Abstract. We define a universal version of the Knizhnik-Zamolodchikov-Bernard (KZB) connection in genus 1. This is a flat connection over a principal bundle on the moduli space of elliptic curves with marked points. It restricts to a flat connection on configuration spaces of points on elliptic curves, which can be used for proving the formality of the pure braid groups on genus 1 surfaces. We study the monodromy of this connection and show that it gives rise to a relation between the KZ associator and a generating series for iterated integrals of Eisenstein forms. We show that the universal KZB connection realizes as the usual KZB connection for simple Lie algebras, and that in the $\mathfrak{sl}_n$ case this realization factors through the Cherednik algebras. This leads us to define a functor from the category of equivariant $D$-modules on $\mathfrak{sl}_n$ to that of modules over the Cherednik algebra, and to compute the character of irreducible equivariant $D$-modules over $\mathfrak{sl}_n$ which are supported on the nilpotent cone.

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INTRODUCTION

The KZ system was introduced in [KZ] as a system of equations satisfied by correlation functions in conformal field theory. It was then realized that this system has a universal version ([Dr3]). The monodromy of this system leads to representations of the braid groups, which can be used for proving the formality of the configuration spaces of $\mathbb{C}$, i.e., the fact that the fundamental groups of these spaces are formal (i.e., their Lie algebras are isomorphic with their associated graded, which is the holonomy Lie algebra and thus has an explicit presentation). This fact was first proved in the framework of minimal model theory ([Su, Ko]). These results gave rise to Drinfeld’s theory of associators and quasi-Hopf algebras ([Dr2, Dr3]); one of the purposes of this work was to give an algebraic construction of the formality isomorphisms, and indeed one of its by-products is the fact that these isomorphisms can be defined over $\mathbb{Q}$. 
In the case of configuration spaces over surfaces of genus $\geq 1$, similar Lie algebra isomorphisms were constructed by Bezrukavnikov ([Bez]), using results of Kriz ([Kr]). In this series of papers, we will show that this result can be reproved using a suitable flat connection over configuration spaces. This connection is a universal version of the KZB connection ([Be1, Be2]), which is the higher genus analogue of the KZ connection.

In this paper, we focus on the case of genus 1. We define the universal KZB connection (Section 1), and rederive from there the formality result (Section 2). As in the integrable case of the KZB connection, the universal KZB connection extends from the configuration spaces $C(E_r, n)/\Sigma_n$ to the moduli space $\mathcal{M}_{1,[n]}$ of elliptic curves with $n$ unordered marked points (Section 3). This means that: (a) the connection can be extended to the directions of variation of moduli, and (b) it is modular invariant.

This connection then gives rise to a monodromy morphism $\gamma_n : \Gamma_{1,[n]} \to G_n \times S_n$, which we analyze in Section 4. The images of most generators can be expressed using the KZ associator, but the image $\widetilde{\Theta}$ of the $S$-transformation expresses using iterated integrals of Eisenstein series. The relations between generators give rise to relations between $\Theta$ and the KZ associator, identities (28). This identity may be viewed as an elliptic analogue of the pentagon identity, as it is a de Rham analogue of the relation 6AS in [HLS] (in [Ma], the question was asked of the existence of this kind of identity).

In Section 5, we investigate how to algebraically construct a morphism $\Gamma_{1,[n]} \to G_n \times S_n$. We show that a morphism $\overline{B}_{1,n} \to \exp(\tilde{\iota}_{1,n} \times S_n)$ can be constructed using an associator only (here $\overline{B}_{1,n}$ is the reduced braid group of $n$ points on the torus). [Dr3] then implies that the formality isomorphism can be defined over $\mathbb{Q}$. In the last part of Section 5, we develop the analogue of the theory of quasitriangular quasibialgebras (QTQBA’s), namely elliptic structures over QTQBA’s. These structures give rise to representations of $\overline{B}_{1,n}$, and they can be modified by twist. We hope that in the case of a simple Lie algebra, and using suitable twists, the elliptic structure given in Section 5.4 will give rise to elliptic structures over the quantum group $U_q(\mathfrak{g})$ (where $q \in \mathbb{C}^\times$) or over the Lusztig quantum group (when $q$ is a root of unity), yielding back the representations of $\overline{B}_{1,n}$ from conformal field theory.

In Section 6, we show that the universal KZB connection indeed specializes to the ordinary KZB connection.

Sections 7-9 are dedicated applications of the ideas of the preceding sections (in particular, Section 6) to representation theory of Cherednik algebras.

More precisely, in Section 7, we construct a homomorphism from the Lie algebra $\tilde{\iota}_{1,n} \times \mathfrak{d}$ to the rational Cherednik algebra $H_n(k)$ of type $A_{n-1}$. This allows us to consider the elliptic KZB connection with values in representations of the rational Cherednik algebra.

The monodromy of this connection then gives representations of the true Cherednik algebra (i.e. the double affine Hecke algebra). In particular, this gives a simple way of constructing an isomorphism between the rational Cherednik algebra and the double affine Hecke algebra, with formal deformation parameters.

In Section 8, we consider the special representation $V_N$ of the rational Cherednik algebra $H_n(k)$, $k = N/n$, for which the elliptic KZB connection is the KZB connection for (holomorphic) $n$-point correlation functions of the WZW model for $\text{SL}_N(\mathbb{C})$ on the elliptic curve, when the marked points are labeled by the vector representation $\mathbb{C}^N$. This representation is realized in the space of equivariant polynomial functions on $\mathfrak{sl}_N$ with values in $(\mathbb{C}^N)^{\otimes n}$, and we show that it is irreducible, and calculate its character.

In Section 9, we generalize the construction of Section 8, by replacing, in the construction of $V_N$, the space of polynomial functions on $\mathfrak{sl}_N$ with an arbitrary $D$-module on $\mathfrak{sl}_N$. This gives rise to an exact functor from the category of (equivariant) $D$-modules on $\mathfrak{sl}_N$ to the category of representations of $H_n(N/n)$. We study this functor in detail. In particular, we show that this functor maps $D$-modules concentrated on the nilpotent cone to modules from category $\mathcal{O}_- \times$ of highest weight modules over the Cherednik algebra, and is closely related...
to the Gan-Ginzburg functor, \cite{GG1}. Using these facts, we show that it maps irreducible \(D\)-modules on the nilpotent cone to irreducible representations of the Cherednik algebra, and determine their highest weights. As an application, we compute the decomposition of cuspidal \(D\)-modules into irreducible representations of \(\text{SL}_N(\mathbb{C})\). Finally, we describe the generalization of the above result to the trigonometric case (which involves \(D\)-modules on the group and trigonometric Cherednik algebras), and point out several directions for generalization.

1. Bundles with flat connections on (reduced) configuration spaces

1.1. The Lie algebras \(t_{1,n}\) and \(\hat{t}_{1,n}\). Let \(n \geq 1\) be an integer and \(k\) be a field of characteristic zero. We define \(t_{1,n}^k\) as the Lie algebra with generators \(x_i, y_i (i = 1, \ldots, n)\) and \(t_{ij} (i \neq j \in \{1, \ldots, n\})\) and relations

\[
t_{ij} = t_{ij}, \quad [t_{ij}, t_{jk} + t_{kj}] = 0, \quad [t_{ij}, t_{ki}] = 0, \quad (1)
\]

\[
[x_i, y_j] = t_{ij}, \quad [x_i, x_j] = [y_i, y_j] = 0, \quad [x_i, y_k] = -\sum_{j \neq i} t_{ij},
\]

\[\quad [x_i, t_{jk}] = [y_i, t_{jk}] = 0, \quad [x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0.\]

(i, j, k, l are distinct). In this Lie algebra, \(\sum_i x_i\) and \(\sum_i y_i\) are central; we then define \(t_{1,n}^k := t_{1,n}^k/(\sum_i x_i, \sum_i y_i)\). Both \(t_{1,n}^k\) and \(\hat{t}_{1,n}^k\) are positively graded, where \(\deg(x_i) = \deg(y_i) = 1\).

The symmetric group \(S_n\) acts by automorphisms of \(t_{1,n}^k\) by \(\sigma(x_i) := x_{\sigma(i)}, \sigma(y_i) := y_{\sigma(i)}\), \(\sigma(t_{ij}) := t_{\sigma(i)\sigma(j)}\); this induces an action of \(S_n\) by automorphisms of \(\hat{t}_{1,n}^k\).

We will set \(t_{1,n} := t_{1,n}^c, \hat{t}_{1,n} := \hat{t}_{1,n}^c\) in Sections 1 to 4.

1.2. Bundles with flat connections over \(C(E, n)\) and \(\hat{C}(E, n)\). Let \(E\) be an elliptic curve, \(C(E, n)\) be the configuration space \(E^n - \{\text{diagonals}\} (n \geq 1)\) and \(\hat{C}(E, n) := C(E, n)/E\) be the reduced configuration space. We will define \(\exp(t_{1,n})\)-principal bundle with a flat (holomorphic) connection \((P_{E,n}, \nabla_{E,n}) \to C(E, n)\). For this, we define a \(\exp(t_{1,n})\)-principal bundle with a flat connection \((P_{E,n}, \nabla_{E,n}) \to C(E, n)\). Its image under the natural morphism \(\exp(\hat{t}_{1,n}) \to \exp(t_{1,n})\) is a \(\hat{t}_{1,n}\)-bundle with connection \((P_{E,n}, \nabla_{E,n}) \to C(E, n)\), and we then prove that \((P_{E,n}, \nabla_{E,n})\) is the pull-back of a pair \((P_{E,n}, \nabla_{E,n})\) under the canonical projection \(C(E, n) \to \hat{C}(E, n)\).

For this, we fix a uniformization \(E \simeq E_\tau\), where for \(\tau \in \mathfrak{H}, \mathfrak{H} := \{\tau \in \mathbb{C}| \Im(\tau) > 0\}\), \(E_\tau := C/\Lambda_\tau\) and \(\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau\).

We then have \(C(E_\tau, n) = (\mathbb{C}^n - \text{Diag}_{n, \tau})/\Lambda_\tau^n\), where \(\text{Diag}_{n, \tau} := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n|z_i = z_j \in \Lambda_\tau\text{ for some }i \neq j\}\). We define \(P_{E,n}\) as the restriction to \(C(E_\tau, n)\) of the bundle over \(\mathbb{C}^n/\Lambda_\tau^n\) for which a section on \(U \subset \mathbb{C}^n/\Lambda_\tau^n\) is a regular map \(f : \pi^{-1}(U) \to \exp(t_{1,n})\), such that \(f(z + \delta_i) = f(z), f(z + \tau \delta_i) = e^{-2\pi i x_i} f(z)\) (here \(\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda_\tau^n\) is the canonical projection and \(\delta_i\) is the \(i\)th vector of the canonical basis of \(\mathbb{C}^n\)).

The bundle \(P_{E,n} \to C(E_\tau, n)\) derived from \(P_{E,n}\) is the pull-back of a bundle \(\tilde{P}_{E,n} \to \hat{C}(E_\tau, n)\) since the \(e^{-2\pi i x_i}\) in \(\exp(t_{1,n})\) commute pairwise and their product is 1. Here \(x \mapsto \tilde{x}\) is the map \(t_{1,n} \to \hat{t}_{1,n}\).

A flat connection \(\nabla_{E,n}\) on \(P_{E,n}\) is then the same as an equivariant flat connection over the trivial bundle over \(\mathbb{C}^n - \text{Diag}_{n, \tau}\), i.e., a connection of the form

\[
\nabla_{\tau} := d - \sum_{i=1}^n K_i(\tau) dz_i,
\]

where \(K_i(-|\tau) : \mathbb{C}^n \to \hat{t}_{1,n}\) is holomorphic on \(\mathbb{C}^n - \text{Diag}_{n, \tau}\), such that:

\footnote{We will denote by \(\hat{g}\) or \(g^{\hat{\cdot}}\) the degree completion of a positively graded Lie algebra \(g\).}

\footnote{We set \(i := \sqrt{-1}\), leaving \(i\) for indices.
Lemma 1.1. \( K_i(z + \delta_j | \tau) = K_i(z|\tau) \) and \( K_i(z + \tau \delta_j | \tau) = e^{-2\pi i \text{ad}(x_j)} (K_i(z|\tau)) \), i.e., the \( K_i(z|\tau) \) satisfy condition (a).

Proof. We have \( K_i(z, x|\tau) = k(z, x|\tau) \) for any \( j \), \( K_i(z + \delta_j | \tau) = K_i(z|\tau) \). We have \( k(z \pm \tau, x|\tau) = e^{\mp 2\pi i x} k(z, x|\tau) + (e^{\mp 2\pi i x} - 1)/x \), so if \( j \neq i \), \( K_i(z + \tau \delta_j | \tau) = \sum_{j' \neq i} K_{ij'}(z_{ij'} | \tau) + e^{2\pi i \text{ad}(x_i)} K_{ij}(z_{ij} | \tau) + \frac{e^{2\pi i \text{ad}(x_i)} - 1}{\text{ad}(x_i)} (t_{ij}) - y_i \). Then

\[
e^{2\pi i \text{ad}(x_i)} (K_{ij}(z_{ij} | \tau)) = e^{-2\pi i \text{ad}(x_i)} (K_{ij}(z_{ij} | \tau)) \) and for \( j' \neq i, j \), \( K_{ij'}(z_{ij'} | \tau) = e^{-2\pi i \text{ad}(x_j)} (K_{ij'}(z_{ij'} | \tau)) \), so \( K_i(z + \tau \delta_j | \tau) = e^{-2\pi i \text{ad}(x_j)} (K_i(z|\tau)) \).

Now \( K_i(z + \tau \delta_i | \tau) = -\sum_i y_i - \sum_{j \neq i} K_j(z + \tau \delta_i | \tau) = -\sum_i y_i - e^{-2\pi i \text{ad}(x_i)} \left( \sum_{j \neq i} K_j(z|\tau) \right) = e^{-2\pi i \text{ad}(x_i)} \left( -\sum_i y_i - \sum_{j \neq i} K_j(z|\tau) \right) = e^{-2\pi i \text{ad}(x_j)} K_i(z|\tau) \) (the first and last equality follow from the proof of (c), the second equality has just been proved, the third equality follows from the centrality of \( y_i \)).

\( K_i(z|\tau) \) satisfy condition (b).

Proposition 1.2. \( \partial / \partial z_i - K_i(z|\tau), \partial / \partial z_j - K_j(z|\tau) = 0 \), i.e., \( K_i(z|\tau) \) satisfy condition (b).

Proof. For \( i \neq j \), let us set \( K_{ij} := K_{ij}(z_{ij} | \tau) \). Recall that \( K_{ij} + K_{ji} = 0 \), therefore if \( \partial_i := \partial / \partial z_i \)

\[ \partial_i K_{ij} - \partial_j K_{ji} = 0, \quad [y_i - K_{ij}, y_j - K_{ji}] = -[K_{ij}, y_i + y_j]. \]
Moreover, if \( i, j, k, \ell \) are distinct, then \([K_{ik}, K_{ij}] = 0\). It follows that if \( i \neq j\),
\[
[K_{ij} + [K_{ij}, K_{ij}], [K_{ij} + [K_{ij}, K_{ij}], [K_{ij} + [K_{ij}, K_{ij}]]]],
\]

Let us assume for a while that if \( k \notin \{i, j\} \), then

\[
[y_i, K_{jk}] - [y_j, K_{ki}] - [y_k, K_{ij}] + [K_{ij}, K_{ki}] + [K_{ik}, K_{ij}] + [K_{ik}, K_{jk}] = 0
\]

(this is the universal version of the classical dynamical Yang-Baxter equation).

Then (2) implies that

\[
[y_i + y_j, K_{ij}] + \sum_{k \neq i, j} [y_k, K_{ij}] = 0
\]

(as \( \sum_k y_k \) is central), which proves the proposition.

Let us now prove (2). If \( f(x) \in \mathbb{C}[x] \), then

\[
[y_k, f(\text{ad} x_i)(t_{ij})] = \frac{f(\text{ad} x_i) - f(-\text{ad} x_i)}{\text{ad} x_i + \text{ad} x_j} [-t_{ki}, t_{ij}],
\]

\[
[y_i, f(\text{ad} x_j)(t_{jk})] = \frac{f(\text{ad} x_j) - f(-\text{ad} x_j)}{\text{ad} x_j + \text{ad} x_k} [-t_{ij}, t_{jk}],
\]

\[
[y_j, f(\text{ad} x_k)(t_{ki})] = \frac{f(\text{ad} x_k) - f(-\text{ad} x_k)}{\text{ad} x_k + \text{ad} x_i} [-t_{jk}, t_{ki}].
\]

The first identity is proved as follows:

\[
[y_k, (\text{ad} x_i)^n(t_{ij})] = - \sum_{s=0}^{n-1} (\text{ad} x_i)^s (\text{ad} t_{ki}) (\text{ad} x_i)^{n-1-s}(t_{ij}) = - \sum_{s=0}^{n-1} (\text{ad} x_i)^s (\text{ad} t_{ki}) (-\text{ad} x_i)^{n-1-s}(t_{ij})
\]

\[
= - \sum_{s=0}^{n-1} (\text{ad} x_i)^s (-\text{ad} x_j)^{n-1-s}(\text{ad} t_{ki})(t_{ij}) = f(\text{ad} x_i, -\text{ad} x_j)(-\text{ad} x_j)^{n-1-s}(t_{ij}),
\]

where \( f(u, v) = (u^n - v^n)/(u - v) \). The two next identities follow from this one and from the fact that \( x_i + x_j + x_k \) commutes with \( t_{ij}, t_{ik}, t_{jk} \).

Then, if we write \( k(z, x) \) instead of \( k(z, x(r) \), the l.h.s. of (2) is equal to

\[
(k(z_{ij}, x_{ij} + x_{ij}) - k(z_{ij}, x_{ij}) k(z_{ij}, x_{ij} + x_{ij}) + k(z_{ik}, x_{ik}) k(z_{ik}, x_{ik} + x_{ik}) + \frac{k(z_{ij}, x_{ij} + x_{ij})}{\text{ad} x_j} \frac{k(z_{ik}, x_{ik} + x_{ik})}{\text{ad} x_j} \frac{k(z_{ij}, x_{ij} - x_{ij})}{\text{ad} x_j} \)[t_{ij}, t_{ik}].
\]

So (2) follows from the identity

\[
k(z, u + v) k(z' + z, u + v) = k(z, u) k(z' + z, u + v) + k(z', u) k(z', z + v)
\]

\[
+ \frac{k(z' - z, u + v)}{u + v} + \frac{k(z' + u + v)}{u} = 0,
\]

where \( u, v \) are formal variables, which is a consequence of the theta-functions identity

\[
(k(z, v - u) - \frac{1}{u + v}) (k(z', u + v) + \frac{1}{u + v}) - (k(z, u) + \frac{1}{u}) (k(z' - z, u + v) + \frac{1}{u + v})
\]

\[
+ (k(z', u) + \frac{1}{u}) (k(z' - z, v) + \frac{1}{v}) = 0.
\]

We have therefore proved:
Theorem 1.3. \((P_{\tau,n}, \nabla_{\tau,n})\) is a flat connection on \(C(E_{\tau}, n)\), and the induced flat connection \((P_{\tau,n}, \nabla_{\tau,n})\) is the pull-back of a unique flat connection \((P_{\tau,n}, \nabla_{\tau,n})\) on \(C(E_{\tau}, n)\).

1.3. Bundles with flat connections on \(C(E, n)/S_n\) and \(\tilde{C}(E, n)/S_n\). The group \(S_n\) acts freely by automorphisms of \(C(E, n)\) by \(\sigma(z_1, \ldots, z_n) := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})\). This descends to a free action of \(S_n\) on \(C(E, n)\). We set \(C(E, [n]) := C(E, n)/S_n\), \(\tilde{C}(E, [n]) := C(E, n)/S_n\).

We will show that \((P_{\tau,n}, \nabla_{\tau,n})\) induces a bundle with flat connection \((P_{\tau,[n]}, \nabla_{\tau,[n]})\) on \(C(E_{\tau}, [n])\) with group exp\(U_{\tau,n}\) \(\times S_n\), and similarly \((\tilde{P}_{\tau,n}, \tilde{\nabla}_{\tau,n})\) induces \((\tilde{P}_{\tau,[n]}, \tilde{\nabla}_{\tau,[n]})\) on \(\tilde{C}(E_{\tau}, [n])\) with group exp\(U_{\tau,[n]}\) \(\times S_n\).

We define \(P_{\tau,[n]} : C(E_{\tau}, [n])\) by the condition that a section of \(U \subset C(E_{\tau}, [n])\) is a regular map \(\pi^{-1}(U) \rightarrow \exp(U_{\tau,n}) \times S_n\), satisfying again \(f(z + \delta_i) = f(z)\), \(f(z + \tau \delta_i) = e^{-2\pi i x_i} f(z)\) and the additional requirement \(f(\sigma z) = \sigma f(z)\) (where \(\pi : C_n - \text{Diag}_{\tau} \rightarrow C(E_{\tau}, [n])\) is the canonical projection). It is clear that \(\nabla_{\tau,n}\) is \(S_n\)-invariant, which implies that it defines a flat connection \(\nabla_{\tau,[n]}\) on \(C(E_{\tau}, [n])\).

The bundle \(P(E_{\tau}, [n]) \rightarrow \tilde{C}(E_{\tau}, [n])\) is defined by the additional requirement \(f(z + u(\sum_i \delta_i)) = f(z)\) and \(\nabla_{\tau,n}\) then induces a flat connection \(\nabla_{\tau,[n]}\) on \(C(E_{\tau}, [n])\).

2. Formality of pure braid groups on the torus

2.1. Reminders on Malcev Lie algebras. Let \(k\) be a field of characteristic 0 and \(\mathfrak{g}\) be a pronilpotent \(k\)-Lie algebra. Set \(\mathfrak{g}^1 = \mathfrak{g}\); \(\mathfrak{g}^k = \mathfrak{g}^{k-1} + [\mathfrak{g}, \mathfrak{g}^{k-1}]\); then \(\mathfrak{g} = \bigoplus_{k \geq 1} \mathfrak{g}^k\). We also consider its completion \(\hat{\mathfrak{g}} = \bigoplus_{k \geq 1} \mathfrak{g}^k\). We say that \(\mathfrak{g}\) is formal if there exists an isomorphism of filtered Lie algebras \(\mathfrak{g} \simeq \mathfrak{g}\). The associated graded Lie algebra is \(\text{gr}(\mathfrak{g}) = \bigoplus_{k \geq 1} \mathfrak{g}^k/\mathfrak{g}^{k+1}\).

If \(\Gamma\) is a finitely generated group, there exists a unique pair \((\Gamma(k), i_{\Gamma})\) of pronilpotent algebraic group \(\Gamma(k)\) and a group morphism \(i_{\Gamma} : \Gamma \rightarrow \Gamma(k)\), which is initial in the category of all pairs \((U, j)\), where \(U\) is an algebraic \(k\)-algebraic group and \(j : \Gamma \rightarrow U\) is a group morphism.

We denote by \(\text{Lie}(\Gamma)_k\) the Lie algebra of \(\Gamma(k)\). Then we have \(\text{Lie}(\Gamma)_k = \exp(\text{Lie}(\Gamma)_k)\); \(\text{Lie}(\Gamma)_k\) is a pronilpotent Lie algebra. We have \(\text{Lie}(\Gamma)_k = \text{Lie}(\Gamma)_\mathbb{Q} \otimes k\). We say that \(\Gamma\) is formal if \(\text{Lie}(\Gamma)_\mathbb{Q}\) is formal (one can show that this implies that \(\text{Lie}(\Gamma)_\mathbb{Q}\) is formal).

When \(\Gamma\) is generated by \(g_1, \ldots, g_n\) and relations \(R_i(g_1, \ldots, g_n)\) \((i = 1, \ldots, p)\), \(\text{Lie}(\Gamma)_\mathbb{Q}\) is the quotient of the topologically free Lie algebra \(\mathfrak{f}_n\) generated by \(\gamma_1, \ldots, \gamma_n\) by the topological ideal generated by \(\log(R_i(e^{\gamma_1}, \ldots, e^{\gamma_n}))\) \((i = 1, \ldots, p)\).

The decreasing filtration of \(\mathfrak{f}_n\) is \(\mathfrak{f}_n = (\mathfrak{f}_n)^1 \supset (\mathfrak{f}_n)^2 \supset \ldots\), where \((\mathfrak{f}_n)^k\) is the part of \(\mathfrak{f}_n\) of degree \(\geq k\) in the generators \(\gamma_1, \ldots, \gamma_n\). The image of this filtration by the projection is map is the decreasing filtration \(\text{Lie}(\Gamma)_\mathbb{Q} = \text{Lie}(\Gamma)^1 \supset \text{Lie}(\Gamma)^2 \supset \ldots\) of \(\text{Lie}(\Gamma)_\mathbb{Q}\).

2.2. Presentation of \(\text{PB}_1\). For \(\tau \in \mathfrak{g}\), let \(U_{\tau} \subset C^n - \text{Diag}_{n,\tau}\) be the open subset of all \(z = (z_1, \ldots, z_n)\), of the form \(z_i = a_i + \tau b_i\), where \(0 < a_1 < \ldots < a_n < 1\) and \(0 < b_1 < \ldots < b_n < 1\). If \(z_0 = (z_0^1, \ldots, z_0^n) \in U_{\tau}\), its image \(z_0\) in \(E^n_{\tau}\) actually belongs to the configuration space \(C(E_{\tau}, n)\).

The pure braid group of \(n\) points on the torus \(\text{PB}_1\) may be viewed as \(\text{PB}_1 = \pi_1(C(E_{\tau}, n), \mathbb{Z}_0)\). Denote by \(X_i, Y_i \in \text{PB}_1\) the classes of the projection of the paths \([0, 1] \ni t \mapsto z_0 - t b_i\) and \([0, 1] \ni t \mapsto z_0 - t \tau \delta_i\).

Set \(A_i := X_i \cdots X_n, B_i := Y_i \cdots Y_n\) for \(i = 1, \ldots, n\). According to [Bil], \(A_i, B_i\) \((i = 1, \ldots, n)\) generate \(\text{PB}_1\), and a presentation of \(\text{PB}_1\) is, in terms of these generators:

\[
(A_i, A_j) = (B_i, B_j) = 1 \text{ (any } i, j), \quad (A_1, B_j) = (B_1, A_j) = 1 \text{ (any } j),
\]
(B_k, A_k A_j^{-1}) = (B_k B_j^{-1}, A_k) = C_{jk} (j \leq k), \quad (A_i, C_{jk}) = (B_i, C_{jk}) = 1 (i \leq j \leq k),
\text{where } (g, h) = ghg^{-1}h^{-1}.

2.3. Alternative presentations of $t_{1,n}$. We now give two variants of the defining presentation of $t_{1,n}$. Presentation (A) below is the original presentation in [Bez], and presentation (B) will be suited to the comparison with the above presentation of $PB_{1,n}$.

**Lemma 2.1.** $t_{1,n}$ admits the following presentations:

(A) generators are $x_i, y_i$ ($i = 1, \ldots, n$), relations are $[x_i, y_j] = [x_j, y_i] = 0$ (any $i, j$), $\{x_i, x_j \mid i \neq j \}$, $\{y_i, y_j \mid i \neq j \}$. \[\text{Similarly, } \{x_i, x_j \mid i \neq j \} \text{ is distinct).} \]

(B) generators are $a_i, b_i$ ($i = 1, \ldots, n$), relations are $[a_i, b_j] = [b_i, a_j] = 0$ (any $i, j$), $\{a_i, b_j \mid i \neq j \}$ are distinct; $\{a_i, b_k \mid i \leq j \leq k \}$, $\{a_i, c_{jk} \mid i \leq j \leq k \}$, where $c_{jk} = [b_k, a_k - a_j]$.

The isomorphism of presentations (A) and (B) is $a_i = \sum_{j=1}^{n} x_j, b_i = \sum_{j=1}^{n} y_j$.

**Proof.** Let us prove that the initial relations for $x_i, y_i, t_{ij}$ imply the relations (A) for $x_i, y_i$. Let us assume the initial relations. If $i \neq j$, then $[x_i, y_j] = t_{ij}$ and $t_{ij} = t_{ji}$. The relations $[x_i, x_j] = [y_i, y_j] = 0$ (any $i, j$) are contained in the initial relations. For any $i$, since $[x_i, y_i] = -\sum_{j \neq i} t_{ij}$ and $[x_j, y_i] = t_{ij}$ (any $i, j, k$), we get $\sum_{j \neq i} t_{ij} = 0$. Similarly, $\sum_{k, j} y_j x_i = 0$ (any $i$). If $i, j, k$ are distinct, since $[x_i, y_k] = t_{jk}$ and $[x_j, t_{jk}] = 0$, we get $[x_i, [y_j, x_k]] = 0$ and similarly we prove $[x_i, [y_j, x_k]] = 0$.

Let us now prove the relations (A) for $x_i, y_i$ imply the initial relations for $x_i, y_i$ and $t_{ij} := [x_i, y_j]$ (i $\neq j$). Assume the relations (A). If $i \neq j$, then $[x_i, y_j] = [x_j, y_i]$ we have $t_{ij} = t_{ji}$. The relation $t_{ij} = [x_i, y_j]$ (i $\neq j$) is clear and $[x_i, x_j] = [y_i, y_j] = 0$ (any $i, j$) are already in relations (A). Since for any $i$, $\sum_{j} x_j y_i = 0$, we get $x_i y_i = -\sum_{j \neq i} [x_i, y_j] = -\sum_{j \neq i} t_{ij} = \sum_{j \neq i} t_{ij}$. If $i, j, k$ are distinct, the relations $[x_i, y_k] = [y_i, y_k] = 0$ (any $i, j$) imply $t_{ij} = [y_i, t_{jk}] = 0$. If $i, j, k$ are distinct, we have shown that $t_{ij} = 0$ and $[t_{ij}, x_i + x_j] = 0$, which implies $[t_{ij}, [x_i + x_j, y_k]] = 0$, i.e., $[t_{ij}, t_{ik} + t_{jk}] = 0$.

We let prove that the relations (A) for $x_i, y_i$ imply relations (B) for $a_i := \sum_{j=1}^{n} x_j, b_i := \sum_{j=1}^{n} y_j$. Summing up the relations $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j'] = [x_j', y_i']$ for $i' = i, \ldots, n$ and $j' = j, \ldots, n$, we get $[a_i, b_j] = [b_i, a_j] = 0$ and $[a_i, b_j] = [a_i, b_j]$ (for any $i, j$).

Summing up $\sum_{j} x_j y_i' = \sum_{j} y_j x_i'$ for $i' = i, \ldots, n$, we get $[a_i, b_i'] = [a_i, b_i] = 0$ (for any $i$). Finally, $c_{jk} = \sum_{k=1}^{n-1} \sum_{j \neq k} t_{alpha}^b$ (in terms of the initial presentation) so the relations $[x_i', t_{alpha}] = 0$ for $i' \neq alpha$ and $[x_i + x_j', [t_{alpha}] = 0$ imply $[a_i, c_{jk}] = 0$ for $i \leq j \leq k$. Similarly, one shows $[b_i, c_{jk}] = 0$ for $i \leq j \leq k$.

