split.

We define an $(H \times H)$-action on H^1 by setting

$$(h_1, h_2) \cdot h := h_1 h h_2^{-1}. \tag{7.7.1}$$

The wonderful compactification.
Let $\mathcal{V}_H$ be the scheme over κ whose points are the $\dim(H)$-dimensional Lie-subalgebras of $\mathrm{Lie}(H \times H)$. This is a closed subscheme of the Grassmannian of $\dim(H)$-dimensional subspaces of $\mathrm{Lie}(H \times H)$ and therefore it is projective. It is endowed with an action by $H \times H$ via

$$(h_1, h_2) \cdot \mathfrak{g} := \mathrm{ad}(h_1, h_2)(\mathfrak{g}). \tag{7.7.2}$$

For every $h \in H^1$ we define a subgroup $H_h \subset H \times H$ as

$$H_h := \{ (h_1, h_2) \in H \times H \mid h_1 = hh_2h^{-1} \}.$$

Then $\dim(H_h) = \dim(H)$ and $H_{(h_1,h_2)\cdot h} = \mathrm{int}(h_1,h_2)(H_h)$ and therefore

$$i\colon H^1 \hookrightarrow \mathcal{V}_H, \qquad h \mapsto \mathrm{Lie}(H_h)$$

is a well defined $(H\times H)$-equivariant morphism with respect to the actions (7.7.1) and (7.7.2). As H is adjoint, ι is an immersion.

We define $\overline{H^1}$ as the closure of $i(H^1)$ in $\mathcal{V}_H$. This is a closed $(H\times H)$-invariant subscheme of $\mathcal{V}_H$ and therefore it is an integral projective scheme over κ with an $(H\times H)$-action.

Definition 7.6. *The projective scheme $\overline{H^1}$ is called the* wonderful compactification *of H^1.*

We collect some facts about the wonderful compactification.

Theorem 7.7. *$\overline{H^1}$ is a smooth projective κ-scheme and $\overline{H^1} \setminus H^1$ is a divisor with normal crossings in $\overline{H^1}$.*

Recall that we identified the set $\mathcal{C}_H$ of conjugacy classes of parabolic subgroups of H with the set of subsets of I, where I is the set of simple reflections of the Coxeter system (W, I) (7.5.1).

Theorem 7.8. *There is a bijection*

$$\begin{aligned} \mathcal{C}_H = 2^I &\leftrightarrow \left\{ \begin{matrix} (H(k)\times H(k))\text{-orbits} \\ \text{of } \overline{H^1}(k) \end{matrix} \right\}, \\ J &\mapsto \overline{H^1_J}, \end{aligned} \tag{7.7.3}$$

such that

(1) *$\overline{H^1_I} = H^1$ and $\overline{H^1_J}$ is contained in the closure of $\overline{H^1_{J'}}$ if and only if $J \subset J'$.*

(2) *The codimension of $\overline{H^1_J}$ in $\overline{H^1}$ is $\#(I \setminus J)$.*

(3) *The intersection of the closure of $\overline{H^1_J}$ and the closure of $\overline{H^1_{J'}}$ is $\overline{H^1_{J\cap J'}}$.*

Proof of Theorems 7.7 and 7.8. If $H^1 = H$, theorems 7.7 and 7.8 are known (see [dCP] and [St]). We will reduce to this case.

We will use the following diction. If X and Y are varieties with an action by algebraic groups G and H, respectively, and if $\alpha\colon G \to H$ is a homomorphism of algebraic groups, we say that a morphism $\varphi\colon X \to Y$ is *α-equivariant* if $\varphi(g\cdot x) = \alpha(g)\cdot\varphi(x)$ for all $g \in G$ and $x \in X$.

Now choose $h^1 \in H^1(\kappa)$. Then

$$\begin{tikzcd} H \arrow[rr, "\sim", "\varphi\colon h\mapsto h^1 h"'] \arrow[d, "i"'] & & H^1 \arrow[d, "i"] \\ \mathcal{V}_H \arrow[rr, "\sim", "\bar{\varphi}\colon \mathfrak{g}\mapsto \mathrm{ad}(h^1,1)(\mathfrak{g})"'] & & \mathcal{V}_H \end{tikzcd} \tag{7.7.4}$$

is a commutative diagram. Moreover, if we define

$$\alpha\colon H \times H \to H \times H, \qquad (h_1, h_2) \mapsto (h^1 h_1 (h^1)^{-1}, h_2),$$

the morphisms φ and $\bar{\varphi}$ are both α-equivariant. Therefore $\bar{\varphi}$ defines an α-equivariant isomorphism $\overline{H} \to \overline{H^1}$. As α is an automorphism of $H \times H$, we can replace H^1 by H for the proof. □

Example 7.9. For $H = \mathrm{PGL}_n$ it follows that the $H \times H$-orbits of $\overline{H}$ are in bijection with subsets of $J \subset \{\tau_1, \dots, \tau_{n-1}\}$ (7.5.2) or also with flag types $(n_1, \dots, n_r)$ (7.4.2). We denote the corresponding orbit either by $\overline{H}_J$ or by $\overline{H}_{(n_1,\dots,n_r)}$.

The diagonal action.
Wo consider now a Frobenius-linear version of the diagonal action of H on $\overline{H^1}$. Therefore assume that $\kappa = \mathbb{F}_p$. Let $F\colon \hat{H} \to \hat{H}$ be the Frobenius endomorphism. If $\hat{H}$ is an algebraic subgroup of GL_n (and every affine algebraic group is isomorphic to an algebraic subgroup of GL_n), F is given on k-valued points by $(a_{ij}) \mapsto (a_{ij}^p)$ for a matrix $(a_{ij}) \in \hat{H}(k) \subset \mathrm{GL}_n(k)$.

Consider the closed embedding

$$H \hookrightarrow H \times H, \quad h \mapsto (h, F(h)) \tag{7.7.5}$$

and denote by H_F the image of H in $H \times H$. Then by restricting the $(H \times H)$-action to H_F we obtain an action of H_F on $\overline{H^1}$.

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