Let us prove that the relations (B) for $a_i, b_i$ imply relations (A) for $x_i := a_i - a_{i+1}$, $y_i := b_i - b_{i+1}$ (with the convention $a_{n+1} = b_{n+1} = 0$). As before, $[a_i, b_j] = [b_i, a_j] = 0$, $[a_i, b_j] = [a_i, b_j]$ imply $[x_i, x_j] = [y_i, y_j] = 0$ (for any $i, j$). We set $t_{ij} := [x_i, y_j]$ for $i \neq j$, then we have $t_{ij} = t_{ji}$. We have for $j < k, t_{jk} = c_{jk} - c_{jk+1} - c_{j+1,k} + c_{j+1,k+1}$ (we set $c_{i,n+1} = 0$), so $[a_i, c_{jk}] = 0$ implies $\sum_{j=i}^{n} x_{j'}, t_{jk} = 0$ for $i \leq j < k$. When $i < j < k$, the difference between this relation and its analogue of $(i+1, j, k)$ gives $[x_i, t_{jk}] = 0$ for $i < j < k$. This can be rewritten $[x_i, [x_j, y_k]] = 0$ and since $[x_i, x_j] = 0$, we get $[x_k, [x_i, y_k]] = 0$, so $[x_i, t_{jk}] = 0$ and by changing indices, $[x_i, t_{jk}] = 0$ for $j < i < k$. Rewriting again $[x_i, t_{jk}] = 0$ for $i < j < k$ as $[x_i, [y_j, x_k]] = 0$ and using $[x_i, x_k] = 0$, we get $[x_k, [x_i, y_j]] = 0$ i.e., $[x_k, t_{jk}] = 0$, which we rewrite $[x_i, t_{jk}] = 0$ for $j < k < i$. Finally, $[x_i, t_{jk}] = 0$ for $i < j < k$ and $i \notin \{j, k\}$, which implies $[x_i, t_{jk}] = 0$ for $i, j, k$ different. One proves similarly $[y_i, t_{jk}] = 0$ for $i, j, k$ different.
2.4. The formality of $\text{PB}_{1,n}$. The flat connection $d - \sum_{i=1}^n K_i(z^\tau) d z_i$ gives rise to a monodromy representation $\mu_{z_0, \tau} : \text{PB}_{1,n} = \pi_1(C, z_0) \to \exp(\hat{t}_{1,n})$, which factors through a morphism $\mu_{z_0, \tau}(C) : \text{PB}_{1,n}(C) \to \exp(\hat{t}_{1,n})$. Let $\text{Lie}(\mu_{z_0, \tau}) : \text{Lie}(\text{PB}_{1,n}) \to \hat{t}_{1,n}$ be the corresponding morphism between pronilpotent Lie algebras.

**Proposition 2.2.** $\text{Lie}(\mu_{z_0, \tau})$ is an isomorphism of filtered Lie algebras, so that $\text{PB}_{1,n}$ is formal.

**Proof.** As we have seen, $\text{Lie}(\text{PB}_{1,n})$ (denoted $\text{Lie}(\text{PB}_{1,n})$ in this proof) is the quotient of the topologically free Lie algebra generated by $a_i, \beta_i$ ($i = 1, \ldots, n$) by the topological ideal generated by $[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\alpha_i, \beta_j], [\beta_i, \alpha_j], \log(e^{\beta_i} e^\alpha), \log(\log e^{\beta_i} e^\alpha), [\alpha_i, \gamma_{jk}], [\beta_i, \gamma_{jk}]$ where $\gamma_{jk} = \log(e^{\beta_k} e^{\alpha_j})$.

This presentation and the above presentation (B) of $t_{1,n}$ imply that there is a morphism of graded Lie algebras $p_n : t_{1,n} \to \text{gr} \text{Lie}(\text{PB}_{1,n})$ defined by $a_i \mapsto [\alpha_i], b_i \mapsto [\beta_i]$, where $\alpha \mapsto [\alpha]$ is the projection map $\text{Lie}(\text{PB}_{1,n}) \to \text{gr} \text{Lie}(\text{PB}_{1,n})$.

$p_n$ is surjective because $\text{gr} \text{Lie} \Gamma$ is generated in degree 1 (as the associated graded of any quotient of a topologically free Lie algebra).

There is a unique derivation $\Delta_0 \in \text{Der}(\hat{t}_{1,n})$, such that $\hat{\Delta}_0(x_i) = y_i$ and $\hat{\Delta}_0(y_i) = 0$. This derivation gives rise to a one-parameter group of automorphisms of $\text{Der}(t_{1,n})$, defined by $\exp(s\Delta_0)(x_i) := x_i + sy_i, \exp(s\Delta_0)(y_i) = y_i$.

$\text{Lie}(\mu_{z_0, \tau})$ induces a morphism $\text{gr} \text{Lie}(\mu_{z_0, \tau}) : \text{gr} \text{Lie}(\text{PB}_{1,n}) \to t_{1,n}$. We will now prove that

$$\text{gr} \text{Lie}(\mu_{z_0, \tau}) \circ p_n = \exp(-\frac{\tau}{2\pi i} \hat{\Delta}_0) \circ w;$$

where $w$ is the automorphism of $t_{1,n}$ defined by $w(a_i) = -b_i, w(b_i) = 2\pi i a_i$.

$\mu_{z_0, \tau}$ is defined as follows. Let $F_{z_0}(z)$ be the solution of $(\partial/\partial z_i) F_{z_0}(z) = K_i(z^\tau) F_{z_0}(z)$, $F_{z_0}(z_0) = 1$ on $U_\tau$; set $H_z := \{z = (z_1, \ldots, z_n) | z_i = a_i + \tau b_i, 0 < a_1 < \ldots < a_n < 1\}$ and $V_z := \{z = (z_1, \ldots, z_n) | z_i = a_i + \tau b_i, 0 < b_1 < \ldots < b_n < 1\}$; let $F^H_{z_0}$ and $F^V_{z_0}$ be the analytic prolongations of $F_{z_0}$ to $H_z$ and $V_z$; then

$$F^H_{z_0} (z + \delta_i) = F^V_{z_0} (z) \mu_{z_0, \tau}(X_i), \quad e^{2\pi i x_i} F^V_{z_0} (z + \tau \delta_i) = F^V_{z_0} (z) \mu_{z_0, \tau}(Y_i).$$

We have $\log F^H_{z_0}(z) = -\sum_i (z_i - z^0_i) y_i + \text{terms of degree } 2$, where $t_{1,n}$ is graded by $\text{deg}(x_i) = \text{deg}(y_i) = 1$, which implies that $\log \mu_{z_0, \tau}(X_i) = -y_i + \text{terms of degree } 2$, $\log \mu_{z_0, \tau}(Y_i) = 2\pi i x_i - \tau y_i + \text{terms of degree } 2$. Therefore $\text{Lie}(\mu_{z_0, \tau}(a_i)) = \log \mu_{z_0, \tau}(A_i) = -b_i + \text{terms of degree } 2, \text{Lie}(\mu_{z_0, \tau}(\beta_i)) = \log \mu_{z_0, \tau}(B_i) = 2\pi i a_i - \tau b_i + \text{terms of degree } 2$. So $\text{gr} \text{Lie}(\mu_{z_0, \tau})(a_i) = -b_i, \text{gr} \text{Lie}(\mu_{z_0, \tau})(\beta_i) = 2\pi i a_i - \tau b_i$.

It follows that $\text{gr} \text{Lie}(\mu_{z_0, \tau}) \circ p_n$ is the endomorphism $a_i \mapsto -b_i, b_i \mapsto 2\pi i a_i - \tau b_i$ of $t_{1,n}$, which is the automorphism $\exp(-\frac{\tau}{2\pi i} \hat{\Delta}_0) \circ w$; this proves (4).

Since we already proved that $p_n$ is surjective, it follows that $\text{gr} \text{Lie}(\mu_{z_0, \tau})$ and $p_n$ are both isomorphisms. As $\text{Lie}(\text{PB}_{1,n})$ and $\hat{t}_{1,n}$ are both complete and separated, $\text{Lie}(\mu_{z_0, \tau})$ is bijective, and since it is a morphism, it is an isomorphism of filtered Lie algebras. 

2.5. The formality of $\overline{\text{PB}}_{1,n}$. Let $z_0 \in U_\tau$ and $[z_0] \in \overline{C}(E_r, n)$ be its image. We set $\overline{\text{PB}}_{1,n} := \pi_1(\overline{C}(E_r, n), [z_0])$. Then $\overline{\text{PB}}_{1,n}$ is the quotient of $\text{PB}_{1,n}$ by its central subgroup (isomorphic to $Z^2$) generated by $A_1$ and $B_1$. We have $\mu_{z_0, \tau}(A_1) = e^{-\sum y_i}$ and $\mu_{z_0, \tau}(B_1) = e^{2\pi i \sum x_i - \tau y_i}$, so $\text{Lie}(\mu_{z_0, \tau})(a_1) = -a_1, \text{Lie}(\mu_{z_0, \tau})(\beta_1) = 2\pi i a_i - \tau b_i$, which implies that $\text{Lie}(\mu_{z_0, \tau})$ induces an isomorphism between $\text{Lie}(\overline{\text{PB}}_{1,n}) \subset \hat{t}_{1,n}$. In particular, $\overline{\text{PB}}_{1,n}$ is formal.

**Remark 2.3.** Let $\text{Diag}_n := \{(z, \tau) \in C^n \times \hat{\tau} | z \in \text{Diag}_n\}$ and let $U \subset (C^n \times \hat{\tau}) - \text{Diag}_n$ be the set of all $(z, \tau)$ such that $z \in U_\tau$. Each element of $U$ gives rise to a Lie algebra isomorphism $\mu_{z, \tau} : \text{Lie}(\text{PB}_{1,n}) \cong t_{1,n}$. For an infinitesimal $(d z, d \tau)$, the composition $\mu_{z + d z, \tau + d \tau} \circ \mu_{z, \tau}^{-1}$ is then an infinitesimal automorphism of $t_{1,n}$. This defines a flat connection over $U$ with
values in the trivial Lie algebra bundle with Lie algebra $\text{Der}(\mathfrak{t}_{1,n})$. When $d\tau = 0$, the infinitesimal automorphism has the form $\exp(\sum_i K_i(\tau) d z_i)$, so the connection has the form $d - \sum_i \text{ad}(K_i(\tau)) d z_i - \Delta(z|\tau) d\tau$, where $\Delta: U \rightarrow \text{Der}(\mathfrak{t}_{1,n})$ is a meromorphic map with poles at $\text{Diag}_n$. In the next section, we determine a map $\Delta: (C^n \times \mathfrak{g}) - \text{Diag}_n \rightarrow \text{Der}(\mathfrak{t}_{1,n})$ with the same flatness properties as $\Delta(z|\tau)$.

2.6. The isomorphisms $B_{1,n}(C) \simeq \exp(\mathfrak{t}_{1,n}) \rtimes S_n$, $B_{1,n}(C) \simeq \exp(\mathfrak{t}_{1,n}) \rtimes S_n$. Let $z_0$ be as above; we define $B_{1,n} := \pi_1(C(E_\tau, [n]), [z_0])$ and $\overline{B}_{1,n} := \pi_1(C(E_\tau, [n]), [z_0])$, where $x \mapsto [x]$ is the canonical projection $C(E_\tau, n) \rightarrow C(E_\tau, [n])$ or $\tilde{C}(E_\tau, n) \rightarrow \tilde{C}(E_\tau, [n])$.

We have an exact sequence $1 \rightarrow PB_{1,n} \rightarrow B_{1,n} \rightarrow S_n \rightarrow 1$. We then define groups $B_{1,n}(C)$ fitting in an exact sequence $1 \rightarrow PB_{1,n}(C) \rightarrow B_{1,n}(C) \rightarrow S_n \rightarrow 1$ as follows: the morphism $B_{1,n} \rightarrow \text{Aut}(PB_{1,n})$ extends to $B_{1,n} \rightarrow \text{Aut}(PB_{1,n}(C))$; we then construct the semidirect product $PB_{1,n}(C) \rtimes B_{1,n}$; then $PB_{1,n}$ embeds diagonally as a normal subgroup of this semidirect product, and $B_{1,n}(C)$ is defined as the quotient $(PB_{1,n}(C) \rtimes B_{1,n})/PB_{1,n}$.

The monodromy of $\nabla_{\tau,[n]}$ then gives rise to a group morphism $B_{1,n} \rightarrow \exp(\mathfrak{t}_{1,n}) \rtimes S_n$, which factors through $B_{1,n}(C) \rightarrow \exp(\mathfrak{t}_{1,n}) \rtimes S_n$. Since this map commutes with the natural morphisms to $S_n$ and using the isomorphism $PB_{1,n}(C) \simeq \exp(\mathfrak{t}_{1,n})$, we obtain that $B_{1,n}(C) \rightarrow \exp(\mathfrak{t}_{1,n}) \rtimes S_n$ is an isomorphism.

Similarly, starting from the exact sequence $1 \rightarrow PB_{1,n} \rightarrow B_{1,n} \rightarrow S_n \rightarrow 1$ one defines a group $\overline{B}_{1,n}(C)$ fitting in an exact sequence $1 \rightarrow PB_{1,n} \rightarrow \overline{B}_{1,n}(C) \rightarrow S_n \rightarrow 1$ together with an isomorphism $\overline{B}_{1,n}(C) \rightarrow \exp(\mathfrak{t}_{1,n}) \rtimes S_n$.

3. Bundles with flat connection on $M_{1,n}$ and $M_{1,[n]}$

We first define Lie algebras of derivations of $\mathfrak{t}_{1,n}$ and a related group $G_n$. We then define a principal $G_n$-bundle with flat connection of $M_{1,n}$ and a principal $G_n \rtimes S_n$-bundle with flat connection on the moduli space $M_{1,[n]}$ of elliptic curves with $n$ unordered marked points.

3.1. Derivations of the Lie algebras $\mathfrak{t}_{1,n}$ and $\mathfrak{t}_{1,n}$ and associated groups. Let $\mathfrak{d}$ be the Lie algebra with generators $\Delta_0$, $d$, $X$ and $\delta_{2m}$ ($m \geq 1$), and relations:

$[d, X] = 2X$, $[d, \Delta_0] = -2\Delta_0$, $[X, \Delta_0] = d$,

$[\delta_{2m}, X] = 0$, $[d, \delta_{2m}] = 2m\delta_{2m}$, $\text{ad}(\Delta_0)^{2m+1}(\delta_{2m}) = 0$.

Proposition 3.1. We have a Lie algebra morphism $\mathfrak{d} \rightarrow \text{Der}(\mathfrak{t}_{1,n})$, denoted by $\xi \mapsto \dot{\xi}$, such that

$\dot{d}(x_i) = x_i, \dot{d}(y_i) = -y_i, \dot{d}(t_{ij}) = 0$, $\dot{X}(x_i) = 0, \dot{X}(y_i) = x_i, \dot{X}(t_{ij}) = 0$,

$\dot{\Delta}_0(x_i) = y_i, \dot{\Delta}_0(y_i) = 0, \dot{\Delta}_0(t_{ij}) = 0$,

$\dot{\delta}_{2m}(x_i) = 0, \dot{\delta}_{2m}(t_{ij}) = [t_{ij}, (\text{ad} x_i)^{2m}(t_{ij})], \dot{\delta}_{2m}(y_i) = \sum_{j \neq i} \frac{1}{2} \sum_{q=2m-1} \sum_{p=0} ([\text{ad} x_i]^p(t_{ij}), (\text{ad} x_i)^q(t_{ij})]$. This induces a Lie algebra morphism $\mathfrak{d} \rightarrow \text{Der}(\mathfrak{t}_{1,n})$.

Proof. The fact that $\Delta_0, \dot{d}, \dot{X}$ are derivations and commute according to the Lie bracket of $\mathfrak{sl}_2$ is clear.

Let us prove that $\dot{\delta}_{2m}$ is a derivation. We have $\dot{\delta}_{2m}(t_{ij}) = [t_{ij}, \sum_{j \neq i} (\text{ad} x_i)^{2m}(t_{ij})]$, which implies that $\dot{\delta}_{2m}$ preserves the infinitesimal pure braid identities. It clearly preserves the relations $[x_i, x_j] = 0$, $[x_i, y_j] = t_{ij}$, $[x_k, t_{ij}] = 0$, $[x_i + x_j, t_{ij}] = 0$. 

Let us prove that $\tilde{\delta}_{2m}$ preserves the relation $[y_k, t_{ij}] = 0$, i.e., that $[\tilde{\delta}_\varphi(y_k), t_{ij}] + [y_k, \tilde{\delta}_\varphi(t_{ij})] = 0$.

$$[\tilde{\delta}_{2m}(y_k), t_{ij}] = \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [([ad x_k]^p(t_{ki}), [ad x_k]^q(t_{kj})], [ad x_k]^p(t_{kj}), [ad x_k]^q(t_{ki})), t_{ij}]$$

$$= \frac{1}{2} \sum_{p+q=2m-1} (-1)^{q+1} [[([ad x_k]^p(t_{ki}), [ad x_k]^q(t_{kj})), ([ad x_k]^p(t_{kj}), [ad x_k]^q(t_{ki})), t_{ij}]]$$

$$= \sum_{p+q=2m-1} (-1)^{q+1} [[([ad x_k]^p(t_{ki}), [ad x_k]^q(t_{kj})], t_{ij}]) = \sum_{p+q=2m-1} (-1)^q ([ad x_i]^p([ad x_j]^q([t_{ki}, t_{kj}]))].$$

On the other hand, $[y_k, \tilde{\delta}_{2m}(t_{ij})] = [y_k, [t_{ij}, ([ad x_i]^{2m}(t_{ij})] = [t_{ij}, [y_k, ([ad x_i]^{2m}(t_{ij})]]].$ Now

$$[y_k, ([ad x_i]^{2m}(t_{ij})] = - \sum_{\alpha+\beta=2m-1} ([ad x_i]^\alpha([t_{ki}, ([ad x_i]^\beta(t_{ij})])$$

$$= - \sum_{\alpha+\beta+1=2m-1} (ad x_i)^\alpha([t_{ki}, (ad x_j)^\beta(t_{ij})]) = - \sum_{\alpha+\beta+1=2m-1} (ad x_i)^\alpha(-ad x_j)^\beta([t_{ki}, t_{ij}])$$

$$= \sum_{p+q=2m-1} (-1)^{p+1}(ad x_i)^p(ad x_j)^q([t_{ki}, t_{kj}]).$$

Hence $[\tilde{\delta}_{2m}(y_k), t_{ij}] + [y_k, \tilde{\delta}_{2m}(t_{ij})] = 0$.

Let us prove that $\tilde{\delta}_{2m}$ preserves the relation $[y_i, y_j] = 0$, i.e., that $[\tilde{\delta}_{2m}(y_i), y_j] + [y_i, \tilde{\delta}_{2m}(y_j)] = 0$.

We have

$$[y_i, \tilde{\delta}_{2m}(y_j)] = \frac{1}{2} [y_i, \sum_{p+q=2m-1} (-1)^q [([ad x_j]^p(t_{ji}), [ad x_j]^q(t_{ji})]$$

$$+ \frac{1}{2} \sum_{k \neq i, j} [y_i, \sum_{p+q=2m-1} (-1)^q [([ad x_j]^p(t_{jk}), [ad x_j]^q(t_{jk})]$$

Now

$$\frac{1}{2} [y_i, \sum_{p+q=2m-1} (-1)^q [([ad x_j]^p(t_{ji}), [ad x_j]^q(t_{ji})]) - (i \leftrightarrow j)$$

$$= - \frac{1}{2} [y_i + y_j, \sum_{p+q=2m-1} (-1)^q [([ad x_i]^p(t_{ij}), [ad x_j]^q(t_{ij})])$$

$$= \sum_{p+q=2m-1} (-1)^{q+1} [[y_i + y_j, ([ad x_i]^p(t_{ij}), [ad x_j]^q(t_{ij}))$$

A computation similar to the above computation of $[y_k, ([ad x_i]^{2m}(t_{ij})]$ yields

$$[y_i + y_j, ([ad x_i]^p(t_{ij})] = (-1)^p \sum_{\alpha+\beta=p-1} ([ad x_k]^\alpha(t_{ik}), [ad x_j]^\beta(t_{jk})],$$

so

$$[y_i, \tilde{\delta}_{2m}(y_j)] = \sum_{\alpha+\beta+\gamma=2m-2} [([ad x_i]^\alpha(t_{ij}), [ad x_k]^\beta(t_{ik}), [ad x_j]^\gamma(t_{jk})]].$$

If now $k \neq i, j$, then

$$[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [([ad x_j]^p(t_{jk}), [ad x_j]^q(t_{jk})])$$

$$= \sum_{p+q=2m-1} (-1)^q [([y_i, ([ad x_j]^p(t_{jk})], [ad x_j]^q(t_{jk})].$$
As we have seen,
\[
[ y_j, (\text{ad} x_i)^p(t_{ik}) ] = (-1)^p \sum_{\alpha + \beta = p - 1} (-\text{ad} x_i)^\alpha (\text{ad} x_k)^\beta [t_{ij}, t_{ik}]
\]
\[
= (-1)^{p+1} \sum_{\alpha + \beta = p - 1} [(-\text{ad} x_i)^\alpha(t_{ij}), (\text{ad} x_k)^\beta(t_{jk})]
\]
So we get
\[
[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [\text{ad} x_j]^p(t_{jk}), \text{ad} x_j]^q(t_{jk})]
\]
\[
= \sum_{\alpha + \beta + \gamma = 2m - 2} [[[\text{ad} x_i]^\alpha(t_{ij}), (\text{ad} x_k)^\beta(t_{ik})], (\text{ad} x_j)^\gamma(t_{jk})]
\]
therefore
\[
[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [\text{ad} x_j]^p(t_{jk}), \text{ad} x_j]^q(t_{jk})] - (i \leftrightarrow j)
\]
\[
= \sum_{\alpha + \beta + \gamma = 2m - 2} [[[\text{ad} x_i]^\alpha(t_{ij}), (\text{ad} x_k)^\beta(t_{ik})], (\text{ad} x_j)^\gamma(t_{jk})].
\]
Therefore \([y_i, [\tilde{\delta}_{2m}(y_j)], y_j] = 0\).

Since \(\tilde{\delta}_{2m}(\sum x_i) = \sum x_i = 0\) and \(\sum y_i\) are central, \(\tilde{\delta}_{2m}\) preserves the relations \(\sum x_i, t_{ij} = 0\) and \(\sum y_i, t_{ij} = 0\). It follows that \(\tilde{\delta}_{2m}\) preserves the relations \(x_i + x_j, t_{ij} = [y_i + y_j, t_{ij}] = 0\) and \([x_i, y_i] = -\sum y_i, t_{ij}\). All this proves that \(\tilde{\delta}_{2m}\) is a derivation.

Let us show that \(\text{ad}(\tilde{\Delta}_0)^{2m+1}(\tilde{\delta}_{2m}) = 0\) for \(m \geq 1\). We have
\[
\text{ad}(\tilde{\Delta}_0)^{2m+1}(\tilde{\delta}_{2m})(x_i) = -(2m + 1)\tilde{\Delta}_0^m \circ \tilde{\delta}_{2m} \circ \tilde{\Delta}_0(x_i) = -(2m + 1)\tilde{\Delta}_0^{2m} \circ \tilde{\delta}_{2m}(y_i)
\]
\[
= -(2m + 1)\tilde{\Delta}_0^{2m} \sum_{j \neq i} \frac{1}{2} \sum_{p+q=2m-1} [(\text{ad} x_i)^p(t_{ij}), (-\text{ad} x_i)^q(t_{ij})] = 0;
\]
the last part of this computation implies that \(\text{ad}(\tilde{\Delta}_0)^{2m+1}(\tilde{\delta}_{2m})(y_i) = 0\), therefore \(\text{ad}(\tilde{\Delta}_0)^{2m+1}(\tilde{\delta}_{2m}) = 0\).

We have clearly \([X, [\tilde{\delta}_{2m}]] = 0\) and \([d, [\tilde{\delta}_{2m}]] = 2m [\tilde{\delta}_{2m}].\) It follows that we have a Lie algebra morphism \(\tilde{\mathcal{D}} \rightarrow \text{Der}(t_{1,n}).\) Since \(\tilde{\mathcal{D}}, \tilde{\Delta}_0, X\) and \(\tilde{\delta}_{2m}\) all map \(\mathbb{C}(\sum x_i) \oplus \mathbb{C}(\sum y_i)\) to itself, this induces a Lie algebra morphism \(\tilde{\mathcal{D}} \rightarrow \text{Der}(t_{1,n}).\) \(\square\)

Let \(e, f, h\) be the standard basis of \(\mathfrak{sl}_2\). Then we have a Lie algebra morphism \(\tilde{\mathcal{D}} \rightarrow \mathfrak{sl}_2\), defined by \(\tilde{\delta}_{2m} \mapsto 0\), \(d \mapsto h\), \(X \mapsto e\), \(\Delta_0 \mapsto f\). We denote by \(\mathfrak{d}_+ \subset \tilde{\mathcal{D}}\) its kernel.

Since the morphism \(\tilde{\mathcal{D}} \rightarrow \mathfrak{sl}_2\) has a section (given by \(e, f, h \mapsto X, \Delta_0, d\)), we have a semidirect product decomposition \(\tilde{\mathcal{D}} = \mathfrak{d}_+ \ltimes \mathfrak{sl}_2\).

We then have
\[
\tilde{t}_{1,n} \ltimes \tilde{\mathcal{D}} = (\tilde{t}_{1,n} \ltimes \mathfrak{d}_+) \ltimes \mathfrak{sl}_2.
\]

**Lemma 3.2.** \(\tilde{t}_{1,n} \ltimes \mathfrak{d}_+\) is positively graded.

**Proof.** We define compatible \(\mathbb{Z}^2\)-gradings of \(\tilde{\mathcal{D}}\) and \(\tilde{t}_{1,n}\) by \(\text{deg}(\Delta_0) = (-1, 1),\) \(\text{deg}(d) = (0, 0),\) \(\text{deg}(X) = (1, -1),\) \(\text{deg}(\tilde{\delta}_{2m}) = (2m + 1, 1),\) \(\text{deg}(x_i) = (1, 0),\) \(\text{deg}(y_i) = (0, 1),\) \(\text{deg}(t_{ij}) = (1, 1).\)

We define the support of \(\tilde{\mathcal{D}}\) (resp., \(\tilde{t}_{1,n}\)) as the subset of \(\mathbb{Z}^2\) of indices for which the corresponding component of \(\tilde{\mathcal{D}}\) (resp., \(\tilde{t}_{1,n}\)) is nonzero.

Since the \(\tilde{x}_i\) on one hand, the \(\tilde{y}_i\) on the other hand generate abelian Lie subalgebras of \(\tilde{t}_{1,n}\), the support of \(\tilde{t}_{1,n}\) is contained in \(\mathbb{N}^2_{>0} \cup \{(1, 0), (0, 1)\}\).

On the other hand, \(\mathfrak{d}_+\) is generated by the \(\text{ad}(\Delta_0)^p(\tilde{\delta}_{2m})\), which all have degrees in \(\mathbb{N}^2_{>0}\). It follows that the support of \(\mathfrak{d}_+\) is contained in \(\mathbb{N}^2_{>0}\).
Lemma 3.3. \( \hat{\mathfrak{t}}_{1,n} \times \mathfrak{d}_+ \) is a sum of finite dimensional \( \mathfrak{sl}_2 \)-modules; \( \mathfrak{d}_+ \) is a sum of irreducible odd dimensional \( \mathfrak{sl}_2 \)-modules.

Proof. A generating space for \( \hat{\mathfrak{t}}_{1,n} \) is \( \sum_i (C x_i \oplus C \tilde{y}_i) \), which is a sum of finite dimensional \( \mathfrak{sl}_2 \)-modules, so \( \hat{\mathfrak{t}}_{1,n} \) is a sum of finite dimensional \( \mathfrak{sl}_2 \)-modules.

A generating space for \( \mathfrak{d}_+ \) is the sum over \( m \geq 1 \) of its \( \mathfrak{sl}_2 \)-submodules generated by the \( \delta_{2m} \), which are zero or irreducible odd dimensional, therefore \( \mathfrak{d}_+ \) is a sum of odd dimensional \( \mathfrak{sl}_2 \)-modules. (In fact, the \( \mathfrak{sl}_2 \)-submodule generated by \( \delta_{2m} \) is nonzero, as it follows from the construction of the above morphism \( \mathfrak{d}_+ \to \text{Der}(\hat{\mathfrak{t}}_{1,n}) \) that \( \delta_{2m} \neq 0 \).)

It follows that \( \hat{\mathfrak{t}}_{1,n}, \mathfrak{d}_+ \) and \( \hat{\mathfrak{t}}_{1,n} \times \mathfrak{d}_+ \) integrate to \( \text{SL}_2(\mathbb{C}) \)-modules (while \( \mathfrak{d}_+ \) even integrates to a \( \text{PSL}_2(\mathbb{C}) \)-module).

We can form in particular the semidirect products

\[
\mathbf{G}_n := \exp((\hat{\mathfrak{t}}_{1,n} \times \mathfrak{d}_+)') \times \text{SL}_2(\mathbb{C})
\]

and \( \exp(\mathfrak{d}_+) \times \text{PSL}_2(\mathbb{C}) \); we have morphisms \( \mathbf{G}_n \to \exp(\mathfrak{d}_+) \times \text{PSL}_2(\mathbb{C}) \) (this is a 2-covering if \( n = 1 \) since \( \hat{\mathfrak{t}}_{1,1} = 0 \)).

Observe that the action of \( \mathbf{S}_n \) by automorphisms of \( \hat{\mathfrak{t}}_{1,n} \) extends to an action on \( \hat{\mathfrak{t}}_{1,n} \times \mathfrak{d}_+ \), where the action on \( \mathfrak{d}_+ \) is trivial. This gives rise to an action of \( \mathbf{S}_n \) by automorphisms of \( \mathbf{G}_n \).

3.2. Bundle with flat connection on \( \mathcal{M}_{1,n} \). The semidirect product \( ((\mathbb{Z}^n)^2 \times \mathbb{C}) \times \text{SL}_2(\mathbb{Z}) \) acts on \( (\mathbb{C}^n \times \mathfrak{H}) - \text{Diag}_n \) by \( (n, m, u) \ast (z, \tau) := (n + \tau n + u(\sum_i \delta_i), \tau) \) for \( (n, m, u) \in (\mathbb{Z}^n)^2 \times \mathbb{C} \) and \( (\alpha \beta) \ast (z, \tau) := \left( \frac{x}{\tau + y}, \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) \) for \( (\alpha \beta) \in \text{SL}_2(\mathbb{Z}) \) (here \( \text{Diag}_n := \{ (z, \tau) \in \mathbb{C}^n \times \mathfrak{H} \} \) for some \( i \neq j, z_{ij} \in \Lambda_r \}). The quotient then identifies with the moduli space \( \mathcal{M}_{1,n} \) of elliptic curves with \( n \) marked points.

Set \( \mathbf{G}_n := \exp((\hat{\mathfrak{t}}_{1,n} \times \mathfrak{d}_+)') \times \text{SL}_2(\mathbb{C}) \). We will define a principal \( \mathbf{G}_n \)-bundle with flat connection \( (\mathcal{P}_n, \nabla_{\mathcal{P}_n}) \) over \( \mathcal{M}_{1,n} \).

For \( u \in \mathbb{C}^n \), \( u^d := \left( \begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{C}) \subset \mathbf{G}_n \) and for \( v \in \mathbb{C} \), \( e^{v \chi} := \left( \begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{C}) \subset \mathbf{G}_n \).

Since \([X, \tilde{x}_i] = 0\), we consistently set \( \exp(aX + \sum_i b_i \tilde{x}_i) := \exp(aX) \exp(\sum_i b_i \tilde{x}_i) \).

Proposition 3.4. There exists a unique principal \( \mathbf{G}_n \)-bundle \( \mathcal{P}_n \) over \( \mathcal{M}_{1,n} \), such that a section of \( U \subset \mathcal{M}_{1,n} \) is a function \( f : \pi^{-1}(U) \to \mathbf{G}_n \) (where \( \pi : (\mathbb{C}^n \times \mathfrak{H}) - \text{Diag}_n \to \mathcal{M}_{1,n} \) is the canonical projection), such that \( f(z + \delta_i | \tau) = f(z + u(\sum_i \delta_i) | \tau) = f(z | \tau), f(z + \tau \delta_i | \tau) = e^{-2\pi i \tilde{x}_i} f(z | \tau), f(z | \tau + 1) = f(z | \tau) \) and \( f(z | \tau) = \tau^d \exp(\frac{2\pi i}{\tau}(\sum_i z_i \tilde{x}_i + X)) \).

Proof. Let \( c_\mathbf{G} : (\mathbb{C}^n \times \mathfrak{H}) \to \mathbf{G}_n \) be a family of holomorphic functions (where \( \tilde{g} \in ((\mathbb{Z}^n)^2 \times \mathbb{C}) \times \text{SL}_2(\mathbb{Z})) \) satisfying the cocycle condition \( c_{gg} (z | \tau) = c_g (\tilde{g}' (z | \tau)) c_{\tilde{g}'} (z | \tau) \). Then there exists a unique principal \( \mathbf{G}_n \)-bundle over \( \mathcal{M}_{1,n} \) such that a section of \( U \subset \mathcal{M}_{1,n} \) is a function \( f : \pi^{-1}(U) \to \mathbf{G}_n \) such that \( f(g \ast (z | \tau)) = c_g (z | \tau) f(z | \tau) \).

We now prove that there is a unique cocycle such that \( c_{(0,0,0)} = c_{(0,\delta_i,0)} = 1, c_{(0,\delta_i,\delta_i)} = e^{-2\pi i \tilde{x}_i}, c_{S} = 1 \) and \( c_{T} (z | \tau) = \tau^d \exp(\frac{2\pi i}{\tau}(\sum_i z_i \tilde{x}_i + X)), \) where \( S = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \).

Such a cocycle is the same as a family of functions \( c_G : (\mathbb{C}^n \times \mathfrak{H}) \to \mathbf{G}_n \) (where \( g \in \text{SL}_2(\mathbb{Z}) \)), satisfying the cocycle conditions \( c_{gg} (z | \tau) = c_g (g' \ast (z | \tau)) c_{g'} (z | \tau) \) for \( g, g' \in \text{SL}_2(\mathbb{Z}) \), and \( c_g (z + \delta_i | \tau) = e^{2\pi i \tilde{x}_i} c_g (z | \tau) \), \( c_g (z + \tau \delta_i | \tau) = e^{-2\pi i \tilde{x}_i} c_g (z | \tau) c_{S} (z | \tau) \), \( c_g (z + u(\sum_i \delta_i) | \tau) = c_g (z | \tau) \) for \( g = \left( \begin{smallmatrix} a & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \).

Lemma 3.5. There exists a unique family of functions \( c_G : (\mathbb{C}^n \times \mathfrak{H}) \to \mathbf{G}_n \) such that \( c_{gg'} (z | \tau) = c_g (g' \ast (z | \tau)) c_{g'} (z | \tau) \) for \( g, g' \in \text{SL}_2(\mathbb{Z}) \), with \( c_S(z | \tau) = 1, c_T (z | \tau) = \tau^d e^{(2\pi i \tau)(\sum_i z_i \tilde{x}_i + X)} \).
Proof. \( SL_2(\mathbb{Z}) \) is the group generated by \( \tilde{S} \), \( \tilde{T} \) and relations \( \tilde{T}^4 = 1, \tilde{(ST)}^3 = \tilde{T}, \tilde{S}T^2 = \tilde{T}^2S \). Let \((\tilde{S}, \tilde{T})\) be the free group with generators \( \tilde{S}, \tilde{T} \); then there is a unique family of maps \( c_g : \mathbb{C}^n \times \mathfrak{s} \to \mathfrak{g}_n \) given \( g \in (\tilde{S}, \tilde{T}) \) satisfying the cocycle conditions (w.r.t. the action of \((\tilde{S}, \tilde{T})\) on \( \mathbb{C}^n \times \mathfrak{s} \) through its quotient \( SL_2(\mathbb{Z}) \)) and \( c_S = c_S, c_T = c_T \). It remains to show that \( c_{\tilde{T}^4} = 1, c_{\tilde{(ST)}^3} = c_{\tilde{T}^2} \) and \( c_{\tilde{S}T^2} = c_{\tilde{S}T} \).

For this, we show that \( c_{\tilde{T}^4}(z|\tau) = (-1)^d \). We have \( c_{\tilde{T}^2}(z|\tau) = c_T(z|\tau - 1/\tau)c_T(z|\tau) = (-\tau)^{-d}\exp(-2\pi i \tau \sum z_j x_j + X)\tau^d \exp(2\pi i \sum z_j x_j + X) = (-1)^d \) since \( \tau^d X \tau^{-d} = \tau^2 X, \tau^d x_j \tau^{-d} = \tau x_j \).

Since \((-1)^d)^2 = 1 \) and \( -1 \) we get \( c_{\tilde{T}^4} = 1 \). Since \( c_{\tilde{S}T} \) and \( c_{\tilde{T}^2} \) are both constant and commute, we also get \( c_{\tilde{S}T^2} = c_{\tilde{S}T} \).

We finally have \( c_{\tilde{(ST)}^3}(z|\tau) = c_T(z|\tau) \) while \( \tilde{T}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \tilde{(ST)^2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) so
\[
c_{\tilde{(ST)}^3}(z|\tau) = c_T(\frac{z}{\tau - \frac{1}{\tau}}) = \frac{\frac{z}{\tau - \frac{1}{\tau}}}{\frac{2\pi i}{\tau}} \sum z_j x_j + \frac{2\pi i}{\tau} X \tau^d \exp(2\pi i \sum z_j x_j + X) = (-1)^d \exp(\frac{2\pi i}{\tau} \sum z_j x_j + X) \exp(\frac{2\pi i}{\tau} \sum z_j x_j + X) = (-1)^d,
\]
so \( c_{\tilde{(ST)}^3} = c_{\tilde{T}^2} \).

End of proof of Proposition 3.4. We now check that the maps \( c_g \) satisfy the remaining conditions, i.e., \( c_g(z + u(\sum \delta_i)|\tau) = c_g(z|\tau), c_g(z + \delta_i|\tau) = e^{2\pi i \frac{1}{\tau} \delta_i} c_g(z|\tau), c_g(\tau + \delta_i|\tau) = e^{-2\pi \frac{1}{\tau} \delta_i} c_g(z|\tau) \) for \( \delta_i, \delta_j \). The cocycle identity \( c_g(z|\tau) = c_g(g' \ast (z|\tau))c_g(z|\tau) \) implies that it suffices to prove these identities for \( g = S \) and \( g = T \). They are trivially satisfied if \( g = S \). When \( g = T \), the first identity follows from \( \sum \tilde{x}_i = 0 \), the third identity follows from the fact that \( (X, \tilde{x}_1, ..., \tilde{x}_n) \) is a commutative family, the second identity follows from the same fact together with \( \tau^d x_i \tau^{-d} = \tau x_i \).

Therefore
\[
g(z, x|\tau) := \sum_{i<j} \gamma \left(\begin{array}{c} i \\ j \end{array}\right) \frac{\partial^\prime}{\partial \tau} f(z, x|\tau)
\]
is a meromorphic function \( \mathbb{C}^n \times \mathfrak{s} \to \mathfrak{l}_{1,n} \) with only poles at \( \text{Diag}_n \).

We set
\[
\Delta(z|\tau) := -\frac{1}{2\pi i} \Delta_0 - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n} + \frac{1}{2\pi i} g(z|\tau),
\]
where \( a_{2n} = -(2n+1)B_{2n+2}(2 \pi)^{2n+2}/(2n+2)! \) and \( B_n \) are the Bernoulli numbers given by \( x/(e^x - 1) = \sum_{n \geq 1} B_n x^n/n! \). This is a meromorphic function \( \mathbb{C}^n \times \mathfrak{s} \to \mathfrak{l}_{1,n} \) with only poles at \( \text{Diag}_n \).
Theorem 3.6. There is a unique flat connection $\nabla_{\Gamma_n}$ on $\mathcal{P}_n$, whose pull-back to $(\mathbb{C}^n \times S) - \text{Diag}_n$ is the connection
\[ d - \tilde{\Delta}(z|\tau) \, d \tau - \sum_i K_i(z|\tau) \, dz_i \]
on the trivial $G_n$-bundle.

Proof. We should check that the connection $d - \tilde{\Delta}(z|\tau) \, d \tau - \sum_i K_i(z|\tau) \, dz_i$ is equivariant and flat, which is expressed as follows (taking into account that we already checked the equivariance and flatness of $d - \sum_i K_i(z|\tau) \, dz_i$ for any $\tau$):

(equivariance) for $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$
\[ \frac{1}{\gamma \tau + \delta} \tilde{K}_i(z|\tau) = \text{Ad}(c_g(z|\tau))(\tilde{K}_i(z|\tau)) + (\partial/\partial z_i) c_g(z|\tau))c_g(z|\tau)^{-1}, \quad (6) \]

\[ \tilde{\Delta}(z + \delta|\tau) = \tilde{\Delta}(z + u(\sum_i \delta_i)|\tau) = \tilde{\Delta}(z|\tau), \quad \tilde{\Delta}(z + \tau \delta|\tau) = e^{-2\pi i \text{ad} x_i(\tilde{\Delta}(z|\tau) - \tilde{K}_i(z|\tau))}, \quad (7) \]

\[ \frac{1}{(\gamma \tau + \delta)^2} \tilde{\Delta}(z|\tau) = \text{Ad}(c_g(z|\tau))(\tilde{\Delta}(z|\tau)) + \frac{\gamma}{\gamma \tau + \delta} \sum_i \text{Ad}(c_g(z|\tau))(\tilde{K}_i(z|\tau)) \]
\[ + \left((\partial/\partial \tau) + \frac{\gamma}{\gamma \tau + \delta} \sum_i \text{ad}(z_i \tau) c_g(z|\tau))c_g(z|\tau)^{-1}, \quad (8) \]

(flattness) $\partial/\partial \tau - \tilde{\Delta}(z|\tau), \partial/\partial z_i - \tilde{K}_i(z|\tau) = 0$.

Let us now check the equivariance identity (6) for $\tilde{K}_i(z|\tau)$. The cocycle identity $c_{g'g}(z|\tau) = c_g(g'(z|\tau))c_{g'}(z|\tau)$ implies that it suffices to check it when $g = S$ and $g = T$. When $g = S$, this is the identity $\tilde{K}_i(z|\tau + 1) = \tilde{K}_i(z|\tau)$, which follows from the identity $\theta(z|\tau + 1) = \theta(z|\tau)$. When $g = T$, we have to check the identity
\[ \frac{1}{T} \tilde{K}_i(z|\tau) - \frac{1}{T} = \text{Ad}(\tau^{d} e^{2\pi i (\sum_i z_i x_i + X)}(\tilde{K}_i(z|\tau)) = 2\pi i x_i. \quad (9) \]

We have
\[ 2\pi i \bar{x}_i - \text{Ad}(e^{2\pi i (\sum_i z_i \bar{x}_i + X)}(\bar{y}_i/\tau) \]
\[ = - \text{Ad}(e^{2\pi i (\sum_i z_i \bar{x}_i)}(\bar{y}_i/\tau) \quad \text{as Ad}(e^{2\pi i \tau X})((\bar{y}_i/\tau) = (\bar{y}_i/\tau + 2\pi i \bar{x}_i) \]
\[ = - \frac{\bar{y}_i}{\tau} - \frac{e^{2\pi i \text{ad}(\sum_k z_k \bar{x}_k)} - 1}{\text{ad}(\sum_k z_k \bar{x}_k)} \sum_j \left( z_j \bar{x}_j \frac{\bar{y}_i}{\tau} \right) \]
\[ = - \frac{\bar{y}_i}{\tau} + \sum_{j \neq i} \frac{e^{2\pi i \text{ad}(\sum_k z_k \bar{x}_k)} - 1}{\text{ad}(\sum_k z_k \bar{x}_k)} \left( z_j \bar{x}_j \frac{\bar{y}_i}{\tau} \right) \]
therefore
\[ \frac{1}{T} \sum_j \left( \frac{e^{2\pi i \text{ad}\bar{x}_i} - 1}{\text{ad}\bar{x}_i} \right) (\bar{y}_i) = - \text{Ad}(\tau^{d} e^{2\pi i (\sum_i z_i \bar{x}_i + X)}(\bar{y}_i) + 2\pi i \bar{x}_i. \quad (10) \]

We have $\theta(z|\tau - 1/\tau) = (1/\tau) e^{(\pi i/\tau) z^2} \theta(z|\tau)$, therefore
\[ \frac{1}{T} \bar{k}(\bar{z}, x|x| - 1/\tau) = e^{2\pi i z x} \bar{k}(z, \tau x|\tau) + \frac{e^{2\pi i \bar{z} x} - 1}{x \tau}, \quad (11) \]
Substituting \((z, x) = (z_{ij}, \text{ad } \bar{x}_i)\) \((j \neq i)\), applying to \(\bar{t}_{ij}\), summing over \(j\) and adding up identity (10), we get

\[
\frac{1}{\tau} \left( \sum_{j \neq i} k \left( \frac{z_{ij}}{\tau}, \text{ad } \bar{x}_i - \frac{1}{\tau} \right)(\bar{t}_{ij}) - \bar{y}_i \right)
\]

\[
= \sum_{j \neq i} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij}) - \text{Ad}(\tau^{d}e^{\frac{2\pi i}{\tau}(\sum_{i \neq j} z_{ij} \bar{x}_j + \bar{X}))(\bar{y}_i) + 2\pi i \bar{x}_i.\]

Since \(e^{2\pi i z_{ij} \text{ad } \bar{x}_i} k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij}) = \text{Ad}(\tau^{d}e^{\frac{2\pi i}{\tau}(\sum_{i \neq j} z_{ij} \bar{x}_j + \bar{X}))(k(z_{ij}, \text{ad } \bar{x}_i)(\bar{t}_{ij})),\) this implies (9). This ends the proof of (6).

Let us now check the shift identities (7) in \(\tilde{\Delta}(z|\tau)\). The first part is immediate; let us check the last identity. We have \(k(z + \tau, x|\tau) = e^{-2\pi i x} g(z, x|\tau) + (e^{-2\pi i x} - 1) / x,\)

therefore \(g(z + \tau, x|\tau) = e^{-2\pi i x} g(z, x|\tau) - 2\pi i e^{-2\pi i x} k(z, x|\tau) + \frac{1}{\tau} \left(1 - e^{-2\pi i x} - 2\pi i e^{-2\pi i x}\right).

Substituting \((z, x) = (z_{ij}, \text{ad } \bar{x}_i)\) \((j \neq i)\), applying to \(\bar{t}_{ij}\), summing up and adding up \(\sum_{k,j} \bar{k}_{kl}, \text{ad } \bar{x}_k|\bar{t}_{kl}\), we get

\[
g(z + \tau \delta_i |\tau) = e^{-2\pi i \text{ad } \bar{x}_i} (g(z|\tau)) - 2\pi i e^{-2\pi i \text{ad } \bar{x}_i}(\bar{K}_i(z|\tau) + \bar{y}_i) + \sum_{j \neq i} \frac{1}{\text{ad } \bar{x}_i} \left(1 - e^{-2\pi i \text{ad } \bar{x}_i} - 2\pi i e^{-2\pi i \text{ad } \bar{x}_i}\right)(\bar{t}_{ij})\]

\[
eq e^{-2\pi i \text{ad } \bar{x}_i} (g(z|\tau)) - 2\pi i e^{-2\pi i \text{ad } \bar{x}_i}(\bar{K}_i(z|\tau) + \bar{y}_i) - \frac{1}{\text{ad } \bar{x}_i} \left(1 - e^{-2\pi i \text{ad } \bar{x}_i} - 2\pi i e^{-2\pi i \text{ad } \bar{x}_i}\right)(\bar{y}_i)\]

\[
eq e^{-2\pi i \text{ad } \bar{x}_i} (g(z|\tau)) - 2\pi i e^{-2\pi i \text{ad } \bar{x}_i}(\bar{K}_i(z|\tau)) - 1 - e^{-2\pi i \text{ad } \bar{x}_i}(\bar{y}_i);\]

on the other hand, we have \(e^{-2\pi i \text{ad } \bar{x}_i}(\Delta_0) = \Delta_0 + \frac{1 - e^{-2\pi i \text{ad } \bar{x}_i}}{\text{ad } \bar{x}_i}(\bar{y}_i)\) \((\text{as } [\Delta_0, \bar{x}_i] = \bar{y}_i)\), therefore \(g(z + \tau \delta_i |\tau) = e^{-2\pi i \text{ad } \bar{x}_i} (\Delta_0 - \Delta_0 - 2\pi i \bar{K}_i(z|\tau))\). Since the \(\delta_{2n}\) commute with \(\bar{x}_i\), we get \(\Delta(z + \tau \delta_i |\tau) = e^{-2\pi i \text{ad } \bar{x}_i}(\Delta(z|\tau) - \bar{K}_i(z|\tau)),\) as wanted.

Let us now check the equivariance identities (8) for \(\Delta(z|\tau)\). As above, the cocycle identities imply that it suffices to check (8) for \(g = S, T\). When \(g = S\), this identity follows from \(\sum_i \bar{K}_i(z|\tau) = 0\). When \(g = T\), it is written

\[
\frac{1}{\tau} \Delta \left( \frac{z}{\tau} - 1 \right) = \text{Ad}(c_T(z|\tau))(\Delta(z|\tau)) + \frac{1}{\tau} \sum_i z_i \bar{K}_i(z|\tau)) + \frac{d}{\tau} - 2\pi i X. \tag{12}\]

The modularity identity (11) for \(k(z, x|\tau)\) implies that

\[
\frac{1}{\tau^2} g(z \frac{x}{\tau}, x - 1) = e^{2\pi i z x} g(z, \tau x|\tau) + \frac{2\pi i x}{\tau} e^{2\pi i z x} k(z, \tau x|\tau) + \frac{1}{\tau^2} e^{2\pi i z x} + 2\pi i x e^{2\pi i z x}.\]

This implies

\[
\frac{1}{\tau^2} \sum_{i < j} \frac{g(z_{ij}}{\tau}, \text{ad } \bar{x}_i - \frac{1}{\tau} \right)(\bar{t}_{ij}) = \sum_{i < j} \frac{e^{2\pi i z_{ij} \text{ad } \bar{x}_i} g(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij})}{\tau^2} + \sum_{i < j} \frac{2\pi i z_{ij}}{\tau^2} \left(1 - e^{2\pi i z_{ij} \text{ad } \bar{x}_i} - 2\pi i e^{2\pi i z_{ij} \text{ad } \bar{x}_i}\right)(\bar{t}_{ij}).\]

We compute as above

\[
\sum_{i < j} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} g(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij}) = \text{Ad}(\tau^{d}e^{\frac{2\pi i}{\tau}(\sum_{i \neq j} z_{ij} \bar{x}_j + \bar{X}))(g(z|\tau))\]

\[
\sum_{i < j} \frac{2\pi i z_{ij}}{\tau} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij}) = \sum_i \frac{2\pi i z_i}{\tau} \left(\sum_{j \neq i} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij})\right)\]

\[
= \sum_i \frac{2\pi i z_i}{\tau} \left(\sum_{j \neq i} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} \right) k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij})\]

\[
+ \sum_{i < j} \frac{2\pi i z_{ij}}{\tau} e^{2\pi i z_{ij} \text{ad } \bar{x}_i} k(z_{ij}, \tau \text{ad } \bar{x}_i)(\bar{t}_{ij}) + \sum_{i < j} \frac{1}{\tau^2} \left(1 - e^{2\pi i z_{ij} \text{ad } \bar{x}_i} - 2\pi i e^{2\pi i z_{ij} \text{ad } \bar{x}_i}\right)(\bar{t}_{ij}).\]
(using $k(z, x|\tau) + k(-z, -x|\tau) = 0$) and
\[
\sum_{i<j} e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i k(z_{ij}, \tau \text{ad} \tilde{x}_i)(\tilde{t}_{ij}) = \text{Ad}(\tau^d e^{2\pi i \sum z_i \tilde{x}_i}(X))(K_i(z|\tau) + \tilde{y}_i).
\]
Therefore
\[
\frac{1}{\tau^2} g(\frac{z}{\tau}| - \frac{1}{\tau}) = \text{Ad}(c_T(z|\tau))(g(z|\tau) + \frac{2\pi i}{\tau} \sum_i z_i \tilde{K}_i(z|\tau) + \frac{2\pi i}{\tau} \sum_i z_i \tilde{y}_i)
+ \sum_{i<j} \left(1 - \frac{e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2(ad \tilde{x}_i)^2} + \frac{2\pi i z_{ij} e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2 \text{ad} \tilde{x}_i}\right)(\tilde{t}_{ij}),
\]
which implies
\[
\frac{1}{\tau^2} \Delta(\frac{z}{\tau}| - \frac{1}{\tau}) = \text{Ad}(c_T(z|\tau))(\Delta(z|\tau) + \frac{1}{\tau} \sum_i \tilde{K}_i(z|\tau))
+ \text{Ad}(c_T(z|\tau))(\frac{1}{\tau} \sum_i z_i \tilde{y}_i) + \frac{1}{2\pi i} \sum_{i<j} \left(1 - \frac{e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2(ad \tilde{x}_i)^2} + \frac{2\pi i z_{ij} e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2 \text{ad} \tilde{x}_i}\right)(\tilde{t}_{ij})
+ \frac{1}{2\pi i} \left(\text{Ad}(c_T(z|\tau))(\Delta(\phi^{|\tau|}) - \frac{1}{\tau^2} \Delta(\phi^{|\tau| - 1|\tau|})\right) = \frac{d}{\tau} - 2\pi i X. \tag{13}
\]

To prove (12), it then suffices to prove
\[
\text{Ad}(c_T(z|\tau))(\frac{1}{\tau} \sum_i z_i \tilde{y}_i) = \frac{1}{\tau^2} \sum_i z_i \tilde{y}_i + \frac{1}{2\pi i} \sum_{i<j} \left(1 - \frac{e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2(ad \tilde{x}_i)^2} + \frac{2\pi i z_{ij} e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\tau^2 \text{ad} \tilde{x}_i}\right)(\tilde{t}_{ij})
+ \frac{1}{2\pi i} \left(\text{Ad}(c_T(z|\tau))(\Delta(\phi^{|\tau|}) - \frac{1}{\tau^2} \Delta(\phi^{|\tau| - 1|\tau|})\right) = \frac{d}{\tau} - 2\pi i X.
\]

We compute
\[
\text{Ad}(c_T(z|\tau))(\frac{1}{\tau} \sum_i z_i \tilde{y}_i) = \frac{1}{\tau^2} \sum_i z_i \tilde{y}_i + \frac{1}{2\pi i} \sum_i z_i \tilde{x}_i + \sum_{i<j} \left(1 - \frac{1}{\tau^2}\right) z_{ij} \frac{e^{2\pi i z_{ij}} \text{ad} \tilde{x}_i}{\text{ad} \tilde{x}_i} - \frac{1}{\text{ad} \tilde{x}_i}(\tilde{t}_{ij}).
\]

We also have \(\text{Ad}(c_T(z|\tau))(E_{2n+2}(\tau)\delta_{2n}) = \frac{1}{\tau^2} E_{2n+2}(-\frac{1}{\tau})\delta_{2n}\) since \([\delta_{2n}, \tilde{x}_i] = [\delta_{2n}, X] = 0\) and \([d, \delta_{2n}] = 2n\delta_{2n},\) and since \(E_{2n+2}(-\tau) = \tau^{2n+2} E_{2n+2}(\tau).\) This implies
\[
\text{Ad}(c_T(z|\tau))(\delta(\phi^{|\tau|})) = \delta(\phi^{|\tau| - 1|\tau|}).
\]

We now compute \(\text{Ad}(c_T(z|\tau))(\Delta_0) = (1/\tau^2)\Delta_0.\) We have \(\text{Ad}(c_T(z|\tau))(\Delta_0) = \text{Ad}(e^{2\pi i \sum z_i \tilde{x}_i}) \text{Ad}(\tau^d e^{(2\pi i /\tau)X})(\Delta_0),\) and \(\text{Ad}(\tau^d e^{(2\pi i /\tau)X})(\Delta_0) = (1/\tau^2)\Delta_0 + (2\pi i /\tau)d - (2\pi i )^2 X.\) Now \(\text{Ad}(e^{2\pi i \sum z_i \tilde{x}_i})(X) = X,\) \(\text{Ad}(e^{2\pi i \sum z_i \tilde{x}_i})(d) = d - 2\pi i \sum z_i \tilde{x}_i.\) We now compute
\[
\text{Ad}(e^{2\pi i \sum z_i \tilde{x}_i})(\Delta_0) = \Delta_0 + e^{2\pi i \sum z_i \tilde{x}_i} - \frac{1}{2\pi i \text{ad}(\sum z_i \tilde{x}_i)}(\sum z_i \tilde{y}_i, \Delta_0)
= \Delta_0 - \frac{e^{2\pi i \sum z_i \tilde{x}_i} - 1}{\text{ad}(\sum z_i \tilde{x}_i)}(\sum z_i \tilde{y}_i, \Delta_0) = \Delta_0 - \sum_{i<j} \left(\frac{1}{\text{ad}(\sum z_i \tilde{x}_i)}(e^{2\pi i \sum z_i \tilde{x}_i} z_i \text{ad} \tilde{x}_i - 2\pi i )((\sum z_i \tilde{x}_j, z_i \tilde{y}_j))\right)
= \Delta_0 - \sum_{i} 2\pi i z_i \tilde{y}_i - \sum_{i \neq j} \left(\frac{1}{\text{ad}(\tilde{x}_j)}(e^{2\pi i z_j \text{ad} \tilde{x}_j} - 2\pi i )((z_i \tilde{t}_{ij}))\right);
invariant under the permutation of \( z; x \). This implies (13). This proves (12) and therefore (8).

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(Recall that the last sum decomposes as

\[
\sum_{i,j} \frac{1}{\text{ad}(x_j)} \left( e^{2\pi i z_i \text{ad} x_j} - 2\pi i (z_i \tilde{f}_{ij}) + \sum_{i,j} \frac{1}{\text{ad}(x_j)} \left( e^{2\pi i z_i \text{ad} x_j} - 2\pi i (z_i \tilde{f}_{ij}) \right) \right)
\]

which implies (13). This proves (12) and therefore (8).

We then prove that flatness identity \( \partial / \partial \tau - \Delta(z, \tau), \partial / \partial z_i - \tilde{K}(z, \tau) = 0 \). For this, we will prove that \( (\partial / \partial \tau) \tilde{K}(z, \tau) = (\partial / \partial \tau) \Delta(z, \tau) \), and that \( [\Delta(z, \tau), \tilde{K}(z, \tau)] = 0 \).

Let us first prove

\[
(\partial / \partial \tau) \tilde{K}(z, \tau) = (\partial / \partial z_i) \Delta(z, \tau).
\]

(14)

We have \( (\partial / \partial \tau) \tilde{K}(z, \tau) = \sum_{j \neq i} (\partial_{z_j} \tilde{K}(z, \tau) \text{ad} (x_i) \tilde{f}_{ij}) \) and \( (\partial / \partial z_i) \Delta(z, \tau) = (2\pi i)^{-1} \sum_{j \neq i} (\partial_{z_j} g)(z_i, \text{ad} (x_i) \tilde{f}_{ij}) \), where \( \partial_{z_j} := \partial / \partial z_j, \partial_{z} := \partial / \partial z \) so it suffices to prove the identity

\[
(\partial / \partial z_i)(z, x | \tau) = (2\pi i)^{-1}(\partial_{z_j} g)(z, x | \tau),
\]

i.e., \( \partial_{z_j} k(z, x | \tau) = (2\pi i)^{-1}(\partial_{z_j} k)(z, x | \tau) \). In this identity, \( k(z, x | \tau) \) may be replaced by \( \tilde{k}(z, x | \tau) := k(z, x | \tau) + 1/x = \theta(z + x | \tau) / (\theta(z | \theta)(x | \tau)) \).

Dividing by \( \tilde{k}(z, x | \tau) \), the wanted identity is rewritten as

\[
2\pi i \left( \frac{\partial}{\partial \theta} \phi(x + z | \tau) - \frac{\partial}{\partial \theta} \phi(x | \tau) \right) = \left( \frac{\partial'}{\partial \theta} \phi(x + z | \tau) - \frac{\partial'}{\partial \theta} \phi(x | \tau) \right)
\]

(recall that \( f'(z | \tau) = \partial_{x} f(z | \tau) \)), or taking into account the heat equation \( 4\pi i \partial_{\tau} \theta / \theta (z | \tau) = (\theta'' / \theta)(z | \tau) - 12\pi i \partial_{\eta} \eta (\tau) \), as follows

\[
2 \left( \frac{\theta'}{\theta} (z | \tau)^{\theta'}(x + z | \tau) - \frac{\theta'}{\theta} (z | \tau)^{\theta'}(x | \tau) - \frac{\theta'}{\theta} (z | \tau) \frac{\theta'}{\theta} (z + x | \tau) \right) + \frac{\theta''}{\theta} (z | \tau) + \frac{\theta''}{\theta} (x + z | \tau) + \frac{\theta''}{\theta} (z | \tau) - 12\pi i \partial_{\tau} \eta (\tau) = 0
\]

(15)

Let us prove (15). Denote its l.h.s. by \( F(z, x | \tau) \). Since \( \theta(z | \tau) \) is odd w.r.t. \( z, F(z, x | \tau) \) is invariant under the permutation of \( z, x, -z, -x \). The identities \( (\theta'' / \theta)(z + x | \tau) = (\theta'' / \theta)(z | \tau) - 2\pi i \) and \( (\theta'' / \theta)(z + x | \tau) = (\theta'' / \theta)(z | \tau) - 4\pi i \partial_{\tau} \theta / \theta (z | \tau) + (2\pi i)^2 \) imply that \( F(z, x | \tau) \) is elliptic in \( z, x \) (w.r.t. the lattice \( \Lambda_{\tau} \)). The possible poles of \( F(z, x | \tau) \) as a function of \( z \) are simple at \( z = 0 \) and \( z = -x \) (mod \( \Lambda_{\tau} \)), but one checks that \( F(z, x | \tau) \) is regular at these points, so it is constant in \( z \). By the \( G_{\delta_{3}} \)-symmetry, it is also constant in \( x \), hence it is a function of \( \tau \) only: \( F(z, x | \tau) = F(\tau) \).

To compute this function, we compute \( F(z, 0 | \tau) = -2(\theta'' / \theta)' - 2(\theta'' / \theta)^2 + 20'' / \theta (z | \tau) + (\theta'' / \theta)(0 | \tau) - 12\pi i(\partial_{\eta} \eta (\tau) / \theta (0 | \tau)) \), hence \( F(\tau) = (\theta'' / \theta)(0 | \tau) - 12\pi i(\partial_{\eta} \eta (\tau) / \theta (0 | \tau)) \). The above heat equation then implies that \( F(\tau) = 4\pi i(\partial_{\eta} \eta (\tau) / \theta (0 | \tau)) \). Now \( \theta'(0 | \tau) = 1 \) implies that \( \theta(z | \tau) \) has the expansion \( \theta(z | \tau) = z + \sum_{n \geq 2} a_n(z) z^n \) as \( z \to 0 \), which implies \( (\partial_{\tau} \theta / \theta)(0 | \tau) = 0 \). So \( F(\tau) = 0 \), which implies (15) and therefore (14).
We now prove

$$[\Delta(z|\tau), K_i(z|\tau)] = 0. \quad (16)$$

Since $\tau$ is constant in what follows, we will write $k(z, x)$, $g(z, x)$, $\varphi$ instead of $k(z, x|\tau)$, $g(z, x|\tau)$, $\varphi(*|\tau)$. For $i \neq j$, let us set $g_{ij} := g(z_{ij}, ad x_i)(t_{ij})$. Since $g(z, x|\tau) = g(-z, -x|\tau)$, we have $g_{ij} = g_{ji}$. Recall that $K_{ij} = k(z_{ij}, ad x_i)(t_{ij})$.

We have

$$2\pi i [\Delta(z|\tau), K_i(z|\tau)] = [-\Delta, \sum_{i,j} g_{ij}, -\bar{y}_i + \sum_{j \neq i} \bar{K}_{ij}] \quad (17)$$

$$= [\Delta, \bar{y}_i] + \sum_{j \neq i} ( - [\Delta, \bar{K}_{ij}] + [\bar{y}_i, g_{ij}] + [g_{ij}, \bar{K}_{ij}] ) + \sum_{j,k, \neq i, \neq j, \neq k} ([\bar{y}_i, g_{jk}] + [g_{ik} + g_{jk}, \bar{K}_{ij}] + [g_{ij} + g_{jk}, \bar{K}_{ik}]).$$

One computes

$$[\Delta, \bar{y}_i] = \sum_{\alpha} [f_{\alpha}(ad x_i)(\bar{t}_{ij}), g_{\alpha}(-ad x_i)(\bar{t}_{ij})], \quad \text{where} \quad \sum_{\alpha} f_{\alpha}(u)g_{\alpha}(v) = \frac{1}{2} \frac{\varphi(u) - \varphi(v)}{u - v} \quad (18)$$

If $f(x) \in \mathbb{C}[x]$, then

$$[\Delta_0, f(ad x_i)(\bar{t}_{ij})] - [\bar{y}_i, f'(ad x_i)(\bar{t}_{ij})] = \sum_{\alpha} [h_{\alpha}(ad x_i)(\bar{t}_{ij}), k_{\alpha}(ad x_i)(\bar{t}_{ij})] + \sum_{k \neq i, j} \frac{f(ad x_i) - (f(-ad x_j) - f'(ad x_j)(ad x_i + ad x_j))(\bar{t}_{ij}, \bar{t}_{ik})}{(ad x_i + ad x_j)^2},$$

where

$$\sum_{\alpha} h_{\alpha}(u)k_{\alpha}(v) = \frac{1}{2} \left( \frac{1}{v^2} (f(u + v) - f(u) - v f'(u)) - \frac{1}{u^2} (f(u + v) - f(v) - u f'(v)) \right).$$

Since $g(z, x) = k_x(z, x)$, we get

$$- [\Delta_0, \bar{K}_{ij}] + [\bar{y}_i, g_{ij}] = - \sum_{\alpha} [f^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij}), g^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij})] \quad (19)$$

$$+ \sum_{k \neq i, j} \frac{k(z_{ij}, ad x_i) - k(z_{ij}, -ad x_i) - (ad x_i + ad x_j)k_x(z_j, -ad x_j)(\bar{t}_{ij}, \bar{t}_{jk})}{(ad x_i + ad x_j)^2},$$

where

$$\sum_{\alpha} f^{ij}_{\alpha}(u)g^{ij}_{\alpha}(v) = \frac{1}{2} \left( \frac{1}{v^2} (k(z_{ij}, u + v) - k(z_{ij}, u) - v k_x(z_{ij}, u)) - \frac{1}{u^2} (k(z_{ij}, u + v) - k(z_{ij}, v) - u k_x(z_{ij}, v)) \right).$$

For $f(x) \in \mathbb{C}[x]$, we have

$$[\delta_\varphi, f(ad x_i)(\bar{t}_{ij})] = \sum_{\alpha} [l^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij}), m^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij})], \quad \text{where} \quad \sum_{\alpha} l^{ij}_{\alpha}(u)m^{ij}_{\alpha}(v) = f(u + v)\varphi(v),$$

therefore

$$- [\delta_\varphi, \bar{K}_{ij}] = - \sum_{\alpha} [l^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij}), m^{ij}_{\alpha}(ad x_i)(\bar{t}_{ij})], \quad \text{where} \quad \sum_{\alpha} l^{ij}_{\alpha}(u)m^{ij}_{\alpha}(v) = k(z_{ij}, u + v)\varphi(v). \quad (20)$$

For $j, k \neq i$ and $j < k$, we have

$$[\bar{y}_i, g_{jk}] + [g_{ik} + g_{jk}, \bar{K}_{ij}] + [g_{ij} + g_{jk}, \bar{K}_{ik}] = [\bar{y}_i, g_{jk}] - [g_{ki}, \bar{K}_{ji}] - [g_{ji}, \bar{K}_{ki}] + [g_{jk}, \bar{K}_{ij}] + [g_{jk}, \bar{K}_{ik}],$$

where

$$f(x) = f(x) + \sum_{\alpha} \frac{1}{2} \frac{\varphi(u) - \varphi(v)}{u - v}.$$
and since for any \( f(x) \in \mathbb{C}[x] \), \( \tilde{y}_i, f(\text{ad} \tilde{x}_i) (\tilde{t}_{ijk}) = -\frac{f(\text{ad} \tilde{x}_i) - f(-\text{ad} \tilde{x}_i)}{\text{ad} \tilde{x}_i + \text{ad} \tilde{x}_i} (\tilde{t}_{ijk}, \tilde{t}_{ijk}) \), we get

\[
[\tilde{y}_i, g_{jk}] + [g_{ik} + g_{jk}, \tilde{K}_{ij}] + [g_{ij} + g_{jk}, \tilde{K}_{ik}] = \left( -\frac{g(z_{jk}, \text{ad} \tilde{x}_j) - g(z_{jk}, -\text{ad} \tilde{x}_j)}{\text{ad} \tilde{x}_j + \text{ad} \tilde{x}_j} - g(z_{ki}, \text{ad} \tilde{x}_k) k(z_{ji}, \text{ad} \tilde{x}_j) + g(z_{ji}, \text{ad} \tilde{x}_j) k(z_{ki}, \text{ad} \tilde{x}_k)
\right)
\]

Summing up (18), (19), (20) and (21), (17) gives

\[
2\pi i [\tilde{\Delta}(z|\tau), \tilde{K}_i(z|\tau)] = \sum_{j \neq i} \sum_{\alpha} F_{ij}^\alpha (\text{ad} \tilde{x}_i)(\tilde{t}_{ij}), G_{ij}^\alpha (\text{ad} \tilde{x}_i)(\tilde{t}_{ij}) + \sum_{j,k \neq i, k \neq i} H(z_{ij}, z_{ik}, -\text{ad} \tilde{x}_j, -\text{ad} \tilde{x}_k)(\tilde{t}_{ij}, t_{jk}),
\]

where \( \sum_{\alpha} F_{ij}^\alpha (u) G_{ij}^\alpha (v) = L(z_{ij}, u, v), \)

\[
L(z, u, v) = \frac{1}{2} \varphi(u) - \varphi(v) + \frac{1}{2} k(z, u + v)(\varphi(u) - \varphi(v)) + \frac{1}{2} g(z, u)(k(z, v) - k(z, u)g(z, v))
\]

\[
- \frac{1}{2} \left( k(z, u + v) - k(z, u) - v k_x(z, u) - \frac{1}{u^2} \left( k(z, u + v) - k(z, v) - u k_x(z, v) \right) \right)
\]

and

\[
H(z, z', u, v) = \frac{1}{2} k(z, u + v) - k(z, u) - v k_x(z, u) - \frac{1}{u^2} \left( k(z', u + v) - \frac{1}{u^2} \left( k(z', v) - k(z', u) - v k_x(z', v) \right) \right)
\]

Explicit computation shows that \( H(z, z', u, v) = 0 \), which implies that \( L(z, u, v) = 0 \) since \( L(z, u, v) = -\frac{1}{2} H(z, z, u, v) \). This proves (16). \( \square \)

**Remark 3.7.** Define \( \Delta(z|\tau) \) by the same formula as \( \tilde{\Delta}(z|\tau) \), replacing \( \tilde{x}_i, \tilde{y}_i \) by \( x_i, y_i \). Then \( d - \Delta(z|\tau) d \tau - \sum_i K_i(z|\tau) d z_i \) is flat. This can be interpreted as follows.

Let \( N_+ \subset \text{SL}_2(\mathbb{C}) \) be the connected subgroup with Lie algebra \( \mathbb{C} \Delta_0 \). Set \( \tilde{N}_n := \text{exp} \left( (t_{1_n} \times \mathfrak{d}_+)^\wedge \right) \times N_+, \) \( N_n := \text{exp} \left( (t_{1_n} \times \mathfrak{d}_+)^\wedge \right) \times N_+ \) and \( G_n := \text{exp} \left( (t_{1_n} \times \mathfrak{d}_+)^\wedge \right) \times \text{SL}_2(\mathbb{C}) \). Then we have a diagram of groups

\[
\begin{array}{ccc}
\tilde{N}_n & \rightarrow & N_n \\
d & \downarrow & d \\
G_n & \rightarrow & G_n
\end{array}
\]

The trivial \( N_n \)-bundle on \( (\mathfrak{g} \times \mathbb{C}^n) - \text{Diag}_n \) with flat connection \( d - \tilde{\Delta}(z|\tau) d \tau - \sum_i K_i(z|\tau) d z_i \) admits a reduction to \( N_n \), where the bundle is again trivial and the connection is \( d - \Delta(z|\tau) d \tau - \sum_i K_i(z|\tau) d z_i \).

\((\mathbb{Z}^n \times \mathbb{C}) \times \text{SL}_2(\mathbb{Z}) \) contains the subgroups \( (\mathbb{Z}^n)^2, (\mathbb{Z}^n)^2 \times \mathbb{C}, (\mathbb{Z}^n)^2 \times \text{SL}_2(\mathbb{Z}) \). We denote the corresponding quotients of \( (\mathbb{C}^n \times \tilde{\mathfrak{g}}) - \text{Diag}_n \) by \( C(n), \tilde{C}(n), \hat{M}_{1,n} \). These fit in the diagram

\[
\begin{array}{ccc}
\tilde{C}(n) & \rightarrow & C(n) \\
\downarrow & & \downarrow \\
\hat{M}_{1,n} & \rightarrow & M_{1,n}
\end{array}
\]

The pair \( (P_n, \nabla_{P_n}) \) can be pulled back to \( G_n \)-bundles over these covers of \( M_{1,n} \). These pull-backs admit \( G \)-structures, where \( G \) is the corresponding group in the above diagram of groups.

We have natural projections \( C(n) \rightarrow \tilde{\mathfrak{g}}, \tilde{C}(n) \rightarrow \tilde{\mathfrak{g}} \). The fibers of \( \tau \in \tilde{\mathfrak{g}} \) are respectively \( C(E_\tau, n) \) and \( \tilde{C}(E_\tau, n) \). The pair \( (P_n, \nabla_n) \) can be pulled back to \( C(E_\tau, n) \) and \( \tilde{C}(E_\tau, n) \);
these pull-backs admit $G$-structures, where $G = \exp(t_{1,n})$ and $\exp(t_{1,n})$, which coincide with $(P_{n,\tau}, \nabla_{P_{n,\tau}})$ and $(P_{n,\tau}, \nabla_{P_{n,\tau}})$.

### 3.3. Bundle with flat connection over $\mathcal{M}_{1,\{n\}}$.

The semidirect product $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes (\text{SL}_2(\mathbb{C}) \times S_n)$ acts on $(\mathbb{C}^n \times \mathcal{H})$—$\text{Diag}_n$, as follows: the action of $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \text{SL}_2(\mathbb{C})$ is as above and the action of $S_n$ is $\sigma * (z_1, \ldots, z_n, \tau) := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}, \tau)$. The quotient then identifies with $\mathcal{M}_{1,\{n\}}$.

We will define a principal $G_n \times S_n$-bundle with a flat connection $(P_{[n]}, \nabla_{P_{[n]}})$ over $\mathcal{M}_{1,\{n\}}$.

**Proposition 3.8.** There exists a unique principal $G_n \times S_n$-bundle $P_{[n]}$ over $\mathcal{M}_{1,\{n\}}$, such that a section of $U \subset \mathcal{M}_{1,\{n\}}$ is a function $f : \tilde{\pi}^{-1}(U) \to G_n \times S_n$, satisfying the conditions of Proposition 3.4 as well as $f(\sigma z|\tau) = \sigma f(z|\tau)$ for $\sigma \in S_n$ (here $\tilde{\pi} : (\mathbb{C}^n \times \mathcal{H}) - \text{Diag}_n \to \mathcal{M}_{1,\{n\}}$ is the canonical projection).

**Proof.** One checks that $\sigma c_{\tilde{g}}(z|\tau)\sigma^{-1} = c_{\tilde{g} \tilde{g}^{-1} \sigma}(\sigma^{-1}z)$, where $\tilde{g} \in ((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \text{SL}_2(\mathbb{Z})$, $\sigma \in S_n$. It follows that there is a unique cocycle $c_{(\tilde{g},\sigma)} : \mathbb{C}^n \times \mathcal{H} \to G_n \times S_n$ such that $c_{(\tilde{g},1)} = c_{\tilde{g}}$ and $c_{(1,\sigma)}(z|\tau) = \sigma$.

**Theorem 3.9.** There is a unique flat connection $\nabla_{P_{[n]}}$ on $P_{[n]}$, whose pull-back to $(\mathbb{C}^n \times \mathcal{H}) - \text{Diag}_n$ is the connection $d - \Delta(z|\tau) d\tau - \sum_i K_i(z|\tau) dz_i$ on the trivial $G_n \times S_n$-bundle.

**Proof.** Taking into account Theorem 3.6, it remains to show that this connection is $S_n$-equivariant. We have already mentioned that $\sum_i K_i(z|\tau) dz_i$ is equivariant; $\Delta(z|\tau)$ is also checked to be equivariant.

### 4. The monodromy morphisms $\Gamma_{1,\{n\}} \to G_n \times S_n$

Let $\Gamma_{1,\{n\}}$ be the mapping class group of genus 1 surfaces with $n$ unordered marked points. It can be viewed as the fundamental group $\pi_1(\mathcal{M}_{1,\{n\}}, *)$, where $*$ is a base point at infinity which will be specified later. The flat connection on $\mathcal{M}_{1,\{n\}}$ introduced above gives rise to morphisms $\gamma_n : \Gamma_{1,\{n\}} \to G_n \times S_n$, which we now study. This study divides in two parts: in the first, analytic part, we show that $\gamma_n$ can be obtained from $\gamma_1$ and $\gamma_2$, and show that the restriction of $\gamma_n$ to $\overline{B}_{1,n}$ can be expressed in terms of the KZ associator only. In the second part, we show that morphisms $\overline{B}_{1,n} \to \exp(t_{1,n}) \times S_n$ can be constructed algebraically using an arbitrary associator. Finally, we introduce the notion of an elliptic structure over a quasi-bialgebra.

### 4.1. The solution $F^{(n)}(z|\tau)$.

The elliptic KZB system is now

$$(\partial/\partial z_1) F(z|\tau) = K_i(z|\tau) F(z|\tau), \quad (\partial/\partial \tau) F(z|\tau) = \Delta(z|\tau) F(z|\tau),$$

where $F(z|\tau)$ is a function $(\mathbb{C}^n \times \mathcal{H}) - \text{Diag}_n \subset U \to G_n \times S_n$ invariant under translation by $\mathbb{C}(\sum_i \delta_i)$. Let $D_n := \{ (z, \tau) \in \mathbb{C}^n \times \mathcal{H} | z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \ldots < a_n < a_1 + 1, b_1 < b_2 < \ldots < b_n < b_1 + 1 \}$. Then $D_n \subset (\mathbb{C}^n \times \mathcal{H}) - \text{Diag}_n$ is simply connected and invariant under $\mathbb{C}(\sum_i \delta_i)$. A solution of the elliptic KZB system on this domain is then unique, up to right multiplication by a constant. We now determine a particular solution $F^{(n)}(z|\tau)$.

Let us study the elliptic KZB system in the region $z_{ij} \ll 1$, $\tau \to \pm \infty$. Then $K_i(z|\tau) = \sum_{ij} (t_{ij} / (z_i - z_j)) + O(1)$.

Now we compute the expansion of $\Delta(z|\tau)$. The heat equation for $\vartheta$ implies the expansion $\vartheta(x|\tau) = \eta(\tau)^3 (x + 2 \pi i \vartheta \log \eta(\tau) x^3 + O(x^5))$, so $\vartheta(x|\tau) = x + 2 \pi i \vartheta \log \eta(\tau) x^3 + O(x^5)$, hence

$$g(0, x|\tau) = \left( \frac{\vartheta^2}{\vartheta} \right)'(x|\tau) + \frac{1}{x^2} = 4 \pi i \vartheta \log \eta(\tau) + O(x) = -\frac{\pi^2}{3} E_2(\tau) + O(x)$$
since $E_2(\tau) = \frac{24}{\pi^2} \partial_\tau \log \eta(\tau)$. We have $g(0, x|\tau) = g(0, 0|\tau) - \varphi(x|\tau)$, so
\[ g(0, x|\tau) = - \sum_{k \geq 0} a_{2k} x^{2k} E_{2k+2}(\tau), \]
where $a_0 = \pi^2/3$. Then
\[ \Delta(z|\tau) = - \frac{1}{2\pi i} \left( \Delta_0 + \sum_{k \geq 0} a_{2k} E_{2k+2}(\tau) (\delta_{2k} + \sum_{i,j|i<j} (\text{ad} \tilde{x}_i)_{2k}(\tilde{i}_{ij})) \right) + o(1) \]
for $z_{ij} \ll 1$ and any $\tau \in \mathfrak{g}$. Since we have an expansion $E_{2k}(\tau) = 1 + \sum_{l>0} a_{k} e^{2\pi i l \tau}$ as $\tau \to \imath \infty$, and using Proposition A.3 with $u_n = z_{n1}$, $u_{n-1} = z_{n-1,1}/z_{n1}$, ..., $u_2 = z_{21}/z_{31}$, $u_1 = q = e^{2\pi i \tau}$, there is a unique solution $F(n)(z|\tau)$ with the expansion
\[ F(n)(z|\tau) \simeq z_{21}^{\tilde{f}_{12} + \tilde{f}_{13} + \cdots + \tilde{f}_{n1}} \cdots \tilde{f}_{n1} \cdots \tilde{f}_{21-n,n} \exp \left( - \frac{\tau}{2\pi i} \left( \Delta_0 + \sum_{k \geq 0} a_{2k} (\delta_{2k} + \sum_{i,j<i} (\text{ad} \tilde{x}_i)_{2k}(\tilde{i}_{ij})) \right) \right) \]
in the region $z_{21} \ll z_{31} \ll \cdots \ll z_{n1} \ll 1$, $\tau \to \imath \infty$, $(z, \tau) \in D_n$ (here $z_{ij} = z_i - z_j$); here the sign $\simeq$ means that any of the ratios of both sides has the form $1 + \sum_{k>0} \sum_{i_{a1},\ldots,i_{an}} z^{k_{a1},\ldots,a_n} (1,1,\ldots,1)$, where the second sum is finite with $a_i \geq 0$, $i \in \{1,\ldots,n\}$, $r_{k_{i;i1,\ldots,a_n}} (u_1,\ldots,u_n)$ has degree $k$, and is $O((\log u_1)^a \cdots (\log u_n)^a)$.

### 4.2. Presentation of $\Gamma_{1,[n]}$.

According to [Bi2], $\Gamma_{1,[n]} = \{ \overline{B}_1 \times \overline{SL_2(\mathbb{Z})} \}/\overline{Z}$, where $\overline{SL_2(\mathbb{Z})}$ is a central extension $1 \to Z \to \overline{SL_2(\mathbb{Z})} \to SL_2(\mathbb{Z}) \to 1$; the action $\alpha : \overline{SL_2(\mathbb{Z})} \to \text{Aut}(\overline{B}_1)$ is such that for $Z$ the central element $1 \in Z \subset \overline{SL_2(\mathbb{Z})}$, $\alpha_Z(x) = Z'x(Z')^{-1}$, where $Z'$ is the image of the generator of the center of $PB_n$ (the pure braid group of $n$ points on the plane) under the natural morphism $PB_n \to \overline{B}_1$; $\overline{B}_1 \times \overline{SL_2(\mathbb{Z})}$ is then $\overline{B}_1 \times \overline{SL_2(\mathbb{Z})}$ with the product $(p,A)(p',A') = (p \circ A(p'), AA')$: this semidirect product is then factored by its central subgroup (isomorphic to $Z$) generated by $(Z')^{-1}, Z$.

$\Gamma_{1,[n]}$ is presented explicitly as follows. Generators are $\sigma_i (i = 1,\ldots,n-1)$, $A_i, B_i (i = 1,\ldots,n), C_{j,k}$ $(1 \leq j < k \leq n)$, $\Theta$ and $\Psi$, and relations are:
\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_{i} \sigma_{i+1} (i = 1,\ldots,n-2), & \sigma_i \sigma_j &= \sigma_j \sigma_i (1 \leq i < j \leq n), \\
\sigma_i^{-1} X_i \sigma_i &= X_{i+1}, & \sigma_i \sigma_i &= Y_{i+1} (i = 1,\ldots,n-1), \\
(\sigma_i, X_j) &= (\sigma_i, Y_j) = 1 (i \in \{1,\ldots,n-1\}, j \in \{1,\ldots,n\}, j \neq i, i+1), \\
\sigma_i^2 &= C_{i+1} C_{i+1,i+2} C_{i+2} (i = 1,\ldots,n-1), \\
(A_i, A_j) &= (B_i, B_j) = 1 \text{ (any } i, j), & A_i &= B_i = 1, \\
(B_k, A_k A_j^{-1}) &= (B_k B_{j+1}, A_k) = C_{j,k} (1 \leq j < k \leq n), & (A_i, C_{j,k}) &= (B_i, C_{j,k}) = 1 (1 \leq i < j \leq k \leq n), \\
\Theta A_i \Theta^{-1} &= B_i^{-1}, & \Theta B_i \Theta^{-1} &= B_i A_i B_i^{-1}, \\
\Psi A_i \Psi^{-1} &= A_i, & \Psi B_i \Psi^{-1} &= B_i A_i, & (\Theta, \sigma_i) &= (\Psi, \sigma_i) = 1, \\
(\Psi, \Theta^2) &= 1, \quad (\Theta \Psi)^3 = \Theta^4 = C_{12} \cdots C_{n-1,n}.
\end{align*}
\]
Here $X_i = A_i A_i^{-1}, Y_i = B_i B_i^{-1}$ for $i = 1,\ldots,n$ (with the convention $A_{n+1} = B_{n+1} = C_{1,n+1} = 1$). The relations imply
\[
C_{j,k} = \sigma_{j+1} \cdots \sigma_{j+n-k,j+n-k+1} \cdots \sigma_{k-1,k-n},
\]
where $\sigma_{i,i+1,j} = \sigma_{j-1,i} \sigma_i$. Observe that $C_{12},\ldots,C_{n-1,n}$ commute with each other.

The group $SL_2(\mathbb{Z})$ is presented by generators $\Theta, \Psi$ and $Z$, and relations: $Z$ is central, $\Theta^4 = (\Theta \Psi)^3 = \Theta$ and $(\Theta, \Theta^2) = 1$. The morphism $\overline{SL_2(\mathbb{Z})} \to SL_2(\mathbb{Z})$ is $\Theta \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Psi \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the morphism $\Gamma_{1,[n]} \to SL_2(\mathbb{Z})$ is given by the same formulas and $A_i, B_i, \sigma_i \mapsto 1$. 
The elliptic braid group $\Omega_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to SL_2(\mathbb{Z})$; it has the same presentation as $\Gamma_{1,[n]}$, except for the omission of the generators $\Theta, \Psi$ and the relations involving them. The "pure" mapping class group $\Gamma_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to S_n$, $A_i, B_i, C_{jk} \mapsto 1$, $\sigma_i \mapsto \sigma_i$; it has the same presentation as $\Gamma_{1,[n]}$, except for the omission of the $\sigma_i$. Finally, recall that $\Omega_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to SL_2(\mathbb{Z}) \times S_n$.

Remark 4.1. The extended mapping class group $\hat{\Gamma}_{1,n}$ of classes of non necessarily orientation-preserving self-homeomorphisms of a surface of type $(1, n)$ fits in a split exact sequence $1 \to \Gamma_{1,n} \to \hat{\Gamma}_{1,n} \to \mathbb{Z}/2\mathbb{Z} \to 1$; it may be viewed as $\{\Omega_{1,n} \times \overline{GL_2(\mathbb{Z})}\}/\mathbb{Z}$; it has the same presentation as $\Gamma_{1,n}$, with the additional generator $\Sigma$ subject to

$$\Sigma^2 = 1, \quad \Sigma \Theta \Sigma^{-1} = \Theta^{-1}, \quad \Sigma \Psi \Sigma^{-1} = \Psi^{-1}, \quad \Sigma A_i \Sigma^{-1} = A_i^{-1}, \quad \Sigma B_i \Sigma^{-1} = A_i B_i A_i^{-1}.$$ 

4.3. The monodromy morphisms $\gamma_n : \Gamma_{1,[n]} \to G_n \times S_n$. Let $F(z|\tau)$ be a solution of the elliptic KZB system defined on $D_n$.

Recall that $D_n := \{ (z, \tau) \in \mathbb{C}^n \times \mathbb{S} | z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \ldots < a_n < a_1 + 1, b_1 < b_2 < \ldots < b_n < b_1 + 1 \}$. The domains $H_n := \{ (z, \tau) \in \mathbb{C}^n \times \mathbb{S} | z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \ldots < a_n < a_1 + 1 \}$ and $D_n := \{ (z, \tau) \in \mathbb{C}^n \times \mathbb{S} | z_i = x_i + b_i \tau, a_i, b_i \in \mathbb{R}, b_1 < b_2 < \ldots < b_n < b_1 + 1 \}$ are also simply connected and invariant, and we denote by $F^H(z|\tau)$ and $F^V(z|\tau)$ the prolongations of $F(z|\tau)$ to these domains.

Then $(z, \tau) \mapsto F^H(z + \sum_{j=1}^n \delta_i |\tau)$ and $(z, \tau) \mapsto e^{2\pi i (\bar{z}_i + \ldots + \bar{x}_n)} F^V(z + \tau (\sum_{j=1}^n \delta_i) |\tau)$ are solutions of the elliptic KZB system on $H_n$ and $D_n$ respectively. We define $A_i^F, B_i^F \in G_n$ by

$$F^H(z + \sum_{j=1}^n \delta_i |\tau) = F^H(z|\tau) A_i^F, \quad e^{2\pi i (\bar{z}_i + \ldots + \bar{x}_n)} F^V(z + \tau (\sum_{j=1}^n \delta_i) |\tau) = F^V(z|\tau) B_i^F.$$ 

The action of $T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is $(z, \tau) \mapsto (-z, -1/\tau)$; this transformation takes $H_n$ to $V_n$. Then $(z, \tau) \mapsto c_{T^{-1}}(z|\tau)^{-1} F^V(-z/\tau - 1/\tau)$ is a solution of the elliptic KZB system on $H_n$ (recall that $c_{T^{-1}}(z|\tau)^{-1} = e^{2\pi i (-\sum_i z_i \bar{x}_i + \tau X)}(-\tau)^d = (-\tau)^d e((2\pi i/\tau)(\sum_i z_i \bar{x}_i + \tau X))$). We define $\Theta^F$ by

$$c_{T^{-1}}(z|\tau)^{-1} F^V(-z/\tau - 1/\tau) = F^H(z|\tau) \Theta^F.$$ 

The action of $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $(z, \tau) \mapsto (z, \tau + 1)$. This transformation takes $H_n$ to itself. Since $c_S(z|\tau) = 1$, the function $(z, \tau) \mapsto F^H(z, \tau + 1)$ is a solution of the elliptic KZB system on $H_n$. We define $\Psi^F$ by

$$F^H(z|\tau + 1) = F^H(z|\tau) \Psi^F.$$ 

Finally, define $\sigma_i^F$ by

$$\sigma_i F(\sigma_i^{-1} z|\tau) = F(z|\tau) \sigma_i^F,$$

where on the l.h.s. $F$ is extended to the universal cover of $(\mathbb{C}^n \times \mathbb{S}) \setminus \text{Diag}_n$ ($\sigma_i$ exchanges $z_i$ and $z_{i+1}$, $z_{i+1}$ passing to the right of $z_i$).

Lemma 4.2. There is a unique morphism $\Gamma_{1,[n]} \to G_{1,n} \times S_n$, taking $X$ to $X^F$, where $X = A_i, B_i, \Theta$ or $\Psi$.

Proof. This follows from the geometric description of generators of $\Gamma_{1,[n]}$; if $(z_0, \tau_0) \in D_n$, then $A_i$ is the class of the projection of the path $[0,1] \ni t \mapsto (z_0 + t \sum_{j=1}^n \delta_j, \tau_0)$, $B_i$ is the class of the projection of $[0,1] \ni t \mapsto (z_0 + t \tau \sum_{j=1}^n \delta_j, \tau_0)$, $\Theta$ is the class of the projection of any path connecting $(z_0, \tau_0)$ to $(-z_0/\tau_0, -1/\tau_0)$ contained in $H_n$, and $\Psi$ is the class of the projection of any path connecting $(z_0, \tau_0)$ to $(z_0, \tau_0 + 1)$ contained in $H_n$.

We will denote by $\gamma_n : \Gamma_{1,[n]} \to G_n \times S_n$ the morphism induced by the solution $F^{(n)}(z|\tau)$. 

4.4. Expression of $\gamma_n: \Gamma_{1,n} \to G_n \times S_n$ using $\gamma_1$ and $\gamma_2$.

Lemma 4.3. There exists a unique Lie algebra morphism $\delta: \mathfrak{sl}_{1,n} \to \mathfrak{g}$, $x \mapsto [x]$, such that $[\delta_{2n}] = \delta_{2n} + \sum_{i,j} (\text{ad}(\tilde{x}_i)) \delta_{1} + \tilde{t}_{ij}$. This induces a group morphism $G_1 \to G_n$, also denoted $g \mapsto [g]$.

Lemma 4.4. For each map $\phi: \{1, \ldots, m\} \to \{1, \ldots, n\}$, there exists a Lie algebra morphism $\mathfrak{t}_{1,n} \to \mathfrak{t}_{1,m}$, $x \mapsto x^\phi$, defined by $(\tilde{x}_i) = \sum_{i' \in \phi^{-1}(i)} \tilde{x}_{i'}$, $(\tilde{y}_i) = \sum_{i' \in \phi^{-1}(i)} \tilde{y}_{i'}$, $(\tilde{t}_{ij}) = \sum_{i' \in \phi^{-1}(i), j' \in \phi^{-1}(j)} \tilde{t}_{ij'j'}$.

It induces a group morphism $\exp(\mathfrak{t}_{1,n}) \to \exp(\mathfrak{t}_{1,m})$, also denoted $g \mapsto g^\phi$.

The proofs are immediate. We now recall the definition and properties of the KZ associator $[\text{Dr3}]$.

We will also use the notation $\Phi = \Phi(t_{12}, t_{23}) \in \exp(t)$ is defined by $G_0(z) = G_1(z) \Phi$, where $G_i : [0, 1] \to \exp(t)$ are the solutions of $G'(z)G(z)^{-1} = t_{12}/z + t_{23}/(z - 1)$ with $G_0(z) \sim z^{t_{12}}$ as $z \to 0$ and $G(z) \sim (1 - z)^{t_{23}}$ as $z \to 1$. The KZ associator satisfies the duality, hexagon and pentagon equation (37), (38) below (where $\lambda = 2\pi i$).

Lemma 4.5. $\gamma_2(A_2)$ and $\gamma_2(B_2)$ belong to $\exp(\mathfrak{t}_{1,2}) \subset G_2$.

Proof. If $F(z|\tau): H_2 \to G_2$ is a solution of the KZB equation for $n = 2$, then $A_2^F = F^H(z + \delta_2|\tau)F^H(z|\tau)^{-1}$ is expressed as the iterated integral, from $z_0 \in D_n$ to $z_0 + \delta_2$, of $K_2(z|\tau) \in \mathfrak{t}_{1,2}$, hence $A_2^F \in \exp(\mathfrak{t}_{1,2})$. Since $\gamma_2(A_2)$ is a conjugate of $A_2^F$, it belongs to $\exp(\mathfrak{t}_{1,2})$ as $\exp(\mathfrak{t}_{1,2}) \subset G_2 \times S_2$ is normal. One proves similarly that $\gamma_2(B_2) \in \exp(\mathfrak{t}_{1,2})$.

Set
$$
\Phi := \Phi^{1 \ldots i - 1 \ldots i + 1 \ldots n} \ldots \Phi^{1 \ldots n - 2 \ldots n - 1 \ldots n} \in \exp(\mathfrak{t}_n).
$$
We denote by $x \mapsto \{x\}$ the morphism $\exp(\mathfrak{t}_n) \to \exp(\mathfrak{t}_{1,2})$ induced by $t_{ij} \mapsto \tilde{t}_{ij}$.

Proposition 4.6. If $n \geq 2$, then
$$
\gamma_n(\Theta) = [\gamma_1(\Theta)]^{\frac{1}{2}} \sum_{i<j} \tilde{t}_{ij}, \quad \gamma_n(\Psi) = [\gamma_1(\Psi)]^{\frac{1}{2}} \sum_{i<j} \tilde{t}_{ij},
$$
and if $n \geq 3$, then
$$
\gamma_n(A_i) = \{\Phi_1\}^{-1}\gamma_2(A_1)^{1 \ldots i - 1 \ldots i + 1 \ldots n}\{\Phi_1\}, \quad \gamma_n(B_i) = \{\Phi_1\}^{-1}\gamma_2(B_1)^{1 \ldots i - 1 \ldots i + 1 \ldots n}\{\Phi_1\}, \quad (i = 1, \ldots, n),
$$
$$
\gamma_n(\sigma_i) = \{\Phi_1^{1 \ldots i - 1 \ldots i + 1 \ldots n}\}^{-1} \big\{\Phi_1^{1 \ldots i - 1 \ldots i + 1}\big\} [\Phi_1^{1 \ldots i - 1 \ldots i + 1}], \quad (i = 1, \ldots, n - 1).
$$

Proof. In the region $z_{21} \ll z_{31} \ll \ldots \ll z_{n1} \ll 1$, $(\mathbf{z}, \tau) \in D_n$, we have
$$
F^{(n)}(\mathbf{z}|\tau) \sim z_{12}^{t_{12}} \ldots z_{n1}^{t_{n-1,n}} \exp\left(-\frac{a_0}{2\pi i} \left(\sum_{i<j} \tilde{t}_{ij}\right)\right) [F(\tau)],
$$

where $F(\tau) = F^{(1)}(\mathbf{z}|\tau)$ for any $\mathbf{z}$. Here $C$ is the constant such that $\int_0^\tau E_2 + C = \tau + o(1)$ as $\tau \to \pm \infty$.\footnote{We will also use the notation $x^{t_1 \ldots t_n}$ for $x^\phi$, where $\phi = \phi^{-1}(i)$.}
We have $F(\tau + 1) = F(\tau)\gamma_1(\Psi)$, $F(-1/\tau) = F(\tau)\gamma_1(\Theta)$. Since $\sum_{i<j} \tilde{t}_{ij}$ commutes with the image of $x \mapsto [x]$, we get $F^{(n)}(\tau \mid z + 1) = F^{(n)}(\tau \mid z) \exp(-\frac{i}{2\pi}(\sum_{i<j} \tilde{t}_{ij})) [\gamma_1(\Psi)]$, so

$$\gamma_n(\Psi) = \exp(i \frac{\pi}{6} \sum_{i<j} \tilde{t}_{ij}) [\gamma_1(\Psi)].$$

In the same region,

$$c_{\tau^{-1}}(z \mid \tau)^{-1} F^{(n)} V(-\frac{z}{\tau} | \frac{1}{\tau}) \simeq \exp(-\frac{a_0}{2\pi} \int_{i}^{\frac{1}{\tau}} \sum_{i<j} \tilde{t}_{ij}) [F(-1/\tau)].$$

Now $E_2(-1/\tau) = \tau^2 E_2(\tau) + (6i / \pi) \tau$, so $\int_{i}^{\frac{-1}{\tau}} E_2 - \int_{i}^{\frac{-1}{\tau}} E_2 = (6i / \pi)[\log(-1/\tau) - \log i]$

(where $\log(re^{i\theta}) = \log r + i \theta$ for $\theta \in \mathbb{R}$).

It follows that

$$c_{\tau^{-1}}(z \mid \tau)^{-1} F^{(n)} (-\frac{z}{\tau} | \frac{1}{\tau}) \simeq e^{2i \pi i (\sum_i \tilde{t}_{i-1}, 1)} \exp(-\frac{a_0}{2\pi} \int_{i}^{\tau} E_2 + C(\sum_{i<j} \tilde{t}_{ij})) \exp(\frac{i\pi}{2} \sum_{i<j} \tilde{t}_{ij})$$

$$\simeq F^{(n)H} (z \mid \tau) [\gamma_1(\Theta)] \exp(\frac{i\pi}{2} \sum_{i<j} \tilde{t}_{ij})$$

(the second $\simeq$ follows from $\sum_i \tilde{t}_{i-1,n} = \sum_{i>1} \tilde{t}_{1,n}$ and $\tilde{t}_{1,n} \rightarrow 0$), so

$$\gamma_n(\Theta) = [\gamma_1(\Theta)] \exp(i \frac{\pi}{2} \sum_{i<j} \tilde{t}_{ij}).$$

Let $G_i(z \mid \tau)$ be the solution of the elliptic KZB system, such that

$$G_i(z \mid \tau)$$

$= z_{i-1,1} \cdots z_{i,n,1} \cdots z_{i,n,n-1} \exp(-\frac{\tau}{2\pi} \left( \Delta_0 + \sum_{n \geq 0} a_2n(\Delta_{2n} + \sum_{i<j} (\tilde{t}_{i,j}(z_{i,j})) \right))$$

when $z_{i,1} \ll \ldots \ll z_{i-1,1} \ll 1$, $z_{i,n-1} \ll \ldots \ll z_{i,n} \ll 1$, $\tau \rightarrow i \infty$ and $(z, \tau) \in D_n$. Then $G_i(z + \sum_{j=1}^{n} \delta_i \tau) = G_i(z \mid \tau) \gamma_2(A_2)^{i-1,i+1}_n$, because in the domain considered $K_i(z \mid \tau)$ is close to $K_i(z, \tau)^{i-1,i+1}_n$ (where $K_i(...)$ corresponds to the 2-point system); on the other hand, $F(z \mid \tau) = G_i(z \mid \tau) \{ \Phi_i \}$, which implies the formula for $\gamma_n(A_i)$. The formula for $\gamma_n(B_i)$ is proved in the same way. Finally, the behavior of $F^{(n)}(z \mid \tau) \tau$ for $\tau \rightarrow i \infty$ and $\tau \rightarrow 1$ is similar to that of a solution of the KZ equations, which implies the formula for $\gamma_n(\sigma_i)$.

**Remark 4.7.** One checks that the composition $\text{SL}_2(\mathbb{Z}) \simeq \Gamma_{1,1} \rightarrow \mathbf{G}_1 \rightarrow \text{SL}_2(\mathbb{C})$ is a conjugation of the canonical inclusion. It follows that the composition $\text{SL}_2(\mathbb{Z}) \subset \Gamma_{1,n} \rightarrow \mathbf{G}_1 \rightarrow \text{SL}_2(\mathbb{C})$ is a conjugation of the canonical projection for any $n \geq 1$.

Let us set $\tilde{A} := \gamma_2(A_2)$, $\tilde{B} := \gamma_2(B_2)$. The image of $A_2 A_3^{-1} = \sigma_1^{-1} A_2^{-1} \sigma_1$ by $\gamma_3$ yields

$$\tilde{A}^{12,3} = e^{i\pi \tilde{t}_{i=1,1}} \{ \Phi \}^{3,1}_3 \tilde{A}^{2,13} \{ \Phi \}^{2,1,3} \tilde{A}^{1,23} \{ \Phi \}^{3,2,1} \tilde{A}^{1,2} \{ \Phi \}^{1,2,3}$$

(22)
and the image of $B_2B_3^{-1} = \sigma_1B_2^{-1}\sigma_1$ yields
\[
\tilde{B}^{12,3} = e^{-i\pi i_{12}}\{\Phi\}^{3,1,2} B^{2,13}\{\Phi\}^{2,1,3} e^{-i\pi i_{12}} \cdot \{\Phi\}^{3,2,1} \tilde{B}^{1,23}\{\Phi\}^{1,2,3}.
\] (23)

Since $(\gamma_3(A_2), \gamma_3(A_3)) = (\gamma_3(B_2), \gamma_3(B_3)) = 1$, we get
\[
\left(\{\Phi\}^{3,2,1} A^{1,23}\{\Phi\}, \tilde{A}^{12,3}\right) = \left(\{\Phi\}^{3,2,1} \tilde{B}^{1,23}\{\Phi\}, \tilde{B}^{12,3}\right) = 1
\] (24)
(this equation can also be directly derived from (22) and (23) by noting that the l.h.s. is invariant $x \mapsto x^{2,1,3}$ and commutes with $e^{\pm i\pi i_{12}}$). We have for $n = 2$, $C_{12} = (B_2, A_2)$, so $\langle A, B \rangle = \gamma_2(C_{12})^{-1}$. Also $\gamma_1(\Theta)^4 = 1$, so $\gamma_2(C_{12}) = \gamma_2(\Theta)^4 = (e^{i\pi i_{12}/2}[\gamma_1(\Theta)])^4 = e^{2\pi i_{12}[\gamma_1(\Theta)^4]} = e^{2\pi i_{12}}$, so
\[
\langle A, B \rangle = e^{-2\pi i_{12}}.
\] (25)

For $n = 3$, we have $\gamma_3(\Theta)^4 = e^{2\pi i(i_{12}+i_{13}+i_{23})} = \gamma_3(C_{12}C_{23})$; since $\gamma_3(C_{12}) = (\gamma_3(B_2), \gamma_3(A_2)) = \{\Phi\}^{-1}(\tilde{B}, \tilde{A})^{1,23}\{\Phi\} = \{\Phi\}^{-1} e^{2\pi i(i_{12}+i_{13})} \{\Phi\}$, we get $\gamma_3(C_{23}) = \{\Phi\}^{-1} e^{2\pi i_{23}} \{\Phi\}$. The image by $\gamma_3$ of $(B_3, A_3 A_2^{-1}) = (B_3 B_2^{-1}, A_3) = C_{23}$ then gives
\[
\left(\tilde{B}^{12,3}, \tilde{A}^{12,3}\{\Phi\}^{-1}(\tilde{A}^{1,23})^{-1}\{\Phi\}\right) = \left(\tilde{B}^{12,3}\{\Phi\}^{-1}(\tilde{B}^{1,23})^{-1}\{\Phi\}, \tilde{A}^{12,3}\right) = \{\Phi\}^{-1} e^{2\pi i_{23}} \{\Phi\}
\] (26)
(applying $x \mapsto x^{\phi,1,2}$, this identity implies (25)).

Let us set $\tilde{\Theta} := \gamma_1(\Theta)$, $\tilde{\Psi} := \gamma_2(\Theta)$. Since $\gamma_1, \gamma_2$ are group morphisms, we have
\[
\tilde{\Theta}^4 = (\tilde{\Theta}\tilde{\Psi})^3 = (\tilde{\Theta}^2, \tilde{\Psi}) = 1,
\] (27)
\[
[\tilde{\Theta}] e^{i \frac{\tau}{2} i_{12}} (\tilde{\Theta}) e^{i \frac{\tau}{2} i_{12}} - 1 = B^{-1},
\] (28)
\[
[\tilde{\Psi}] e^{i \frac{\tau}{2} i_{12}} (\tilde{\Psi}) e^{i \frac{\tau}{2} i_{12}} - 1 = A, \quad [\tilde{\Psi}] e^{i \frac{\tau}{2} i_{12}} (\tilde{\Psi}) e^{i \frac{\tau}{2} i_{12}} - 1 = \tilde{B}\tilde{A}.
\] (29)
(27) (resp., (28), (29)) are identities in $G_1$ (resp., $G_2$); in (28), (29), $x \mapsto [x]$ is induced by the map $\partial \rightarrow \partial \times i_{1,2}$ defined above.

4.5. Expression of $\tilde{\Psi}$ and of $\tilde{A}$ and $\tilde{B}$ in terms of $\Phi$. In this section, we compute $\tilde{A}$ and $\tilde{B}$ in terms of the KZ associator $\Phi$. We also compute $\tilde{\Psi}$.

Recall the definition of $\tilde{\Psi}$. The elliptic KZB system for $n = 1$ is
\[
2\pi i \partial_{\tau} F(\tau) + (\Delta_0 + \sum_{k \geq 1} a_{2k} E_{2k+2}(\tau) \delta_{2k}) F(\tau) = 0.
\]
The solution $F(\tau) := F^{(1)}(z|\tau)$ (for any $z$) is determined by $F(\tau) \approx \exp(-\frac{1}{\Delta_0 + \sum_{k \geq 1} a_{2k} \delta_{2k}})$. Then $\tilde{\Psi}$ is determined by $F(\tau + 1) = F(\tau)\tilde{\Psi}$. We have therefore:

Lemma 4.8. $\tilde{\Psi} = \exp\left(-\frac{1}{\Delta_0 + \sum_{k \geq 1} a_{2k} \delta_{2k}}\right)$.

Recall the definition of $\tilde{\Theta}$ and $\tilde{\Psi}$. The elliptic KZB system for $n = 2$ is
\[
\partial_{\tau} F(z|\tau) = -\left(\frac{\partial}{\partial(\frac{z+ad x}{\tau})} \frac{\tau}{\partial(\frac{z+ad x}{\tau})}\right)(y) \cdot F(z|\tau),
\] (30)
\[
2\pi i \partial_{\tau} F(z|\tau) + (\Delta_0 + \sum_{k \geq 1} a_{2k} E_{2k+2}(\tau) \delta_{2k} - g(z, ad x|\tau)(t)) F(z|\tau) = 0,
\] (31)
where $z = z_2, x = x_2, y = y_2, t = i_{12} = -[x, y]$.

The solution $F(z|\tau) := F^{(2)}(z_1, z_2|\tau)$ is determined by its behavior $F(z|\tau) \approx z \cdot \exp(-\frac{z}{\Delta_0 + \sum_{k \geq 0} a_{2k} \delta_{2k} + (ad x)^{2k}(t)})$ when $z \rightarrow 0^+, \tau \rightarrow i \infty$. We then have $F^H(z + 1|\tau) = F^H(z|\tau)\tilde{A}, e^{2\pi i x^V} F^V(z + \tau|\tau) = F^V(z|\tau)\tilde{B}$. 


where which implies that where \(g\) is chosen with imaginary part in \(\mathbb{C} \setminus \mathbb{R}_-\) and \(x \in \mathfrak{n}\), where \(\mathfrak{n}\) is a pronilpotent Lie algebra, then \(z^z\) is \(\exp(x \log z) \in \exp(\mathfrak{n})\), where \(\log z\) is chosen with imaginary part in \([-\pi, \pi]\).

**Proposition 4.9.** We have
\[
\hat{A} = (2\pi i)^t \Phi(y, t) e^{2\pi i \bar{y}} \Phi(y, t)^{-1} (i/2\pi)^t = (2\pi i)^t i^{-3t} \Phi(-\bar{y} - t, t) e^{2\pi i (\bar{y} + t)} \Phi(-\bar{y} - t, t)^{-1} (2\pi i)^{-t},
\]
where \(\hat{y} = -\frac{ad x}{2\pi i \sin(\pi z)}(y)\).

**Proof.** \(\hat{A} = F^H(z|\tau)^{-1} F^H(z + 1|\tau)\), which we will compute in the limit \(\tau \to i\infty\). For this, we will compute \(F(z|\tau)\) in the limit \(\tau \to i\infty\). In this limit, \(\theta(z|\tau) = (1/\pi) \sin(\pi z)[1 + \mathcal{O}(e^{2\pi i \tau})]\) so the system becomes
\[
\partial_z F(z|\tau) = (\pi \cotg(\pi z) t - \pi \cotg(\pi ad x) \text{ad} x(y) + \mathcal{O}(e^{2\pi i \tau})) F(z|\tau)
\]
(32)
\[
2\pi i \partial_t F(z|\tau) + (\Delta_0 + \sum_{k \geq 1} a_{2k} \delta_{2k} + \left(\frac{\pi^2}{\sin^2(\pi ad x)} - \frac{1}{(ad x)^2}\right) (t) + \mathcal{O}(e^{2\pi i \tau})) F(z|\tau) = 0
\]
where the last equation is
\[
2\pi i \partial_t F(z|\tau) + (\Delta_0 + a_0 t + \sum_{k \geq 1} a_{2k} \delta_{2k} + (ad x)^{2k}(t) + \mathcal{O}(e^{2\pi i \tau})) F(z|\tau) = 0.
\]

We set
\[
\Delta := \Delta_0 + \sum_{k \geq 1} a_{2k} \delta_{2k}, \quad \text{so} \quad \Delta_0 + a_0 t + \sum_{k \geq 1} a_{2k} \delta_{2k} + (ad x)^{2k}(t) = [\Delta] + a_0 t.
\]
The compatibility of this system implies that \([\Delta] + a_0 t\) commutes with \(t\) and \((\pi \text{ad} x) \cotg(\pi \text{ad} x)(y) = i\pi(-t - 2\bar{y})\), hence with \(t\) and \(\bar{y}\); actually \(t\) commutes with each \([\delta_{2k}] = \delta_{2k} + (ad x)^{2k}(t)\).

Equation (30) can be written \(\partial_z F(z|\tau) = (t/2 + \mathcal{O}(1)) F(z|\tau)\). We then let \(F_0(z|\tau)\) be the solution of (30) in \(V := \{ (z, \tau) | \tau \in S, z = a + br, a \in [0, 1], b \in \mathbb{R} \}\) such that \(F_0(z|\tau) \simeq z^t\) when \(z \to 0^+\), for any \(\tau\). This means that the left (equivalently, right) ratio of these quantities has the form \(1 + \sum_{k>0} (\text{degree } k) O(z \log z)^{f(k)}\) where \(f(k) \geq 0\).

We now relate \(F(z|\tau)\) and \(F_0(z|\tau)\). Let \(F(\tau) = F^{(1)}(z|\tau)\) for any \(z\) be the solution of the KZB system for \(n = 1\), such that \(F(\tau) \simeq \exp(-\frac{\pi^2}{2\tau^2} \Delta)\) as \(\tau \to i\infty\) (meaning that the left, or equivalently right, ratio of these quantities has the form \(1 + \sum_{k>0} (\text{degree } k) O(\tau^{f(k)}) e^{2\pi i \tau}\), where \(f(k) \geq 0\).

**Lemma 4.10.** We have
\[
F(z|\tau) = F_0(z|\tau) \exp(-\frac{a_0}{2\pi i} (\int_0^\tau E_2 + C) t)[F(\tau)], \quad \text{where} \quad C = \tau + \mathcal{O}(e^{2\pi i \tau}).
\]

**Proof of Lemma.** \(F(z|\tau) = F_0(z|\tau) X(\tau)\), where \(X : S \to G_2\) is a map. We have \(g(z, ad x)(t) = a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)(ad x)^{2k}(t) + O(z)\) when \(z \to 0^+\) and for any \(\tau\), so (31) is written as
\[
2\pi i \partial_t F(z|\tau) + (\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}] + O(z)) F(z|\tau) = 0
\]
where \(O(z)\) has degree \(> 0\). Since \(\Delta_0, t\) and the \([\delta_{2k}]\) all commute with \(t\), the ratio \(F_0(z|\tau)^{-1} F(z|\tau)\) satisfies
\[
2\pi i \partial_t [F_0^{-1}(z|\tau)](\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}] + \sum_{k>0} (\text{degree } k) O(z \log z)^{h(k)}) (F_0^{-1} F(z|\tau)) = 0
\]
where \(h(k) \geq 0\). Since \(F_0(z|\tau)^{-1} F(z|\tau) = X(\tau)\) is in fact independent on \(z\), we have
\[
2\pi i \partial_t [X(\tau)] + (\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}]) [X(\tau)] = 0,
\]
which implies that \(X(\tau) = \exp(-\frac{a_0}{2\pi i} (\int_0^\tau E_2 + C) t)[F(\tau)] X_0\), where \(X_0\) is a suitable element in \(G_2\). The asymptotic behavior of \(F(z|\tau)\) when \(\tau \to i\infty\) and \(z \to 0^+\) then implies \(X_0 = 1\).

\[\square\]
End of proof of Proposition. We then have $F(z|\tau) = F_0(z|\tau) X(\tau)$, where $X(\tau) \approx \exp(-\frac{\tau}{2\pi i}([\Delta]+a_0t))$ as $\tau \to i\infty$, where this means that the left ratio (equivalently, the right ratio) of these quantities has the form $1 + \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu(k)}e^{2\pi i \tau})$, where $x(k) \geq 0$.

If we set $u := e^{2\pi iz}$, then (30) is rewritten as

$$\partial_u \tilde{F}(u|\tau) = (\tilde{g}/u + t/(u - 1) + O(e^{2\pi i \tau})) \tilde{F}(u|\tau),$$

where $\tilde{F}(u|\tau) = F(z|\tau)$.

Let $D' := \{u||u| \leq 1\} - [0,1]$ be the complement of the unit interval in the unit disc. Then we have a bijection $\{(z,\tau) | \tau \in i\mathbb{R}^2_+, z = a + \tau b, a \in [0,1], b \geq 0\} \to D' \times i\mathbb{R}^2_+$, given by $(z,\tau) \mapsto (u,\tau) := (e^{2\pi iz}, \tau)$.

Let $\tilde{F}_a, \tilde{F}_f$ be the solutions of (33) in $D' \times i\mathbb{R}^2_+$, such that $\tilde{F}_a(u|\tau) \simeq ((u - 1)/(2\pi i))^{t}$ when $u = 1 + 10^t$, and for any $\tau$, and $\tilde{F}_f(u|\tau) \simeq e^{i\pi/2}((1 - u)/(2\pi i))^{t}$ when $u = 1 - 10^t$, for any $\tau$.

Then one checks that $F_0(z|\tau) = \tilde{F}_a(e^{2\pi iz}|\tau), F_0(z - 1|\tau) = \tilde{F}_f(e^{2\pi iz}|\tau)$ when $(z,\tau) \in \{(z,\tau) | \tau \in i\mathbb{R}^2_+, z = a + \tau b, a \in [0,1], b \geq 0\}$.

We then define $\tilde{F}_0, ..., \tilde{F}_e$ as the solutions of (33) in $D' \times i\mathbb{R}^2_+$, such that $\tilde{F}_0(u|\tau) \simeq (1 - u)^{t}$ as $u = 1 - 10^t$, $\delta(u) > 0$ for any $\tau$, $\tilde{F}_e(u|\tau) \simeq u^t$ as $u \to 0^+$, $\delta(u) > 0$ for any $\tau$, $\tilde{F}_d(u|\tau) \simeq u^{i\pi/2}$ as $u \to 0^+$, $\delta(u) < 0$ for any $\tau$, $\tilde{F}_c(u|\tau) \simeq u^{i\pi/2}$ as $u \to 0^+$, $\delta(u) < 0$ for any $\tau$.

Then $\tilde{F}_0 = \tilde{F}_b(-2\pi i)^t, \tilde{F}_c(-\tau) = \tilde{F}_a(\tau) | \Phi(\tilde{y}, t) + O(e^{2\pi i \tau})|, \tilde{F}_d(-\tau) = \tilde{F}_a(\tau) e^{2\pi i \tau}, \tilde{F}_e(-\tau) = \tilde{F}_a(\tau) e^{2\pi i \tau} | \Phi(\tilde{y}, t) - O(e^{2\pi i \tau})|, \tilde{F}_f = \tilde{F}_a(i/2\pi)^t$.

So $\tilde{F}_y(-\tau) = \tilde{F}_a(-\tau) (-2\pi i)^t \Phi(\tilde{y}, t) e^{2\pi i \tau} \Phi(\tilde{y}, t)^{-1} (i/2\pi)^t + O(e^{2\pi i \tau})$. It follows that $F_0(z + 1|\tau) = F_0(z|\tau) A(\tau)$, where

$$A(\tau) = (-2\pi i)^t \Phi(\tilde{y}, t) e^{2\pi i \tau} \Phi(\tilde{y}, t)^{-1} (i/2\pi)^t + O(e^{2\pi i \tau}).$$

Now

$$\tilde{A} = \tilde{F}(z|\tau)^{-1} (z + 1|\tau) = X(\tau)^{-1} A(\tau) X(\tau) = (1 + \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu(k)}e^{2\pi i \tau}))^{-1} \exp(\frac{\tau}{2\pi i}([\Delta]+a_0t)) \exp(-\frac{\tau}{2\pi i}([\Delta]+a_0t)) \exp(-\tau \text{ ad}(\frac{[\Delta]+a_0t}{2\pi i}))(O(e^{2\pi i \tau})) = \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu_1(k)}e^{2\pi i \tau}).$$

As we have seen, $[\Delta] + a_0t$ commutes with $\tilde{y}$ and $t$; on the other hand,

$$\exp(\frac{\tau}{2\pi i}([\Delta]+a_0t)) O(e^{2\pi i \tau}) \exp(-\frac{\tau}{2\pi i}([\Delta]+a_0t)) = \exp(\tau \text{ ad}(\frac{[\Delta]+a_0t}{2\pi i})) O(e^{2\pi i \tau}) = \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu_1(k)}e^{2\pi i \tau})$$

where $n_1(k) \geq 0$, as $[\Delta] + a_0t$ is a sum of terms of positive degree and of $\Delta_0$, which is locally ad-nilpotent.

Then

$$\tilde{A} = (1 + \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu(k)}e^{2\pi i \tau}))^{-1}((-2\pi i)^t \Phi(\tilde{y}, t) e^{2\pi i \tau} \Phi(\tilde{y}, t)^{-1} (i/2\pi)^t$$

$$+ \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu_1(k)}e^{2\pi i \tau})) (1 + \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu_2(k)}e^{2\pi i \tau})).$$

It follows that

$$\tilde{A} = (-2\pi i)^t \Phi(\tilde{y}, t) e^{2\pi i \tau} \Phi(\tilde{y}, t)^{-1} (i/2\pi)^t + \sum_{k \geq 0} (\text{degree } k) O(\tau^{\nu_2(k)}e^{2\pi i \tau}).$$
where \( n_2(k) \geq 0 \), which implies the first formula for \( \tilde{A} \). The second formula either follows from the first one by using the hexagon identity, or can be obtained repeating the above argument using a path \( 1 \to +\infty \to 1 \), winding around 1 and \( \infty \).

We now prove:

**Theorem 4.11.**

\[
\tilde{B} = (2\pi i)^2 \Phi(-\tilde{y} - t, t) e^{2\pi i x} \Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}.
\]

**Proof.** We first define \( F_0(z|\tau) \) as the solution in \( V := \{ a + br | a \in \mathbb{Z}, b \in \mathbb{R} \} \) of (30) such that \( F_0(z|\tau) \sim z^t \) as \( z \to 0^+ \). Then there exists \( B(\tau) \) such that \( e^{2\pi i x} F_0(z + \tau|\tau) = F_0(z|\tau) B(\tau) \). We compute the asymptotics of \( B(\tau) \) as \( \tau \to i \infty \).

We define four asymptotic zones (\( z \) is assumed to remain on the segment \( [0, \tau] \), and \( \tau \) in the line \( i \mathbb{R}_+ \)): (1) \( z \ll 1 < \tau \), (2) \( 1 \ll z \ll \tau \), (3) \( 1 \ll \tau - z \ll 1 \), (4) \( \tau - z \ll 1 \).

In the transition (1)-(2), the system takes the form (32), or if we set \( u := e^{2\pi i z} \), (33).

In the transition (3)-(4), \( G(z'|\tau) := e^{2\pi i x} F(\tau + z'|\tau) \) satisfies (30), so \( G(u'|\tau) = e^{2\pi i x} F(\tau + z'|\tau) \) satisfies (33), where \( u' = e^{2\pi i z'} \).

We now compute the form of the system in the transition (2)-(3). We first prove:

**Lemma 4.12.** Set \( u := e^{2\pi i z} \), \( v := e^{2\pi i (\tau - z)} \). When \( 0 < \Im(z) < \Im(\tau) \), we have \( |u| < 1 \), \( |v| < 1 \). When \( k \geq 0 \), \( (\theta'/(\theta))(z|\tau) = (\pi i)^k + \sum_{s,t \geq 0, s + t > 0} a_{st} u^s v^t \), where the sum in the r.h.s. is convergent in the domain \( |u| < 1 \), \( |v| < 1 \).

**Proof.** This is clear if \( k = 0 \). Set \( q = uv = e^{2\pi i r} \). We have \( \theta(z|\tau) = u^{1/2} \prod_{s>0}(1 - q^s u) \cdot (2\pi i)^{-1} \prod_{s>0}(1 - q^s)^{-2} \cdot \), so

\[
(\theta'/\theta)(z|\tau) = \pi i - 2\pi i \sum_{s>0} v^s u/(1 - q^s u) + 2\pi i \sum_{s>0} q^s u/(1 - q^s u) = \pi i + \sum_{s,t > 0} a_{st} u^s v^t,
\]

where \( a_{st} = 2\pi i \) if \( (s, t) = (k, r, r + 1), k \geq 0, r \geq 0 \), and \( a_{st} = -2\pi i \) if \( (s, t) = (k + 1, r), k > 0, r \geq 0 \). One checks that this series is convergent in the domain \( |u| < 1 \), \( |v| < 1 \). This proves the lemma for \( k = 1 \).

We then prove the remaining cases by induction, using

\[
\frac{\theta^{(k+1)}}{\theta}(z|\tau) = \frac{\theta^{(k)}}{\theta}(z|\tau) \frac{\theta'}{\theta}(z|\tau) = \frac{\partial}{\partial z} \frac{\theta^{(k)}}{\theta}(z|\tau).
\]

Using the expansion

\[
\frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)} = \frac{x}{\theta(x|\tau)} \sum_{k \geq 0} (\theta^{(k)}/\theta)(z|\tau) \frac{x^k}{k!}
\]

\[
= \frac{\pi x}{\sin(\pi x)} (1 + \sum_{n \geq 0} q^n P_n(x)) \left( \sum_{k \geq 0} (-\pi i)^k + \sum_{s + t > 0} a_{st}^{(k)} u^s v^t \frac{x^k}{k!} \right)
\]

\[
= \frac{\pi x}{\sin(\pi x)} e^{-izx} + \sum_{s + t > 0} a_{st}(x) u^s v^t = \frac{2\pi i x}{e^{2\pi x} - 1} + \sum_{s + t > 0} a_{st}(x) u^s v^t,
\]

the form of the system in the transition (2)-(3) is

\[
\partial_z F(z|\tau) = \left( -\frac{2\pi i x ad x}{e^{2\pi x} - 1} (y) + \sum_{s,t | s + t > 0} a_{st} u^s v^t \right) F(z|\tau)
\]

\[
= (2\pi i y + \sum_{s,t | s + t > 0} a_{st} u^s v^t) F(z|\tau),
\]

(34)
where each homogeneous part of $\sum_{s,t} a_{st} u^s v^t$ converges for $|u| < 1, |v| < 1$.

**Lemma 4.13.** There exists a solution $F_c(z|\tau)$ of (34) defined for $0 < \Im(z) < \Im(\tau)$, such that

$$F_c(z|\tau) = u^\theta \left( 1 + \sum_{k > 0} \sum_{s \leq s(k)} \log(u)^s f_{ks}(u, v) \right)$$

($\log u = i\pi z, u^\theta = e^{2\pi i \theta}$), where $f_{ks}(u, v)$ is an analytic function taking its values in the homogeneous part of the algebra of degree $k$, convergent for $|u| < 1$ and $|v| < 1$, and vanishing at $(0, 0)$. This function is uniquely defined up to right multiplication by an analytic function of the form $1 + \sum_{k > 0} a_k(q)$ (recall that $q = uv$), where $a_k(q)$ is an analytic function on $\{q||q| < 1\}$, vanishing at $q = 0$, with values in the degree $k$ part of the algebra.

**Proof of Lemma.** We set $G(z|\tau) := u^{-\theta} F(z|\tau)$, so $G(z|\tau)$ should satisfy

$$\partial_{z} G(z|\tau) = \exp(- \text{ad}(\tilde{\gamma}) \log u) \{ \sum_{s+t > 0} a_{st} u^s v^t \} G(z|\tau),$$

which has the general form

$$\partial_{z} G(z|\tau) = \left( \sum_{k > 0} \sum_{s \leq s(k)} \log(u)^s a_{ks}(u, v) \right) G(z|\tau),$$

where $a_{ks}(u, v)$ is analytic in $|u| < 1, |v| < 1$ and vanishes at $(0, 0)$. We show that this system admits a solution of the form $1 + \sum_{k > 0} \sum_{s \leq s(k)} \log(u)^s f_{ks}(u, v)$, with $f_{ks}(u, v)$ analytic in $|u| < 1, |v| < 1$, in the degree $k$ part of the algebra, vanishing at $(0, 0)$ for $s \neq 0$. For this, we solve inductively (in $k$) the system of equations

$$\partial_{z} \left( \sum_{s} (\log u)^s f_{ks}(u, v) \right) = \sum_{s', s'', k', k'' | k' + k'' = k} (\log u)^{s' + s''} a_{k' k''}(u, v) f_{k' k''}(u, v).$$  \hspace{1cm} (35)

Let $O$ be the ring of analytic functions on $\{(u, v)||u| < 1, |v| < 1\}$ (with values in a finite dimensional vector space) and $m \subset O$ be the subset of functions vanishing at $(0, 0)$. We have an injection $O[X] \to \{ \text{analytic functions in } (u, v), |u| < 1, |v| < 1, u \notin \mathbb{R}_{-} \}$, given by $f(u, v) X^k \mapsto (\log u)^k f(u, v)$. The endomorphism $\frac{\partial}{\partial u} = 2\pi i (u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v})$ then corresponds to the endomorphism of $O[X]$ given by $2\pi i (\frac{\partial}{\partial X} u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v})$. It is surjective, and restricts to a surjective endomorphism of $m[X]$. The latter surjectivity implies that equation (35) can be solved.

Let us show that the solution $G(z|\tau)$ is unique up to right multiplication by functions of $q$ like in the lemma. The ratio of two solutions is of the form $1 + \sum_{k > 0} \sum_{s \leq s(k)} \log(u)^s f_{ks}(u, v)$ and is killed by $\partial_z$. Now the kernel of the endomorphism of $m[X]$ given by $2\pi i (\frac{\partial}{\partial X} u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v})$ is $m^*(m_1)$, where $m^*(m_1) \subset m$ is the set of all functions of the form $a(uv)$, where $a$ is an analytic function on $\{q||q| < 1\}$ vanishing at $0$. This implies that the ratio of two solutions is as above.

**End of proof of Theorem.** Similarly, there exists a solution $F_d(z|\tau)$ of (34) defined in the same domain, such that

$$F_d(z|\tau) = v^{-\theta} \left( 1 + \sum_{k > 0} \sum_{s \leq s(k)} \log(v)^s g_{ks}(u, v) \right),$$

where $b_{ks}(u, v)$ is as above (and $\log v = i\pi (\tau - z), v^{-\theta} = \exp(2\pi i (z - \tau) \tilde{\gamma})$). $F_c(z|\tau)$ is defined up to right multiplication by a function of $q$ as above.

We now study the ratio $F_c(z|\tau)^{-1} F_d(z|\tau)$. This is a function of $\tau$ only, and it has the form

$$q^{-\theta} \left( 1 + \sum_{k > 0} \sum_{s \leq s(k), t \leq t(k)} (\log u)^s (\log v)^t a_{kst}(u, v) \right)$$
where $a_{ks}(u, v) \in \mathfrak{m}$ (as $e^{-\tilde{y}} \left( 1 + \sum_{k>0} \sum_{s \leq s(k)} (\log u)^s c_{ks}(u, v) \right) v^\tilde{y}$ has the form $1 + \sum_{k>0} \sum_{s \leq s(k)} (\log u)^s \delta_k(1)$, where $\delta_k(1)$ vanishes at $u = 0$, then this ratio can be rewritten $q^{-\tilde{y}} \left( 1 + \sum_{k>0} \sum_{s \leq s(k), t \leq t(k)} (\log q)^s (\log q)^t \delta_k(1) \right)$, where $b_{ks}(u, v) \in \mathfrak{m}$, and since the product of this ratio with $q^{\tilde{y}}$ is killed by $\partial_z$ (which identifies with the endomorphism $2\pi i \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \right)$ of $\mathcal{O}[X]$), the ratio is in fact of the form

$$F_{c^{-1}}F_{d}(z|\tau) = q^{\tilde{y}} \left( 1 + \sum_{k>0} \sum_{s \leq s(k)} (\log q)^s a_{ks}(q) \right),$$

where $a_{ks}$ is analytic in $\{ |q| |q| < 1 \}$, vanishing at $q = 0$.

It follows that

$$F_{c^{-1}}F_{d}(z|\tau) = e^{-2\pi i \tau \tilde{y}} \left( 1 + \sum_{k>0} (\text{degree } k) \Phi(e^{2\pi i \tau}) \right).$$

(36)

In addition to $F_c$ and $F_d$, which have prescribed behaviors in zones (2) and (3), we define solutions of (30) in $V$ by prescribing behaviors in the remaining asymptotic zones: $F_a(z|\tau) \simeq z^t$ when $z \to 0^+$ for any $\tau$; $F_b(z|\tau) \simeq (2\pi z / i)^t$ when $z \to 10^+$ for any $\tau$ (in particular in zone (1)); $e^{2\pi i x} F_c(z|\tau) \simeq (2\pi (\tau - z) / i)^t$ when $z = \tau - 10^+$ for any $\tau$; $e^{2\pi i x} F_f(z|\tau) \simeq (z - \tau)^t$ when $z = \tau + 0^+$ for any $\tau$ (in particular in zone (4)).

Then $F_{0}(z|\tau) = F_{a}(z|\tau)$, and $e^{-2\pi i x} F_{b}(z-\tau|\tau) = F_{f}(z|\tau)$. We have $F_{b} = F_{a}(2\pi / i)^t$, $F_{f} = F_{c}(2\pi i)^{-t}$.

Let us now compute the ratio between $F_{0}$ and $F_{c}$. Recall that $u = e^{2\pi iz}$, $v = e^{2\pi i(\tau-z)}$. Set $\tilde{F}(u, v) := F(z|\tau)$. Using the expansion of $\partial(\theta(z|\tau))$, one shows that (30) has the form

$$\partial_u \tilde{F}(u, v) = \left( \frac{A(u, v)}{u} + \frac{B(u, v)}{u-1} \right) \tilde{F}(u, v),$$

where $A(u, v)$ is holomorphic in the region $|v| < 1/2$, $|u| < 2$, and $A(u, 0) = \tilde{y}$, $B(u, 0) = t$. We have $\tilde{F}_{b}(u, v) = (1 - u)^t \left( 1 + \sum_{k} \sum_{s \leq s(k)} \log(1 - u)^k b_{ks}(u, v) \right)$ and $\tilde{F}_{a}(u, v) = u^t \left( 1 + \sum_{k} \sum_{s \leq s(k)} \log(u)^s a_{ks}(u, v) \right)$, with $a_{ks}, b_{ks}$ analytic, and $a_{ks}(0, v) = b_{ks}(1, v) = 0$. The ratio $F_{b}^{-1}F_{c}$ is an analytic function of $q$ only, which coincides with $\Phi(\tilde{y}, t)$ for $q = 0$, so it has the form $\Phi(\tilde{y}, t) + \sum_{k>0} a_{k}(q)$, where $a_{k}(q)$ has degree $k$, is analytic in the neighborhood of $q = 0$ and vanishes at $q = 0$. Therefore

$$F_{c}(z|\tau) = F_{b}(z|\tau) (\Phi(\tilde{y}, t) + O(e^{2\pi i \tau})).$$

In the same way, one proves that

$$F_{c}(z|\tau) = F_{d}(e^{-2\pi i x} \Phi(-\tilde{y} - t, t)^{-1} + O(e^{2\pi i \tau})).$$

Indeed, let us set $G_{d}(u', v') := e^{2\pi i x} F_{d}(\tau + z'|\tau)$, $G_{e}(u', v') := e^{2\pi i x} F_{e}(\tau + z'|\tau)$, where $u' = e^{2\pi i (\tau + z')}$, $v' = e^{-2\pi i z'}$, then $G_{d}(u', v') \simeq (v')^{-\tilde{y}-t} e^{2\pi i x}$ as $u' \to 0^+$ and $G_{e}(u', v') \simeq (1 - v')^{-t}$ as $v' \to 1^-$. Therefore $F_{d} = F_{e} \Phi(-\tilde{y} - t, t) e^{2\pi i x} + O(u')$. Combining these results, we get:


$$B(\tau) \simeq (2\pi i)^t \Phi(-\tilde{y} - t, t) e^{2\pi i x} e^{2\pi \tau \tilde{y}} \Phi(\tilde{y}, t)^{-1} (2\pi i)^{-t},$$

in the sense that the left (equivalently, right) ratio of these quantities has the form $1 + \sum_{k>0} (\text{degree } k) \Phi(e^{2\pi i \tau})$ for $n(k) \geq 0$.

Recall that we have proved:

$$F(z|\tau) = F_{0}(z|\tau) \exp(-\frac{\alpha_0}{2\pi i} \int_{1}^{\tau} E_{2} + C) t |F(\tau)|,$$

where $C$ is such that $\int_{1}^{\tau} E_{2} + C = \tau + O(e^{2\pi i \tau})$.

Set $X(\tau) := \exp(-\frac{\alpha_0}{2\pi i} \int_{1}^{\tau} E_{2} + C) t |F(\tau)|.$
When $\tau \to i\infty$, $X(\tau) = \exp(-\frac{i\tau}{2\pi}([\Delta] + a_0 t))(1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau}))$.

Then

$$\tilde{B} = F(z|\tau)^{-1}e^{2\pi i z}F(z + \tau|\tau) = X(\tau)^{-1}B(\tau)X(\tau)$$

$$= \text{Ad}\left((1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau})^{-1}\exp(\frac{\tau}{2\pi i}([\Delta] + a_0 t)))\right)$$

$$\left((2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}(1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau}))\right),$$

where $\text{Ad}(u)(x) = uxu^{-1}$.

$[\Delta] + a_0 t$ commutes with $\tilde{y}$ and $t$; assume for a moment that $\text{Ad}(\exp(\frac{\tau}{2\pi i}([\Delta] + a_0 t)))(e^{2\pi i x}e^{2\pi i \tau}) = e^{2\pi i x}$ (Lemma 4.15 below), then

$$\text{Ad}(\exp(\frac{\tau}{2\pi i}([\Delta] + a_0 t)))(2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}.$$  

On the other hand, $\text{Ad}(\exp(\frac{\tau}{2\pi i}([\Delta] + a_0 t)))(1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau}))$ has the form $1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau})$, where $n'(k) \geq 0$. It follows that

$$\tilde{B} = \text{Ad}\left((1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau}))\right)$$

$$\left((2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}(1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau}))\right);$$

now

$$\text{Ad}\left((2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}\right)^{-1}(1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau}))$$

$$= 1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau}),$$

so

$$\tilde{B} = \left((2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}\right)(1 + \sum_{k > 0}(\text{degree } k)O(\tau^f(k)e^{2\pi i \tau}))$$

$$(1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau}))$$

$$= \left((2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t}\right)(1 + \sum_{k > 0}(\text{degree } k)O(\tau^n(k)e^{2\pi i \tau}))$$

for $n''(k) \geq 0$. Since $\tilde{B}$ is constant w.r.t. $\tau$, this implies

$$\tilde{B} = (2\pi i)^t\Phi(-\tilde{y} - t, t)e^{2\pi i x}e^{2\pi i \tau}\Phi(\tilde{y}, t)^{-1}(2\pi i)^{-t},$$

as claimed.

We now prove the conjugation used above.

**Lemma 4.15.** For any $\tau \in \mathbb{C}$, we have

$$e^{\frac{\tau}{2\pi i}([\Delta] + a_0 t)}e^{2\pi i x}e^{-\frac{\tau}{2\pi i}([\Delta] + a_0 t)}e^{2\pi i \tau}\tilde{y} = e^{2\pi i x}.$$

**Proof.** We have $[\Delta] + a_0 t = \Delta_0 + \sum_{k \geq 0} a_{2k}(\delta_{2k} + (\text{ad } x)^{2k}(t))$ (where $\delta_0 = 0$), so $[[\Delta] + a_0 t, x] = y - \sum_{k \geq 0} a_{2k}(\text{ad } x)^{2k+1}(t)$. Recall that

$$\sum_{k \geq 0} a_{2k}u^{2k} = \frac{\pi^2}{\sin^2(\pi u)} - \frac{1}{u^2},$$

and

$$e^{\frac{\tau}{2\pi i}([\Delta] + a_0 t)}e^{2\pi i x}e^{-\frac{\tau}{2\pi i}([\Delta] + a_0 t)}e^{2\pi i \tau}\tilde{y} = e^{2\pi i x}.$$
then \([[\Delta] + a_0 t, x] = y - (\text{ad} x)(\frac{\pi^2}{\sin^2(\pi \text{ad} x)} - \frac{1}{(\text{ad} x)^2})(t)\). So
\[
e^{-2\pi i x} \left( \frac{1}{2\pi i} ([\Delta] + a_0 t) \right) e^{2\pi i x} = \frac{1}{2\pi i} \left( (\Delta] + a_0 t) + \frac{e^{-2\pi i \text{ad} x} - 1}{\text{ad} x} \left( [x, \frac{1}{2\pi i} ([\Delta] + a_0 t)] \right) \right) - \frac{\pi^2}{\sin^2(\pi \text{ad} x)} - \frac{1}{(\text{ad} x)^2} \right)(t) = 0.
\]

We have
\[
\begin{multline*}
- \frac{1}{2\pi i} e^{-2\pi i \text{ad} x} - \frac{1}{(\text{ad} x)^2} \left( y - (\text{ad} x)(\frac{\pi^2}{\sin^2(\pi \text{ad} x)} - \frac{1}{(\text{ad} x)^2})(t) \right) = -2\pi i \hat{y},
\end{multline*}
\]
therefore we get
\[
e^{-2\pi i x} \left( \frac{1}{2\pi i} ([\Delta] + a_0 t) \right) e^{2\pi i x} = \frac{1}{2\pi i} \left( [\Delta] + a_0 t \right) - 2\pi i \hat{y}.
\]
Multiplying by \(\tau\), taking the exponential, and using the fact that \([\Delta] + a_0 t\) commutes with \(\hat{y}\), we get
\[
e^{-2\pi i x} e^{-2\pi i ([\Delta] + a_0 t)} e^{2\pi i x} = e^{-2\pi i \tau \hat{y}},
\]
which proves the lemma. \(\square\)

This ends the proof of Theorem 4.11. \(\square\)

5. Construction of morphisms \(\Gamma_{1,[n]} \rightarrow G_n \rtimes S_n\)

In this section, we fix a field \(k\) of characteristic zero. We denote the algebras \(\mathfrak{t}_{1,n}\), \(\mathfrak{t}_n\) simply by \(t_{1,n}\), \(t_n\). The above group \(G_n\) is the set of \(C\)-points of a group scheme defined over \(\mathbb{Q}\), and we now again denote by \(G_n\) the set of its \(k\)-points.

5.1. Construction of morphisms \(\Gamma_{1,[n]} \rightarrow G_n \rtimes S_n\) from a 5-uple \((\Phi_\lambda, \tilde{A}, \tilde{B}, \tilde{\Theta}, \tilde{\Psi})\). Let \(\Phi_\lambda\) be a \(\lambda\)-associtor defined over \(k\). This means that \(\Phi_\lambda \in \exp(t_3)\) (the Lie algebras are now over \(k\)),
\[
\Phi_\lambda^{3,2,1} = \Phi_\lambda^{-1}, \quad \Phi_\lambda^{2,3,4} \Phi_\lambda^{1,23,4} \Phi_\lambda^{1,2,3} = \Phi_\lambda^{1,2,3,4} \Phi_\lambda^{12,3,4},
\]
\[
e^{\lambda \pi^2/2} \Phi_\lambda^{2,3,1} e^{\lambda \pi^2/2} \Phi_\lambda^{12,3,2} e^{\lambda \pi^2/2} \Phi_\lambda^{3,1,2} = e^{\lambda (t_{12} + t_{23} + t_{13})/2}.
\]
E.g., the KZ associator is a \(2\pi i\)-associtor over \(C\).

Proposition 5.1. If \(\tilde{\Theta}, \tilde{\Psi} \in G_1\), and \(\tilde{A}, \tilde{B} \in \exp(\mathfrak{t}_{1,2})\) satisfy: the “\(T_{1,1}\) identities” (27), the “\(T_{1,2}\) identities” (28), (29), and the “\(T_{1,3}\) identities” (23), (22), (26) (with \(2\pi i\) replaced by \(\lambda\)), as well as \(\tilde{A}^{\lambda,1} = \tilde{A}^{1,\lambda} = \tilde{B}^{\lambda,1} = \tilde{B}^{1,\lambda} = 1\), then one defines a morphism \(\Gamma_{1,[n]} \rightarrow G_n \rtimes S_n\) by
\[
\Theta \mapsto [\tilde{\Theta}] e^{\pi i \sum_{i,j} \tilde{t}_{i,j}} \quad \Psi \mapsto [\tilde{\Psi}] e^{\pi i \sum_{i,j} \tilde{t}_{i,j}} \quad \sigma_i \mapsto \{\Phi_\lambda^{1-\tilde{A}_{i-\tilde{t}_{i+1}}(i, i+1)}\}_{i-\tilde{t}_{i+1}}^{-1} e^{\lambda \tilde{t}_{i+1}/2(i, i+1)} \{\Phi_\lambda^{1-\tilde{A}_{i-\tilde{t}_{i+1}}(i, i+1)}\}_{i-\tilde{t}_{i+1}}^{-1},
\]
\[
C_{j,k} \mapsto \{\Phi_\lambda^{1-\tilde{A}_{j-\tilde{t}_{k+1}}(i, i+1)}...\Phi_\lambda^{1-\tilde{A}_{j-\tilde{t}_{k+1}}(i, i+1)}\}_{i-\tilde{t}_{k+1}}^{-1} e^{\lambda \tilde{t}_{i-\tilde{t}_{k+1}}(i, i+1)}...\Phi_\lambda^{1-\tilde{A}_{j-\tilde{t}_{k+1}}(i, i+1)}\}_{i-\tilde{t}_{k+1}}^{-1},
\]
\[
A_{j} \mapsto \{\Phi_\lambda\}_{i-\tilde{t}_{j+1}}^{-1} \tilde{A}_{j-\tilde{t}_{j+1}}(i, i+1) \{\Phi_\lambda\}_{i-\tilde{t}_{j+1}}^{-1}, \quad B_i \mapsto \{\Phi_\lambda\}_{i-\tilde{t}_{j+1}}^{-1} \tilde{B}_{j-\tilde{t}_{j+1}}(i, i+1) \{\Phi_\lambda\}_{i-\tilde{t}_{j+1}}^{-1}.
\]

According to Section 4.4, the representations \(\gamma_n\) are obtained by the procedure described in this proposition from the KZ associator, \(\tilde{\Theta}, \tilde{\Psi}\) arising from \(\gamma_1\), and \(\tilde{A}, \tilde{B}\) arising from \(\gamma_2\).

Note also that the analogue of (22) is equivalent to the pair of equations
\[
e^{\lambda \tilde{t}_{12}/2} \tilde{A}^{3,1,2} e^{\lambda \tilde{t}_{12}/2} \tilde{A} = 1, \quad (e^{\lambda \tilde{t}_{12}/2} \tilde{A})^{3,1,2} (e^{\lambda \tilde{t}_{12}/2} \tilde{A})^{2,3,1} (e^{\lambda \tilde{t}_{12}/2} \tilde{A})^{1,2,3} = 1,
\]
and similarly (23) is equivalent to the same equations, with \(\tilde{A}, \lambda\) replaced by \(\tilde{B}, -\lambda\).
Remark 5.2. One can prove that it $\Phi_\lambda$ satisfies only the pentagon equation and $\Theta, \Psi, \tilde{A}, \tilde{B}$ satisfy the the "\(\Gamma_{1,1}\) identities" (27), the "\(\Gamma_{1,2}\) identities" (28), (29), and the "\(\Gamma_{1,3}\) identities" (24), (26), then the above formulas (removing $\sigma_1$) define a morphism $\Gamma_{1,n} \to G_n$. In the same way, if $\Phi_\lambda$ satisfies all the associator conditions and $\tilde{A}, \tilde{B}$ satisfy the $\Gamma_{1,[3]}$ identities (22), (23), (26), then the above formulas (removing $\Theta, \Psi$) define a morphism $\tilde{\Gamma}_{1,n} \to \exp(\hat{u}_{1,n}) \times S_n$.

**Proof.** Let us prove that the identity $(A_i, A_j) = 1$ $(i < j)$ is preserved. Applying $x \mapsto x_i^{1-1}, i, j-1, j, n$ to the first identity of (24), we get

$$ (A^{1-1, i-1, i, j-1, j, n} \Phi_{\lambda, i, j, n}^{-1} A^{1-1, j, i, j-1, j, n}) = 1. $$

The pentagon identity implies

$$ \Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n} = (\Phi_{\lambda, i+1, j, n} \Phi_{\lambda, i, j-1, n}) \Phi_{\lambda, i, j, n} (\Phi_{\lambda, i, j-1, n} \Phi_{\lambda, i, j, n}^{-1}), $$

so the above identity is rewritten

$$ (\Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n}^{-1} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n}^{-1}) = 1. $$

Now $\Phi_{\lambda, i, j, n}$ \(\Phi_{\lambda, j, k, n}\) commute with $A^{1-1, i, j, n}$ and $\Phi_{\lambda, i, j, n}$ \(\Phi_{\lambda, j, k, n}\) commute with $\Phi_{\lambda, i, j, n}$ \(\Phi_{\lambda, j, k, n}\)$, which implies

$$ (\Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n}^{-1})^{-1} (\Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n}) = 1, $$

so that $(A_i, A_j) = 1$ is preserved. In the same way, one shows that $(B_i, B_j) = 1$ is preserved.

Let us show that $(B_k, A_k A_j^{-1}) = C_{jk}$ is preserved (if $j \leq k$).

$$ (\Phi_{\lambda, i, j, k, n}^{-1} B^{1-1, k, k, n} \Phi_{\lambda, k, n}^{-1} A^{1-1, k, k, n} \Phi_{\lambda, k, n}^{-1} (\Phi_{\lambda, i, j, n})^{-1} \Phi_{\lambda, j, k, n}) $$

$$ = (\Phi_{\lambda, i, j, n})^{-1} (A^{1-1, j, h, n} \Phi_{\lambda, j, k, n}^{-1} A^{1-1, h, j, n})^{-1} (\Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, j, k, n}^{-1})^{-1} (\Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, j, k, n}^{-1}) $$

$$ = (\Phi_{\lambda, i, j, n})^{-1} (A^{1-1, j, h, n} \Phi_{\lambda, j, k, n}^{-1} A^{1-1, h, j, n})^{-1} (\Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, j, k, n}^{-1}) $$

where the second identity uses (39) and the invariance of $\Phi_\lambda$, the third identity uses the fact that $\Phi_{\lambda, i, j, n} \Phi_{\lambda, j, k, n} \Phi_{\lambda, k, n} \Phi_{\lambda, n} \Phi_{\lambda, i, j, n}^{-1}$ commute with $A^{1-1, j, h, n}$ again by the invariance of $\Phi_\lambda$, and the last identity uses (26). So $(B_k, A_k A_j^{-1}) = C_{jk}$ is preserved. One shows similarly that

$$ (\Phi_{\lambda, i, j, k, n}^{-1} B^{1-1, k, k, n} \Phi_{\lambda, k, n}^{-1} A^{1-1, j, j, n})^{-1} (\Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, j, k, n}^{-1}) $$

$$ = (\Phi_{\lambda, i, j, n})^{-1} (A^{1-1, j, h, n} \Phi_{\lambda, j, k, n}^{-1} A^{1-1, h, j, n})^{-1} (\Phi_{\lambda, i, j, n})^{-1} (\Phi_{\lambda, j, k, n}^{-1}) $$

so that $(B_k B_j^{-1}, A_k) = C_{jk}$ is preserved.
Let us show that \((A_i, C_{jk}) = 1\) \((i \leq j \leq k)\) is preserved. We have

\[
\begin{aligned}
&\left(\Phi_{\lambda,i}^{-1} A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right. \\
&\left. \Phi_{\lambda,j}^{-1} \Phi_{\lambda,k}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \right) \left(\Phi_{\lambda,j}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \ldots \Phi_{\lambda,k-n}^{-1} \right) \\
&\quad= \Phi_{\lambda,i}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \ldots \Phi_{\lambda,k-n}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \\
&\quad= \Phi_{\lambda,i}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \ldots \Phi_{\lambda,k-n}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \\
&\quad= \Phi_{\lambda,i}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right) \ldots \Phi_{\lambda,k-n}^{-1} \left( A_{1\ldots-i, j-n} \Phi_{\lambda,i} \right)
\end{aligned}
\]

where the second equality follows from the generalized pentagon identity (39), the third equality follows from the fact that \(\Phi_{\lambda,i}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \) commute with \((e^{2\pi i \theta_{12}}) A_{1\ldots-i, j-n} \Phi_{\lambda,i} \ldots \Phi_{\lambda,k-n}^{-1} \), \(\Phi_{\lambda,j}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \), \(\Phi_{\lambda,j}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \), the fourth equality follows from the fact that \(\Phi_{\lambda,i}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \) commute with \(A_{1\ldots-i, j-n} \Phi_{\lambda,i} \ldots \Phi_{\lambda,k-n}^{-1} \) (as \(\Phi_{\lambda,i} \) is invariant), the last equality follows from the fact that \(\Phi_{\lambda,i}^{-1} \ldots \Phi_{\lambda,k-n}^{-1} \) commutes with \(A_{1\ldots-i, j-n} \Phi_{\lambda,i} \ldots \Phi_{\lambda,k-n}^{-1} \) (again as \(\Phi_{\lambda,i} \) is invariant) and with \((e^{2\pi i \theta_{12}}) A_{1\ldots-i, j-n} \Phi_{\lambda,i} \ldots \Phi_{\lambda,k-n}^{-1} \) (as \(\Phi_{\lambda,i} \) is invariant). Therefore \((A_i, C_{jk}) = 1\) is preserved. One shows similarly that \((B_i, C_{jk}) = 1\) \((i \leq j \leq k)\), \(X_{i+1} = \sigma_i X_i \sigma_i^{-1}\) and \(Y_{i+1} = \sigma_i Y_i \sigma_i^{-1}\) are preserved.

The fact that the relations \(\Theta_{A_i} \Theta^{-1} = B_i^{-1} A_i^{-1} B_i \) hold, \(\Theta B_i \Theta^{-1} = B_i^{-1} A_i^{-1} B_i \) are preserved follows from the identities (28), (29) and that we denote by \(x \mapsto [x]_n\) the morphism \(\Phi \to \Phi \times I_{1,n}\) defined above, then: (a) \(\Phi_i \) commutes with \(\sum_{i<j} t_{ij}\) and with the image of \(\Phi \to \Phi \times I_{1,n}, x \mapsto [x]_n\); (b) for \(x \in \Phi, y \in I_{1,2}\), we have \([[x], y]_{1 \ldots-i, 1 \ldots-j} = [[x], y]_{1 \ldots-i, 1 \ldots-j}\). Let us prove (a): the first part follows from the fact that \(\Phi\) commutes with \(t_{12} + t_{13} + t_{23}\); the second part follows from the fact that \(X, d, \Delta_0, \delta_{2n} + \sum_{k<l} (ad \bar{x}_k) \delta_{2n} (I_{k,l})\) commute with \(t_{ij}\) for any \(i < j\). Let us prove (b): the identity holds for \(x, x'\) whenever it holds for \(x\) and \(x'\), so it suffices to check it for \(x\) a generator of \(\Phi\). Let us denote \(x \in \Phi, y \in I_{1,2}\), then the identity holds because we have

\[
\begin{aligned}
&\delta_{2s} + (ad \bar{x}) \delta_{2s} (I_{12}) \bar{x}_{1 \ldots-i, 1 \ldots-j} = - (ad \bar{x}) \delta_{2s} (I_{12}) \\
&= - \sum_{u' \in I} (ad \bar{x}) \delta_{2s} (I_{12}) \sum_{1 \leq u \leq v \leq n} t_{uv} = \sum_{1 \leq u \leq v \leq n} (ad \bar{x}) \delta_{2s} (I_{12}),
\end{aligned}
\]
while
\[
\begin{align*}
\delta_{2s} + \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^{2s}(\bar{t}_{uv}), \sum_{u' = 1}^{i-1} \bar{x}_{u'} & = \left[ \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^{2s}(\bar{t}_{uv}), \sum_{u' = 1}^{i-1} \bar{x}_{u'} \right] \\
& = - \sum_{1 \leq u < i \leq v \leq n} (\text{ad} \, \bar{x}_u)^{2s+1}(\bar{t}_{uv})
\end{align*}
\]
where the first equality follows from the fact that \((\text{ad} \, \bar{x}_u)^{2s}(\bar{t}_{uv})\) commutes with \(\sum_{u' = 1}^{i-1} \bar{x}_{u'}\) whenever \(u < v < i\) or \(i \leq u < v\). If \(x = \delta_{2s}\) and \(y = \bar{x}_2\), then the identity follows because
\[
[\delta_{2s} + (\text{ad} \, \bar{x}_1)^{2s}(\bar{t}_{12}), x_1 + \bar{x}_2] = 0 \quad \text{and} \quad [\delta_{2s} + \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^{2s}(\bar{t}_{uv}), \sum_{u' = 1}^{n} \bar{x}_{u'}] = 0.
\]
If \(x = \delta_{2s}\) and \(y = \bar{y}_1\), then
\[
[\delta_{2s} + (\text{ad} \, \bar{x}_1)^{2s}(\bar{t}_{12}), \bar{y}_1] \cdots \cdots = \frac{1}{2} \sum_{p+q=2s-1} \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^p(\bar{t}_{uv}), \sum_{1 \leq u' < i \leq \bar{y}_1} (\text{ad} \, \bar{x}_u)^q(\bar{t}_{uv'})
\]
on the other hand,
\[
[\delta_{2s} + \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^{2s}(\bar{t}_{uv}), \bar{y}_1 + \cdots + \bar{y}_{i-1}]
\]
where the second equality follows from the fact that \([\text{ad} \, \bar{x}_u]^p(\bar{t}_{uv}), (\text{ad} \, \bar{x}_u)^q(\bar{t}_{uv})\]

Then
\[
[\delta_{2s} + (\text{ad} \, \bar{x}_1)^{2s}(\bar{t}_{12}), \bar{y}_1] \cdots \cdots = \frac{1}{2} \sum_{p+q=2s-1} \sum_{1 \leq u < i \leq \bar{y}_1} [\text{ad} \, \bar{x}_u]^p(\bar{t}_{uv}), (\text{ad} \, \bar{x}_u')^q(\bar{t}_{uv'})
\]

\[
\begin{align*}
\frac{1}{2} \sum_{p+q=2s-1} \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^p(\bar{t}_{uv}), (\text{ad} \, \bar{x}_u')^q(\bar{t}_{uv'}) & = \sum_{1 \leq u < v \leq n} [\text{ad} \, \bar{x}_u]^2(\bar{t}_{uv}), \bar{y}_i + \cdots + \bar{y}_{i-1} - \sum_{1 \leq u < v \leq n} [\text{ad} \, \bar{x}_u]^2(\bar{t}_{uv}), \bar{y}_{i} + \cdots + \bar{y}_{i-1} \\
& \quad + \frac{1}{2} \sum_{p+q=2s-1} \sum_{1 \leq u < v \leq n} (\text{ad} \, \bar{x}_u)^p(\bar{t}_{uv}), (\text{ad} \, \bar{x}_u')^q(\bar{t}_{uv'})
\end{align*}
\]
where the second equality follows from the centrality of \( ar{y}_1 + \cdots + \bar{y}_n \), the last equality follows for the fact that \( (\text{ad} \bar{x}_u)^p(\bar{t}_{uv}) \) and \( (\text{ad} \bar{x}_u')^q(\bar{t}_{uv'}) \) commute for \( u, v, u', v' \) all distinct. Since \( p + q \) is odd, it follows that

\[
[\delta_{2s} + (\text{ad} \bar{x}_1)^2(\bar{t}_{12}), \bar{y}_1]^{1, \ldots, i, \ldots, n = 1} = \sum_{1 \leq u < v \leq n}^{n} (\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] = \sum_{1 \leq u < v < i}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] + \sum_{i \leq u < v \leq n}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), (-\text{ad} \bar{x}_u)^q(\bar{t}_{uv})]
\]

and the second equality follows from the fact that \( \bar{x}_u \) and \( \bar{x}_u' \) commute for \( u, v, u', v' \) all distinct. Since \( p + q \) is odd, it follows that

\[
[\delta_{2s} + (\text{ad} \bar{x}_1)^2(\bar{t}_{12}), \bar{y}_1]^{1, \ldots, i, \ldots, n = 1} = \sum_{1 \leq u < v \leq n}^{n} (\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] = \sum_{1 \leq u < v < i}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] + \sum_{i \leq u < v \leq n}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), (-\text{ad} \bar{x}_u)^q(\bar{t}_{uv})]
\]

and the second equality follows from the fact that \( \bar{x}_u \) and \( \bar{x}_u' \) commute for \( u, v, u', v' \) all distinct. Since \( p + q \) is odd, it follows that

\[
[\delta_{2s} + (\text{ad} \bar{x}_1)^2(\bar{t}_{12}), \bar{y}_1]^{1, \ldots, i, \ldots, n = 1} = \sum_{1 \leq u < v \leq n}^{n} (\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] = \sum_{1 \leq u < v < i}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), \bar{y}_1 + \ldots + \bar{y}_{i-1}] + \sum_{i \leq u < v \leq n}^{n} [(\text{ad} \bar{x}_u)^2(\bar{t}_{uv}), (-\text{ad} \bar{x}_u)^q(\bar{t}_{uv})]
\]

and the second equality follows from the fact that \( \bar{x}_u \) and \( \bar{x}_u' \) commute for \( u, v, u', v' \) all distinct. Since \( p + q \) is odd, it follows that
So the product of the images of $C_{12} \ldots C_{n-1,n}$ is $e^{\sum_{i<j} t_{ij}}$.

The relation $(\Theta \Psi)^3 = C_{12} \ldots C_{n-1,n}$ is then preserved because

$$
(\Theta \Psi)^3 e^{2\pi i \sum_{i<j} t_{ij} \cdot [\Psi]} e^{2\pi i \sum_{i<j} t_{ij}} = (\Theta \Psi)^3 e^{2\pi i \sum_{i<j} t_{ij}} = e^{2\pi i \sum_{i<j} t_{ij}},
$$

where the first equality follows from the fact that $\sum_{i<j} t_{ij}$ commutes with the image of $G_1 \to G_n$, $g \mapsto [g]$, the second equality follows from the fact that $g \mapsto [g]$ is a group morphism and the last equality follows from (27). In the same way, one proves that $\Theta^4 = C_{12} \ldots C_{n-1,n}$, $\sigma_i^2 = C_{i,i+1} C_{i+1,i+2} C_{i,i+1}$ and $(\Theta, \sigma_i) = (\Psi, \sigma_i) = 1$ are preserved.

\[\square\]

5.2. Construction of morphisms $\overline{\mathfrak{b}}_{1,n} \to \exp(\widehat{\mathfrak{t}}_{1,n}) \rtimes S_n$ using an associator $\Phi_\lambda$. Let us keep the notation of the previous section. Set $a_{2n}(\lambda) := -(2n+1)B_{2n+2} \lambda^{2n+2}/(2n+2)!$, $\tilde{y}_\lambda := -\frac{\text{ad} \lambda}{\exp \lambda - 1}(y)$,

$$
\tilde{A}_\lambda := \Phi_\lambda(\tilde{y}_\lambda, t) e^{\alpha \lambda} \Phi_\lambda(\tilde{y}_\lambda, t)^{-1} = e^{-\lambda/2} \Phi_\lambda(-\tilde{y}_\lambda - t, t) e^{\lambda(\tilde{y}_\lambda + t)} \Phi_\lambda(-\tilde{y}_\lambda - t, t)^{-1} e^{-\lambda/2},
$$

$$
\tilde{B}_\lambda := e^{\lambda/2} \Phi_\lambda(-\tilde{y}_\lambda - t, t) e^{\alpha \lambda} \Phi_\lambda(\tilde{y}_\lambda, t)^{-1}
$$

(the identity in the definition of $A_\lambda$ follows from the hexagon relation).

**Proposition 5.3.** We have

$$
\tilde{A}_\lambda^{12,3} = e^{\frac{-\lambda}{2}} \{ \Phi_\lambda \}^{3,1,2} \tilde{A}_\lambda^{2,13} \{ \Phi_\lambda \}^{2,1,3} e^{\frac{-\lambda}{2}} \cdot \{ \Phi_\lambda \}^{3,2,1} \tilde{A}_\lambda^{1,23} \{ \Phi_\lambda \}^{1,2,3},
$$

$$
\tilde{B}_\lambda^{12,3} = e^{\frac{-\lambda}{2}} \{ \Phi_\lambda \}^{3,1,2} \tilde{B}_\lambda^{2,13} \{ \Phi_\lambda \}^{2,1,3} e^{\frac{-\lambda}{2}} \cdot \{ \Phi_\lambda \}^{3,2,1} \tilde{B}_\lambda^{1,23} \{ \Phi_\lambda \}^{1,2,3},
$$

so the formulas of Proposition 5.1 (restricted to the generators $A_1, B_1, \sigma_i, C_{jk}$) induce a morphism $\overline{\mathfrak{b}}_{1,n} \to \exp(\widehat{\mathfrak{t}}_{1,n}) \rtimes S_n$ (here $\widehat{\mathfrak{t}}_{1,n}$ is the degree completion of $\overline{\mathfrak{t}}_{1,n}$).

**Proof.** In this proof, we shift the indices of the generators of $t_{n+1}$ by 1, so these generators are now $t_{ij}$, $i \neq j \in \{0, \ldots, n\}$ (recall that $t_{n+1} = \overline{t}_{n+1}$, $\overline{t}_{1,n} = \overline{t}_{1,n}$).

We have a morphism $\alpha_n : t_{n+1} \to \overline{t}_{1,n}$, defined by $t_{ij} \mapsto \overline{t}_{ij}$ if $1 \leq i < j \leq n$ and $t_{0i} \mapsto \overline{y}_i := -\frac{\text{ad} \lambda}{\exp \lambda - 1}(\overline{y}_i)$ if $1 \leq i \leq n$ (it takes the central element $\sum_{0 < i < j < n} t_{ij}$ to 0).

Let $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be a map and $\phi' : \{0, \ldots, m\} \to \{0, \ldots, n\}$ be given by $\phi'(1) = 1$, $\phi'(i) = \phi(i)$ for $i = 1, \ldots, m$. The diagram

$$
\begin{array}{ccc}
t_{n+1} & \xrightarrow{x = x'} & t_{m+1} \\
\alpha_n \downarrow & & \downarrow \alpha_m \\
\overline{t}_{1,n} & \xrightarrow{x = x'} & \overline{t}_{1,m}
\end{array}
$$

is not commutative, we have instead the identity

$$
\alpha_m(x') = \alpha_n(x) - \sum_{i=1}^{n} \xi_i(x) \sum_{i' < j'} \overline{t}_{ij'},
$$

where $\xi_i : \overline{t}_{1,n} \to \mathbf{k}$ is the linear form defined by $\xi_i(t_{0i}) = 1$, $\xi_i$ any other homogeneous Lie polynomial in the $t_{kl}$.

Since the various $\sum_{i' < j'} \frac{\text{ad} \lambda}{\exp \lambda - 1}(\overline{y}_i) t_{ij'}$ commute with each other and with the image of $x \mapsto x'$, this implies

$$
\alpha_m(g) = \alpha_n(g) \prod_{i=1}^{n} e^{-\xi_i(\log \lambda)(\sum_{i' < j'} \overline{t}_{ij'})}
$$

for $g \in \exp(\overline{t}_{n+1})$. 

Set $\tilde{A}_\lambda := \Phi^{0,1,2}_\lambda e^{\lambda_{01}}(\Phi^{0,1,2}_\lambda)^{-1} \in \exp(\mathfrak{t}_1)$. One proves that
\[
\tilde{A}_\lambda^{0,12,3} e^{\lambda_{12}} = e^{\lambda_{12}/2} \Phi^{0,1,2}_\lambda A^{0,1,2,3}_\lambda \Phi^{2,1,3}_\lambda e^{\lambda_{12}/2} \cdot \Phi^{3,2,1}_\lambda \tilde{A}_\lambda^{0,1,23} \Phi^{1,2,3}_\lambda
\]
(relation in $\exp(\mathfrak{t}_1)$). We then have $\alpha_2(\tilde{A}_\lambda) = \tilde{A}_\lambda$, $\alpha_3(\Phi^{0,1,2,3}_\lambda) = \Phi^{1,2,3}_\lambda$, and the relation between the $\alpha_i$ and coproducts implies $\alpha_3(\tilde{A}_\lambda^{0,1,23}) = \tilde{A}_\lambda^{1,23}$ and $\alpha_3(\tilde{A}_\lambda^{0,12,3} e^{\lambda_{12}}) = \tilde{A}_\lambda^{12,3}$.

Taking the image by $\alpha_3$, we get the first identity.

As we have already mentioned, this identity implies $(\Phi^{-1}_\lambda \tilde{A}_\lambda^{1,23} \Phi^{1,2,3}_\lambda, \tilde{A}_\lambda^{12,3}) = 1$.

Let $\exp(\mathfrak{t}_{n+1}) \ast \mathbb{Z}^n / I_n$ be the quotient of the free product of $\exp(\mathfrak{t}_{n+1})$ with $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z} X_i$ by the normal subgroup generated by the ratios of the exponentials of the sides of each of the equations
\[
X_i t_0 X_i^{-1} = \sum_{0 \leq \alpha \leq n, \alpha \neq i} t_{\alpha i} X_i (t_{0j} + t_{ij}) X_i^{-1} = t_{0j}, X_i t_j X_i^{-1} = t_{jk}, X_j X_k t_{jk} (X_j X_k)^{-1} = t_{jk}
\]
where $i, j, k$ are distinct in $\{1, ..., n\}$. Then the morphism $\alpha_n : t_{n+1} \rightarrow \mathfrak{t}_{n}$ extends to $\mathfrak{t}_n$ by $X_i \mapsto e^{\lambda x_i}$.

If $\phi : \{1, ..., m\} \rightarrow \{1, ..., n\}$ is a map, then the Lie algebra morphism $t_{n+1} \rightarrow t_{m+1}$, $x \mapsto x^\phi$ extends to a group morphism $\exp(\mathfrak{t}_{n}) \ast \mathbb{Z}^m / I_m$ by $X_i \mapsto \prod_{i' \in \phi^{-1}(i)} X_{i'}$.

Let $B_{\lambda} := e^{\lambda_{12}/2} \Phi^{0,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda \exp(\mathfrak{t}_3) \ast \mathbb{Z}^2 / I_2$.

then $\alpha_2(B_\lambda) = B_\lambda$.

We will prove that
\[
B_{\lambda}^{0,12,3} = e^{-\lambda_{12}/2} \Phi^{0,1,2,3}_\lambda B_{\lambda}^{0,2,13} \Phi^{2,1,3}_\lambda e^{-\lambda_{12}/2} \cdot \Phi^{3,2,1}_\lambda B_{\lambda}^{0,12,3} \Phi^{1,2,3}_\lambda. \tag{40}
\]

The l.h.s. is
\[
B_{\lambda}^{0,12,3} = e^{\lambda_{12}/2} \Phi^{0,3,12}_\lambda X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \cdot \Phi^{3,2,1}_\lambda B_{\lambda}^{0,12,3} \Phi^{1,2,3}_\lambda
\]
and the r.h.s. is
\[
e^{-\lambda_{12}/2} \Phi^{0,3,12}_\lambda e^{\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda B_{\lambda}^{0,12,3} \Phi^{1,2,3}_\lambda.
\]
The equality between these terms is rewritten as
\[
X_1 X_2 = \Phi^{0,3,12}_\lambda \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda,
\]
or, using the fact that $X_i$ commutes with $t_{jk}$ ($i, j, k$ distinct), as
\[
X_1 X_2 = \Phi^{0,3,12}_\lambda \Phi^{1,2,1,0}_\lambda X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda.
\]
Now $X_2 \Phi^{0,2,3,1}_\lambda = \Phi^{0,3,12}_\lambda X_2$, $X_1 \Phi^{0,12,3}_\lambda = \Phi^{0,2,3,1}_\lambda X_1$ and $X_1 X_2 \Phi^{1,02,3,1}_\lambda = \Phi^{2,1,0,3}_\lambda X_1 X_2$, so the r.h.s. is rewritten as $\Phi^{0,3,12}_\lambda \Phi^{1,2,1,0}_\lambda X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda = \Phi^{2,1,0,3}_\lambda X_1 X_2$. This ends the proof of (40). Taking the image by $\alpha_3$, we then get the second identity of the Proposition.

Let us prove the next identity. We have
\[
(\Phi^{0,12,3}_\lambda e^{\lambda_{12}/2} \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} X_1 \Phi^{0,2,3,1}_\lambda e^{\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda) \tag{40}
\]

Now
\[
X_1 X_2 \Phi^{0,2,3,1}_\lambda e^{\lambda_{12}/2} \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda e^{\lambda_{12}/2} \Phi^{3,2,1}_\lambda \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{0,2,3,1}_\lambda e^{\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda e^{-\lambda_{12}/2} X_2 \Phi^{3,2,1}_\lambda e^{-\lambda_{12}/2} \Phi^{3,2,1}_\lambda X_1 \Phi^{1,2,1,0}_\lambda = \Phi^{2,1,0,3}_\lambda X_1 X_2.
\]
Plugging this in the above expression for \((B_{\lambda}^{-1/2}e^{\lambda t/2}A_{\lambda}^{-1/2}, e^{\lambda t/2})\), one then finds \((B_{\lambda}^{-1/2}e^{\lambda t/2}A_{\lambda}^{-1/2}e^{\lambda t/2}/e^{\lambda t/2}) = e^{\lambda t/2}\). Taking the image by \(\alpha_4\), we then obtain \((B_{\lambda}^{-1/2}e^{\lambda t/2}A_{\lambda}^{-1/2}e^{\lambda t/2/2}) = e^{\lambda t/2\cdot 1/2}\). Let us prove that last identity. For this, we will show

\[\text{We have} \quad e^{-\lambda t/2}X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

Let us prove that last identity. For this, we will show

\[\text{We have} \quad e^{-\lambda t/2}X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

Now

\[X_{\lambda}^{-1} = \exp(\lambda t/2)X_{\lambda}^{-1} = \exp(\lambda t/2)X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

Thus

\[\exp(\lambda t/2)X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

So

\[\exp(\lambda t/2)X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

\[\exp(\lambda t/2)X_{\lambda}^{-1} = \text{the identity operator} \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

In particular, \((A_{\lambda}, B_{\lambda})\) give rise to a morphism \(\mathbb{F}_{1,n} \to \exp(\mathfrak{f}_{1,n}) \times S_n\); one proves as in Section 2 that it induces an isomorphism of filtered Lie algebras \(\text{Lie}(\mathbb{F}_{1,n}) \cong \mathfrak{f}_{1,n}\). Taking \(\Phi_{\lambda}\) to be a rational associator ([Dr3]), we then obtain:

**Corollary 5.4.** We have a filtered isomorphism \(\text{Lie}(\mathbb{F}_{1,n}) \cong \mathfrak{f}_{1,n}\), which can be extended to an isomorphism \(\mathbb{F}_{1,n}(Q) \cong \exp(\mathfrak{f}_{1,n}) \times S_n\).

5.3. Construction of morphisms \(\Gamma_{1,n} \to \text{G}_{1,n} \times S_n\) using a pair \((\Phi_{\lambda}, \tilde{\Psi}_{\lambda})\). Keep the notation of the previous section and set

\[\tilde{\Psi}_{\lambda} := \exp\left(-\frac{1}{\lambda}(\Delta_0 + \sum_{k \geq 1} a_{2k}(\lambda)\delta_{2k})\right).
\]

**Proposition 5.5.** We have

\[\tilde{\Psi}_{\lambda}e^{\lambda t/2}A_{\lambda}^{-1}(\tilde{\Psi}_{\lambda}e^{\lambda t/2})^{-1} = \tilde{A}_{\lambda}, \quad [\tilde{\Psi}_{\lambda}e^{\lambda t/2}B_{\lambda}(\tilde{\Psi}_{\lambda})^{-1}]^{-1} = \tilde{B}_{\lambda}A_{\lambda}.
\]

**Proof.** The first identity follows from the fact that \(\Delta_0 + \sum_{k \geq 1} a_{2k}(\lambda)\delta_{2k} = \lambda^2 t/12\) commutes with \(t\) and \(\tilde{g}_0\); the second identity follows from these facts and the analogue of Lemma 4.15, where \(2m\) is replaced by \(\lambda\). \(\square\)

Assume that \(\tilde{\Theta}_{\lambda} \in \text{G}_{1,n}\) satisfies

\[\tilde{\Theta}_{\lambda}^4 = (\tilde{\Theta}_{\lambda}^2)^2 = (\tilde{\Theta}_{\lambda}^2)^2 \tilde{\Psi}_{\lambda} = 1,
\]

\[\tilde{\Theta}_{\lambda}^{-1}e^{\lambda t/2}A_{\lambda}^{-1}(\tilde{\Theta}_{\lambda}^{-1}e^{\lambda t/2/4})^{-1} = \tilde{B}_{\lambda}^{-1}, \quad [\tilde{\Theta}_{\lambda}^{-1}e^{\lambda t/2}B_{\lambda}(\tilde{\Theta}_{\lambda})^{-1}]^{-1} = \tilde{B}_{\lambda}A_{\lambda}^{-1}.
\]
(one can show that the two last equations are equivalent), then $\Theta \mapsto [\tilde{\Theta}_\lambda] e^{\lambda(\sum_{i<j} l_{ij})/4}$, $\Psi \mapsto [\tilde{\Psi}_\lambda] e^{\lambda(\sum_{i<j} l_{ij})/12}$ extends the morphism defined in Proposition 5.3 to a morphism $\Gamma_{1,[n]} \to G_n \times S_n$.

We do not know whether for each $\Phi_\lambda$ defined over $k$, there exists a $\tilde{\Theta}_\lambda$ defined over $k$, satisfying the above conditions.

5.4. Elliptic structures over QTQBA’s. Let $(H, \Delta_H, R_H, \Phi_H)$ be a quasitriangular quasi-bialgebra (QTQBA). Recall that this means that $([ Dr2])$: $(H, m_H)$ is an algebra, $\Delta_H : H \to H \otimes H$ is an algebra morphism, $R_H \in H \otimes H$ and $\Phi_H \in H \otimes H$ are invertible, and

$$
\Delta_H(x) = R_H \Delta_H(x) R_H^{-1}, \quad (id \otimes \Delta_H) \circ \Delta_H(x) = \Phi_H(\Delta_H \otimes id) \circ \Delta_H(x) \Phi_H^{-1},
$$

$$
R_H^{1,2,3} = \Phi_H^{1,2,3} = \Phi_H^{2,3,1} = \Phi_H^{3,1,2}, \quad R_H^{1,2,3} = \Phi_H^{2,3,1} = \Phi_H^{3,1,2} = \Phi_H^{1,2,3}.
$$

One also assumes the existence of a unit $1_H$ and a counit $\varepsilon_H$.

If $A$ is an algebra and $J_1, J_2 \subseteq A$ are left ideals, define the Hecke bimodule $\mathcal{H}(A, J_1, J_2)$ or $\mathcal{H}(J_1, J_2) = \text{Hom}_A(A/J_1, A/J_2) = (A/J_2)^{J_1}$, where $J_1$ acts on the quotient from the left; we have thus $\mathcal{H}(J_1, J_2) = \{x \in A \mid J_1 x \subseteq J_2\}$. The product of $A$ induces a product $\mathcal{H}(J_1, J_2) \otimes \mathcal{H}(J_2, J_3) \to \mathcal{H}(J_1, J_3)$ when $J_1 \cap J_2 = J$. $\mathcal{H}(J) \equiv \mathcal{H}(J, J)$ is the usual Hecke algebra, and $\mathcal{H}(J_1, J_2)$ is a $(\mathcal{H}(J_1), \mathcal{H}(J_2))$-bimodule. Recall that we have a functor $A$ -mod $\to \mathcal{H}(J)$-mod, $V \to V^J = \{v \in V \mid jv = 0\}$.

If $H$ is an algebra with unit equipped with a morphism $\Delta_H : H \to H \otimes H$ and $a : H \to D$ is a morphism of algebras with unit, we define for each $n \geq 1$ and each pair of words $w, w'$ in the free magma generated by $1, \ldots, n$ containing $1, \ldots, n$ exactly once (recall that a magma is a set with a non-necessarily associative binary operation) the Hecke bimodule

$$
\mathcal{H}^{w,w'}(D, H) := \mathcal{H}(D \otimes H^{\otimes n}, J_w, J_{w'}),
$$

(or simply $\mathcal{H}^{w,w'}$) where $J_w \subset D \otimes H^{\otimes n}$ is the left ideal generated by the image of $(a \otimes \Delta_H^{w'}) \circ \Delta_H : H_+ \to D \otimes H^{\otimes n}$. Here $H_+ = \text{Ker}(H \otimes k)$ and for example $\Delta_H^{(21)} = (21) \circ (\Delta_H \otimes id_H) \circ \Delta_H$, etc. We have products $\mathcal{H}^{w,w'} \otimes \mathcal{H}^{w'',w''''} \to \mathcal{H}^{w,w'''}$. We denote the Hecke algebra $\mathcal{H}^{w,w}$ by $\mathcal{H}^{w}(D; H)$ or $\mathcal{H}^{w}$; we denote by $1_w$ its unit. We denote by $(\mathcal{H}^{w,w'})^{\times}$ the set of invertible elements of $\mathcal{H}^{w,w'}$, i.e., the set of elements $X$ such that for some $X' \in \mathcal{H}^{w',w}$, $X'X = 1_w$, $XX' = 1_w$. The symmetric group $S_n$ acts on the system of bimodules $\mathcal{H}^{w,w'}$ by permuting the factors, so we get maps $\text{Ad}^w(D) : \mathcal{H}^{w,w'} \to \mathcal{H}^{w,\sigma(w)}(w')$ (where $\sigma(w)$ is the word $w$, where $i$ is replaced by $\sigma(i)$). If $w_0 = ((12)\ldots n)$, we define an algebra structure on $\oplus_{\sigma \in S_n} \mathcal{H}^{w_0,\sigma(w_0)} \sigma$ by $(\sum_{\sigma \in S_n} h_\sigma)(\sum_{\tau \in S_n} h'_\tau) := \sum_{\sigma, \tau \in S_n} h_\sigma(\tau) h'_\sigma(\tau)$. Then $\sqcup_{\sigma \in S_n} (\mathcal{H}^{w_0,\sigma(w_0)})^{\times} \subset \oplus_{\sigma \in S_n} \mathcal{H}^{w_0,\sigma(w_0)}$ is a group with unit $1_{w_0}$. We have an exact sequence $1 \to (\mathcal{H}^{w_0})^{\times} \to \sqcup_{\sigma \in S_n} (\mathcal{H}^{w_0,\sigma(w_0)})^{\times} \to S_n$, but the last map is not necessarily surjective (and if it is, does not necessarily split).

If $H$ is a quasibialgebra, then $\Phi_H$ gives rise to an element of $\mathcal{H}^{(23), (123)}(D, H)$, which we also denote $\Phi_H$; similarly $\Phi_H^{-1}$ gives rise to the inverse (w.r.t. composition of Hecke bimodules) element $\Phi_H^{-1} \in \mathcal{H}^{(23), (123)}(D, H)$. We have algebra morphisms $\mathcal{H}^{12}(D, H) \to \mathcal{H}^{(123)}(D, H)$ induced by $X \mapsto X^{0,1,2,3} := (id_H \otimes (\Delta_H \otimes id_H) \circ \Delta_H)(X)$ (0 is the index of $D$) and similarly morphisms $\mathcal{H}^{12}(D, H) \to \mathcal{H}^{(23)}(D, H), X \mapsto X^{0,1,2,3}$, $\mathcal{H}^{12}(D, H) \to \mathcal{H}^{(123)}(D, H), X \mapsto X^{0,1,2,3}$, $\mathcal{H}^{12}(D, H)$, etc. If moreover $H$ is quasitriangular, then $R_H \in \mathcal{H}^{(21), (12)}(D, H)$, $R_H^{-1} \in \mathcal{H}^{(21), (12)}(D, H)$, so in that case $\sqcup_{\sigma \in S_n} \mathcal{H}^{w_0,\sigma(w_0)} \sigma \to S_n$ is surjective, and we have a morphism $B_n \to \sqcup_{\sigma \in S_n} \mathcal{H}^{w_0,\sigma(w_0)} \sigma$ such that the composition $B_n \to \sqcup_{\sigma \in S_n} \mathcal{H}^{w_0,\sigma(w_0)} \sigma \to S_n$ is the canonical projection.

Definition 5.6. If $H$ is a QTQBA, an elliptic structure on $H$ is a triple $(A, D, B)$, where $D$ is an algebra with unit, equipped with an algebra morphism $a : H \to D$, and $A, B \in$
\( \mathcal{H}^{12}(D, H) \) are invertible such that \( A^{0,1,0} = A^{0,0,1} = B^{0,1,0} = B^{0,0,1} = 1 \),
\[
A^{0,1,2} = R_1^{2,1}(\Phi_{1,2,3}^{-1})^{-1} A^{0,2,13} R_1^{1,2,3} \Phi_{1,2,3}^{-1} A^{0,1,23},
\]
\[
B^{0,1,2} = (R_1^{1,2})^{-1}(\Phi_{1,2,3}^{-1})^{-1} B^{0,2,13} \Phi_{1,2,3} (R_1^{1,2})^{-1}(\Phi_{1,2,3}^{-1})^{-1} B^{0,1,23},
\]
and
\[
(B^{0,1,2}, R_1^{2,1}(\Phi_{1,2,3}^{-1})^{-1} A^{0,2,13} \Phi_{1,2,3}^{-1} R_1^{1,2})
= ((R_1^{1,2})^{-1}(\Phi_{1,2,3}^{-1})^{-1} B^{0,2,13} \Phi_{1,2,3} (R_1^{1,2})^{-1}, A^{0,1,2}) = (\Phi_{1,2,3}^{-1})^{-1} R_1^{3,2} R_1^{2,3} \Phi_{1,2,3}
\]
(identities in \( \mathcal{H}^{12}(D, H) \)).

The pair of identities (41), (42) is equivalent to
\[
R_1^{2,1} A^{0,2,1} R_1^{1,2} A^{0,1,2} = 1, \\
R_1^{3,12} A^{0,3,12} \Phi_{3,1,2} R_1^{2,31} A^{0,2,31} \Phi_{2,3,1} R_1^{2,3} A^{1,23} \Phi_{1,2,3} = 1,
\]
and
\[
(R_1^{1,2})^{-1} B^{0,2,1} (R_1^{1,2})^{-1} B^{0,1,2} = 1, \\
(R_1^{1,2})^{-1} B^{0,3,12} \Phi_{1,2,3}^{-1} (R_1^{1,2})^{-1} B^{0,2,31} \Phi_{2,3,1}^{-1} (R_1^{1,2})^{-1} B^{0,1,23} \Phi_{1,2,3}^{-1} = 1,
\]
so the invertibility conditions on \( A, B \) follow from (41), (42).

If \( F \in H^{0,2} \) is invertible with \( (\varepsilon_H \otimes \text{id}_H)(F) = (\varepsilon_H \otimes \varepsilon_H)(F) = 1 \), then the twist of \( H \) by \( F \) is the quasi-Hopf algebra \( \mathcal{F}H \) with product \( m_H \), coproduct \( \Delta_H = F \Delta_H(x) F^{-1} \), \( R \)-matrix \( R_H = F^{21} R H F^{-1} \) and associator \( \Phi_H = F^{23} F^{123} \Phi_H (F^{1,2} F^{12,3})^{-1} \). If \( \varphi : H \to D \) is an algebra morphism, it can be viewed as a morphism \( \varphi : H \to D \), and we have an algebra isomorphism \( \mathcal{H}^{12}(D, H) \to \mathcal{H}^{12}(D, \mathcal{F}H) \), induced by \( X \mapsto \varphi \otimes \varphi \). If \( \mathcal{F}H \) is an algebra isomorphism of the systems of bimodules \( \mathcal{H}^{w,w'}(D, H) \to \mathcal{H}^{w,w'}(D, \mathcal{F}H) \) induced by \( X \mapsto \varphi \otimes \varphi \).

If \( (D, A, B) \) is an elliptic structure on \( H \), then an elliptic structure \( \mathcal{F}H \) is \( \hat{D}, \hat{A}, \hat{B} \), where \( \hat{A} = F^{1,2} F^{0,12} A(F^{1,2}, F^{0,12})^{-1} \) and \( \hat{B} = F^{1,2} F^{0,12} B(F^{1,2}, F^{0,12})^{-1} \).

An elliptic structure \( (D, A, B) \) over \( H \) gives rise to a unique group morphism
\[
\overline{G}_{1,n} \to \cup_{\sigma \in S_n} \mathcal{H}^{w_0, \sigma(w_0)}(D, H),
\]
such that
\[
\sigma_i \mapsto \Phi_{1,2,3}^{-1}(\Phi_{1,2,3}^{-1}(1, i, i+1))^{-1} R_H^{i-1, i+1}(i, i+1) \Phi_{H,i},
\]
\[
A_i \mapsto \Phi_{H,i} A^{0,1} \Phi_{H,i}, \quad B_i \mapsto \Phi_{H,i}^{-1} B^{0,1} \Phi_{H,i},
\]
where
\[
\Phi_{H,i} = \Phi_{H,12, \ldots, i, i+1, \ldots, n-1, n}.
\]
Here we have for example \( x^{(12,3)} = (\Delta_H \otimes \text{id}_H) \circ \Delta_H(x) \) for \( x \in H \).

If \( g \) is a Lie algebra and \( t_g \in S^2(g) \# \) is nondegenerate, then \( H = U(g) \otimes [h] \) is a QTBA, with \( m_H, \Delta_H \) the undeformed product and coproduct, \( R_H = e^{h_{t_g}/2} \) and \( \Phi_H = \Phi(h_{t_g}^{1,2}, h_{t_g}^{2,3}) \), where \( \Phi \) is an \( 1 \)-associator. The results of next Section then imply that \( (D, A, B) \) is an elliptic structure over \( H \), where \( D = D(g) \otimes [h] \) (\( D(g) \) is the algebra of algebraic differential operators on \( g \)) and \( A, B \) are given by the formulas for \( \hat{A}, \hat{B} \) with \( t \) replaced by \( h t_{\hat{g}}^{1,2} \), \( x \) replaced by \( h \sum x \otimes (e^x_1 + e^x_2) \), \( y \) replaced by \( h \sum \partial_x (e^x_1 + e^x_2) \).

Remark 5.7. If \( H \) is a Hopf algebra, we have an isomorphism
\[
\mathcal{H}^{w_0}(D, H) \simeq (D \otimes H^{n-1})^H,
\]
where the right side is the commutant of the diagonal map \( H \to D \otimes H^{n-1} \), \( h \mapsto (a \otimes h^{n-1}) \circ \Delta_H(h) \). This map takes the class of \( d_a \otimes h_1 \otimes \ldots h_n \) to \( da(S_H(h_1^{(1)})) \otimes h_1 S_H(h_1^{(n-1)}) \otimes \ldots \otimes h_1 S_H(h_1^{(1)}) \) (\( S_H \) is the antipode of \( H \)). So \( A, B \) identify with elements \( A, B \in (D \otimes H)^H \); the conditions are then
\[
A^{0,12} = R_1^{2,1} A^{0,2} R_1^{1,2} A^{0,1}, \quad B^{0,12} = (R_1^{1,2})^{-1} B^{0,2} (R_1^{2,1})^{-1} B^{0,1},
\]
$(B^{0,12},R^{2,1}_{H},A^{0,2}_{R}R^{2,2}_{H}) = ((R^{1,2}_{H})^{-1}B^{0,2}(R^{1,1}_{H}),A^{0,12}) = (R^{3,2}_{H}R^{1,1}_{H}R^{0,2}_{H}R^{2,1}_{H}R^{3,0}_{H})$

(conditions in $(D \otimes H^{\otimes 2})^H$), where the superscript $B''_n \times Z^{n-1} \to B_{n-1} \times Z^n$ denotes the map $x_0 \otimes \ldots \otimes x_{n-1} \mapsto S_H(x_0) \otimes S_H(x_1) \otimes \cdots \otimes S_H(x_n)$.

Moreover, the morphism $PB_n \to (H^{\otimes n})^\times \simeq (D \otimes H^{\otimes n-1})^H$ factors through $PB_n \to PB_{n-1} \times Z^{n-1} \to D \otimes H^{\otimes n-1}$, where (a) the first morphism is induced by $Z^{n-1} \times B'_n \to Z^{n-1} \times B_{n-1}$ (where $B'_n = B_n \times S_n$, $S_n$ is the group of braids leaving the last strand fixed), constructed as follows: we have a composition $B'_{n+1} \to \pi_1((\mathbb{P}^1)^{n+1} - \text{diagonals})/S_n \to \pi_1((\mathbb{C}^n - \text{diagonals})/S_n) = B_n$, where the first map is induced by $\mathbb{C} \subset \mathbb{P}^1$, and the middle middle comes from the fibration $\mathbb{C}^n - \text{diagonals} \to (\mathbb{P}^1)^{n+1} - \text{diagonals} \to \mathbb{P}^1$.

Let $\sigma_{n-1} \cdot x := \{R^{1,2}_{H},R^{0,2}_{H}R^{2,1}_{H}R^{3,1}_{H}\} \otimes \ldots \otimes R^{2n-1,1}_{H}\otimes \ldots \otimes x_{n-1} S_H(x_n)$

We have then $\cup_{x \in S_n}(H^{\otimes o,x})^\times \sigma \simeq (D \otimes H^{\otimes n-1})^H \times PB_n$, $B_n$ (the index means that $PB_n \subset B_n$ is identified with its image in $(D \otimes H^{\otimes n-1})^H$).

Then if $(A,B)$ is an elliptic structure over $a : H \to D$, the morphism $B_n \to ((D \otimes H^{\otimes n-1})^H)^H \times PB_n$ extends to a morphism

$\overline{B}_{1,n} \to ((D \otimes H^{\otimes n-1})^H)^H \times PB_n$

via $A_i \mapsto A^{0,1 \ldots i-1}$, $B_i \mapsto B^{0,1 \ldots i-1}$.

This interpretation of $H^{\otimes n}$ and of the relations between $A,B$ can be extended to the case when $H$ is a quasi-Hopf algebra.

Remark 5.8. Let $C$ be a rigid braided monoidal category. We define an elliptic structure on $C$ as a quadruple $(E,A,B,F)$, where $E$ is a category, $F : E \to C$ is a functor, and $A,B$ are functorial automorphisms of $F(?)$ itself, which reduce to the identity if the second factor is the neutral object $1$, and such that the following equalities of automorphisms of $F(M) \otimes (X \otimes Y)$ hold (we write them omitting associativity maps, as they can be put in automatically):

$A_{M,X \otimes Y} = \beta_{Y,X} A_{M,Y} \beta_{X,Y} A_{M,X}$,

$B_{M,X \otimes Y} = \beta_{X,Y}^{-1} B_{M,Y} \beta_{Y,X}^{-1} B_{M,X}$,

$(B_{M,X \otimes Y}^{-1} \beta_{Y,X} A_{M,Y} \beta_{X,Y}^{-1}, A_{M,X \otimes Y}) = (\beta_{Y,X}^{-1} B_{M,Y} \beta_{Y,X}^{-1}, A_{M,X \otimes Y})$

where $\beta_{X,Y} \in \text{Hom}_C(1,X \otimes X^*)$ is the canonical map and the r.h.s. of the last identity is viewed as an element of $\text{End}_C(M \otimes X \otimes Y)$ using its identification with $\text{Hom}_C(1,(M \otimes X \otimes Y) \otimes (M \otimes X \otimes Y^*))$. An elliptic structure on a quasitriangular quasi-Hopf algebra $H$ gives rise to an elliptic structure on $H$-mod. An elliptic structure over a rigid braided monoidal category $C$ gives rise to representations of $\overline{B}_{1,n}$ by $C$-automorphisms of $F(M) \otimes X^{\otimes n-1}$.

6. The KZB connection as a realization of the universal KZB connection

6.1. Realizations of $\overline{B}_{1,n}$. Let $g$ be a Lie algebra and $t_g \in S^2(g)^g$ be nondegenerate. We denote by $(a,b) \mapsto (a,b)$ the corresponding invariant pairing.

Let $D(g)$ be the algebra of algebraic differential operators on $g$. It has generators $x_a, \partial_a, a \in g$, and relations: $a \mapsto x_a, a \mapsto \partial_a$ are linear, $[x_a,x_b] = [\partial_a,\partial_b] = 0, [\partial_a,x_b] = (a,b)$. 


There is a unique Lie algebra morphism $\rho_\mathfrak{g} : \mathfrak{l}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$, $\bar{x}_i \mapsto \sum_{\alpha} x_\alpha \otimes (\sum_i e_{\alpha}^{(i)})$, $\bar{y}_i \mapsto -\sum_{\alpha} \partial_\alpha \otimes e_{\alpha}^{(i)}$, $\bar{t}_{ij} : = t_{ij}^{(i)}$ (we set $x_\alpha := x_{\alpha,\beta}$, $\partial_\alpha := \partial_{\alpha,\beta}$).

**Proposition 6.1.** There is a unique Lie algebra morphism $\rho_\mathfrak{g} : \mathfrak{l}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$, $\bar{x}_i \mapsto \sum_{\alpha} x_\alpha \otimes (\sum_i e_{\alpha}^{(i)})$, $\bar{y}_i \mapsto -\sum_{\alpha} \partial_\alpha \otimes e_{\alpha}^{(i)}$, $\bar{t}_{ij} : = t_{ij}^{(i)}$ (we set $x_\alpha := x_{\alpha,\beta}$, $\partial_\alpha := \partial_{\alpha,\beta}$).

**Proof.** The images of all the generators of $\mathfrak{g}$ in $\mathfrak{A}_n$, therefore also in its normalizer. According to Lemma 2.1, we will use the following presentation of $\mathfrak{l}_{1,n}$. Generators are $\bar{x}_i, \bar{y}_i, \bar{t}_{ij}$, relations are $[\bar{x}_i, \bar{x}_j] = [\bar{y}_i, \bar{y}_j] = 0$, $[\bar{x}_i, \bar{y}_j] = \bar{t}_{ij}$ ($i \neq j$), $\bar{t}_{ij} = \bar{t}_{ji}$, $\sum_i \bar{x}_i = \sum_i \bar{y}_i = 0$, $[\bar{x}_i, \bar{t}_{jk}] = 0$ ($i, j, k$ distinct).

The relations $[\bar{x}_i, \bar{x}_j] = [\bar{y}_i, \bar{y}_j] = 0$, $[\bar{x}_i, \bar{y}_j] = \bar{t}_{ij}$ ($i \neq j$), $\bar{t}_{ij} = \bar{t}_{ji}$ and $[\bar{x}_i, \bar{t}_{jk}] = [\bar{y}_i, \bar{t}_{jk}] = 0$ are obviously preserved. Let us check that $\sum_i \bar{x}_i = \sum_i \bar{y}_i = 0$ are preserved.

We have
\[
\sum_i \rho_\mathfrak{g}(\bar{x}_i) = \sum_{\alpha} x_\alpha \otimes \sum_i e_{\alpha}^{(i)} = \sum_{\alpha} (x_\alpha \otimes 1)(Y_\alpha - X_\alpha \otimes 1) = 0
\]

since $x_\alpha$ commutes with $x_{\alpha,\beta}$ and $\sum_{\beta} e_{\beta} \otimes e_{\beta} = t_\mathfrak{g}$ is invariant. We also have
\[
\sum_i \rho_\mathfrak{g}(\bar{y}_i) = -\sum_{\alpha} \partial_\alpha \otimes (\sum_i e_{\alpha}^{(i)}) = -\sum_{\alpha} (\partial_\alpha \otimes 1)(Y_\alpha - X_\alpha \otimes 1) = \sum_{\alpha} \partial_\alpha X_\alpha \otimes 1
\]

since $t_\mathfrak{g}$ is invariant and $\langle -, - \rangle$ is symmetric, we have $\sum_{\alpha} (e_{\alpha,\beta} \otimes e_{\alpha,\beta}) = 0$ for any $\beta$, and since $\langle \partial_{\alpha,\beta}, \partial_{\alpha,\beta} \rangle = 0$, we have $\sum_{\alpha,\beta} x_{\alpha,\beta} \partial_{\alpha,\beta} \partial_{\alpha,\beta} = 0$, so $\sum_i \rho_\mathfrak{g}(\bar{y}_i) = 0$.

**6.2. Realizations of $\mathfrak{l}_{1,n} \rtimes \mathfrak{d}$.** Let $(\mathfrak{g}, t_\mathfrak{g})$ be as in Subsection 6.1. We keep the same notations.

**Proposition 6.2.** The Lie algebra morphism $\rho_\mathfrak{g} : \mathfrak{l}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$ of Proposition 6.1 extends to a Lie algebra morphism $\rho_\mathfrak{g} : \mathfrak{l}_{1,n} \rtimes \mathfrak{d} \to \mathcal{H}_n(\mathfrak{g})$, defined by $\Delta_0 \mapsto -\frac{1}{2}(\sum_{\alpha} \partial_{\alpha}^2) \otimes 1$, $X \mapsto \frac{1}{2}(\sum_{\alpha} x_{\alpha} \partial_{\alpha} + \partial_{\alpha} x_{\alpha}) \otimes 1$, and
\[
\delta_{2m} \mapsto \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha} x_{\alpha_1} \cdots x_{\alpha_{2m}} \otimes \sum_{i=1}^{n} (\text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}}))(e_{\alpha_i} \otimes e_i^{(i)})
\]

for $m \geq 1$. This morphism further extends to a morphism $U(\mathfrak{l}_{1,n} \rtimes \mathfrak{d}) \rtimes S_n \to \mathcal{H}_n(\mathfrak{g}) \rtimes S_n$ by $\sigma \mapsto \sigma$.

**Proof.** We have
\[
[\rho_\mathfrak{g}(\delta_{2m}), \rho_\mathfrak{g}(\bar{x}_i)] = \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha, \beta} x_{\alpha_1} \cdots x_{\alpha_{2m}} x_{\beta} \otimes \sum_{i=1}^{n} (\text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}}))(e_{\beta}(e_{\alpha_i}) \otimes e_i^{(i)}) = 0
\]

\[
= \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha, \beta} x_{\alpha_1} \cdots x_{\alpha_{2m}} x_{\beta} \otimes \sum_{i=1}^{2m} (\text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}}))(e_{\beta}(e_{\alpha_i}) \otimes e_i^{(i)}) = 0
\]
the second equality follows from the invariance of \( t_\mathfrak{g} \), and the last equality follows from the fact that the first factor is symmetric in \( (\beta, \alpha_i) \) while the second is antisymmetric in \( (\beta, \alpha_i) \).

\( \rho_\mathfrak{g} \) preserves the relation \( [\delta_{2m}, t_{ij}] = [t_{ij}, \text{ad}(x)]^{2m}(t_{ij}) \), because \( \rho_\mathfrak{g}(\delta_{2m} + \sum_{i<j} \text{ad}(\tilde{x}_i)\text{ad}(\tilde{x}_j)) \) belongs to \( D(\mathfrak{g}) \otimes \text{im}(\Delta^{(n)}: \mathfrak{g} \rightarrow U(\mathfrak{g})^{\otimes n}) \), where \( \Delta^{(n)} \) is the \( n \)-fold coproduct and \( U(\mathfrak{g}) \) is equipped with its standard bialgebra structure.

Now

\[
[\rho_\mathfrak{g}(\delta_{2m}), \rho_\mathfrak{g}(\tilde{y}_i)] = \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha, \beta} \left( \sum_j [\partial_\beta, x_{\alpha_1} \cdots x_{\alpha_{2m}}] \otimes e_{\beta}^{(j)} \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(j)} e_{\alpha}ight)
\]

\[
+ x_{\alpha_1} \cdots x_{\alpha_{2m}} \partial_\beta \otimes [e_{\beta}, \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(j)} e_{\alpha}]
\]

\[
= \frac{1}{2} \sum_{l=1}^{2m} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha, \beta} \sum_j (x_{\alpha_1} \cdots \xi_{\alpha_1} \cdots x_{\alpha_{2m}} \otimes e_{\alpha_1}^{(i)} \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(j)} e_{\alpha})
\]

\[
+ x_{\alpha_1} \cdots x_{\alpha_{2m}} \partial_\beta \otimes \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(i)} e_{\alpha}
\]

\[
= \frac{1}{2} \sum_{l=1}^{2m} \sum_{\alpha_1, \ldots, \alpha_{2m}, \alpha, \beta} \sum_j (x_{\alpha_1} \cdots \xi_{\alpha_1} \cdots x_{\alpha_{2m}} \otimes e_{\alpha_1}^{(i)} \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(j)} e_{\alpha})
\]

\[
- x_{\alpha_1} \cdots \xi_{\alpha_1} \cdots x_{\alpha_{2m}} \otimes \text{ad}(e_{\alpha_1}) \cdots \text{ad}(e_{\alpha_{2m}})(e_{\alpha})^{(i)} e_{\alpha}^{(j)} e_{\alpha_1}
\]
6.3. Reductions. Assume that \( \mathfrak{g} \) is finite dimensional and we have a reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \), i.e., \( \mathfrak{h} \subset \mathfrak{g} \) is a Lie subalgebra and \( \mathfrak{n} \subset \mathfrak{g} \) is a vector subspace such that \( \mathfrak{[h, n]} \subset \mathfrak{n} \); assume also that \( t_{\mathfrak{g}} = t_{\mathfrak{h}} + t_{\mathfrak{n}} \), where \( t_{\mathfrak{g}} \in S^2(\mathfrak{h}) \) and \( t_{\mathfrak{n}} \in S^2(\mathfrak{n}) \).

We assume that for a generic \( h \in \mathfrak{h} \), \( \text{ad}(h)|_{\mathfrak{n}} \in \text{End}(\mathfrak{n}) \) is invertible. This condition is equivalent to the nonvanishing of \( P(\lambda) := \det(\text{ad}(\lambda)^{\mathfrak{n}})|_{\mathfrak{n}} \in S^{\dim \mathfrak{n}}(\mathfrak{h}) \), where \( \lambda \mapsto \lambda^V := (\lambda \otimes \text{id})(t_h) \). If \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \), an equivalent condition is that a generic element of \( \mathfrak{g}^* \) is conjugate to some element in \( \mathfrak{h}^* \) (see [EE]).

Let us set, for \( \lambda \in \mathfrak{h}^* \),

\[
\tau(\lambda) := (\text{id} \otimes \text{ad}(\lambda^V)|_{\mathfrak{n}}^{-1})(t_n).
\]

Then \( r : \mathfrak{h}^* \rightarrow \mathfrak{g}^2(\mathfrak{n}) \) is an \( h \)-equivariant map (here \( \mathfrak{h}^* = \{ \lambda \in \mathfrak{h}^* | P(\lambda) \neq 0 \} \)), satisfying the classical dynamical Yang-Baxter (CDYB) equation

\[
\text{CYB}(r) - \text{Alt}(d r) = 0
\]

(see [EE]). Here for \( r = \sum_{\alpha} a_\alpha \otimes b_\alpha \otimes \ell_\alpha \in (\mathfrak{g}^2 \otimes S(\mathfrak{h})[1/P])^{\mathfrak{b}} \), we set \( \text{CYB}(r) = \sum_{\alpha, \alpha'}([a_\alpha, a_{\alpha'}] \otimes b_\alpha \otimes b_{\alpha'} + a_\alpha \otimes [b_\alpha, a_{\alpha'}] \otimes b_{\alpha'} + a_\alpha \otimes a_{\alpha'} \otimes [b_\alpha, b_{\alpha'}]) \otimes \ell_\alpha \ell_{\alpha'} \), \( d r := \sum_{\alpha} a_\alpha \otimes b_\alpha \otimes d \ell_\alpha \), where \( d \) extends \( S(\mathfrak{h}) \rightarrow \mathfrak{g} \otimes S(\mathfrak{h}) \), \( x^k \rightarrow k \times x \times k^{-1} \) and \( \text{Alt}(X \otimes \ell) = (X + X^{2,3,1} + X^{3,1,2}) \otimes \ell \).

We also set

\[
\psi(\lambda) := (\text{id} \otimes (\text{ad}(\lambda^V)|_{\mathfrak{n}})^{-2})(t_n).
\]

We write \( \psi(\lambda) = \sum_{\alpha} A_\alpha \otimes B_\alpha \otimes L_\alpha \).

Let \( D(\mathfrak{h})[1/P] \) be the localization at \( P \) of the algebra \( D(\mathfrak{h}) \) of differential operators on \( \mathfrak{h} \); the latter algebra is generated by \( \tilde{x}_h, \partial_h, h \in \mathfrak{h} \), with relations \( h \mapsto \tilde{x}_h, h \mapsto \partial_h \) linear, \( [\tilde{x}_h, \partial_h] = [\partial_h, \tilde{x}_h] = 0 \), and \( [\partial_h, \tilde{x}_h] = (h, h') \).

Let \( B_n := D(\mathfrak{h})[1/P] \otimes U(\mathfrak{g})^{\otimes n} \). For \( h \in \mathfrak{h} \), we define \( \tilde{X}_h := \sum_{\nu} \tilde{x}_{[h, \nu]} \partial_{\nu} \in D(\mathfrak{h}) \),

\[
t_\nu := \sum_{\nu} h_{\nu} \otimes h_{\nu}.
\]

We then set \( \tilde{Y}_h := \tilde{X}_h + \sum_{i=1}^{n} h^{(i)} \). The map \( h \mapsto B_n \) is a Lie algebra morphism; we denote by \( \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}) \) its image.

We denote by \( \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}) \) the Hecke algebra of \( B_n \) relative to \( h \)-diagonal. Explicitly, \( \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}) = \{ x \in B_n | \forall h \in \mathfrak{h}, \tilde{Y}_h x = B_n \mathfrak{h}_{\text{diag}} / B_n \mathfrak{h}_{\text{diag}} \} \).

Proposition 6.3. There is a unique Lie algebra morphism

\[
\rho_{\mathfrak{g}, h} : \mathfrak{t}_{1,n} \rightarrow \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}),
\]

such that \( \tilde{x}_i \mapsto \sum_{\nu} \tilde{x}_{\nu} \otimes h^{(i)}_{\nu}, \tilde{y}_i \mapsto -\sum_{\nu} \partial_{\nu} \otimes h^{(i)}_{\nu} + \sum_{j} \sum_{\nu} \tilde{x}_j \otimes a_{\nu} b^{(i)}_{j\nu}, \tilde{t}_{ij} \mapsto t^{(i)}_{\nu} \). Here \( \tau(\lambda) = \sum_{\alpha} \xi_{\alpha}(\lambda)(a_\alpha \otimes b_\alpha) \).

If \( V_1, \ldots, V_n \) are \( g \)-modules, then \( S(\mathfrak{g})[1/P] \otimes \otimes V_i \) is a module over \( D(\mathfrak{h})[1/P] \otimes (\otimes V_i) \), and \( (S(\mathfrak{g})[1/P] \otimes \otimes V_i)^{\mathfrak{b}} \) is a module over \( H_n(\mathfrak{g}, \mathfrak{h}) \).

Moreover, we have a restriction morphism \( (S(\mathfrak{g})[1/P] \otimes \otimes V_i)^{\mathfrak{g}} \rightarrow (S(\mathfrak{h})[1/P] \otimes \otimes V_i)^{\mathfrak{g}} \).

Note that \( S(\mathfrak{g})[1/P] \otimes \otimes V_i \) is a \( \mathfrak{t}_{1,n} \)-module using the morphism \( \tilde{t}_{1,n} \rightarrow \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}) \), while \( (S(\mathfrak{h})[1/P] \otimes \otimes V_i)^{\mathfrak{g}} \) is a \( \mathfrak{t}_{1,n} \)-module using the morphism \( \tilde{t}_{1,n} \rightarrow \mathcal{H}_{n}(\mathfrak{g}, \mathfrak{h}) \). Then one checks that the restriction morphism \( (S(\mathfrak{g})[1/P] \otimes \otimes V_i)^{\mathfrak{g}} \rightarrow (S(\mathfrak{h})[1/P] \otimes \otimes V_i)^{\mathfrak{g}} \) is a \( \mathfrak{t}_{1,n} \)-modules morphism.

Proof. The images of the above elements are all \( h \)-invariant. To lighten the notation, we will imply summation over repeated indices and denote elements of \( B_n \) as follows: \( \tilde{x}_i \) by \( \tilde{e}_i \), \( \tilde{y}_i \) by \( \tilde{y}_i \), \( 1 \otimes x^{(i)} \) by \( x^i \). Then \( \rho_{\mathfrak{g}, h}(\tilde{x}_i) = (\lambda^V)^i, \rho_{\mathfrak{g}, h}(\tilde{y}_i) = -h^{(i)} \partial_{\nu} + \sum_{j=1}^{n} r(\lambda)^{ij} \) (here for \( x \otimes y \in g^{2,2}, (x \otimes y)^{ij} := x^{(i)} y^{(j)} \)).

We will use the same presentation of \( \mathfrak{t}_{1,n} \) as in Proposition 6.1. The relations \( [\tilde{x}_i, \tilde{x}_j] = 0 \) and \( \tilde{t}_{ij} = \tilde{t}_{ji} \) are obviously preserved.

Let us check that \( [\tilde{x}_i, \tilde{y}_j] = \tilde{t}_{ij} \) is preserved. We have for \( i \neq j \),

\[
\rho_{\mathfrak{g}, h}(\tilde{x}_i), \rho_{\mathfrak{g}, h}(\tilde{y}_j) = \rho_{\mathfrak{g}, h}(\tilde{t}_{ij}), -\tilde{h}_i \partial_{\nu} \sum_k r(\lambda)^{ik} \]

which vanishes by the same argument as above and \( \sum_i \rho_{\mathfrak{g}, h}(\tilde{y}_i) = \sum_i (\lambda^V)^i \) (by the antisymmetry of \( r(\lambda) \)), which vanishes by the same argument as above.
Let us check that \([\vec{y}_i, \vec{y}_j] = 0\) is preserved, for \(i \neq j\). We have

\[
[\rho_{\mathfrak{g}, \mathfrak{h}}(\vec{y}_i), \rho_{\mathfrak{g}, \mathfrak{h}}(\vec{y}_j)] = \sum_{k \neq i, j} ( - h^i_\nu(\partial_\nu r(\lambda))^{ik} + [r(\lambda)^{ij}, r(\lambda)^{jk}] + [r(\lambda)^{ik}, r(\lambda)^{jk}] + [r(\lambda)^{ik}, r(\lambda)^{ij}])
\]

\[
+ [(h^i_\nu + h^i_{\overline{\nu}})\partial_\nu, r(\lambda)^{ij}] + [h^i_\nu, \partial_\nu, r(\lambda)^{ij}] + [h^i_{\overline{\nu}}, \partial_\nu, r(\lambda)^{ij}] + [r(\lambda)^{ij}, r(\lambda)^{ij}] + [r(\lambda)^{ij}, r(\lambda)^{ij}]
\]

\[
= (\sum_{k \neq i, j} h^k_\nu(\partial_\nu r(\lambda))^{ij}) + [(h^i_\nu + h^i_{\overline{\nu}})\partial_\nu, r(\lambda)^{ij}] + [h^i_\nu, \partial_\nu, r(\lambda)^{ij}] + [h^i_{\overline{\nu}}, \partial_\nu, r(\lambda)^{ij}] + [r(\lambda)^{ij}, r(\lambda)^{ij}]
\]

\[
\equiv (\partial_\nu r(\lambda))^{ij} (h^i_\nu - h^i_{\overline{\nu}} - \vec{X}_\nu) + [(h^i_\nu + h^i_{\overline{\nu}})\partial_\nu, r(\lambda)^{ij}] - h^i_\nu(\partial_\nu r(\lambda))^{ij} + h^i_{\overline{\nu}}(\partial_\nu r(\lambda))^{ij}
\]

\[
+ [r(\lambda)^{ij}, r(\lambda)^{ij}] + [r(\lambda)^{ij}, r(\lambda)^{ij}] + [h^i_\nu, \partial_\nu, r(\lambda)^{ij}] + h^i_{\overline{\nu}}(\partial_\nu r(r(\lambda)^{ij})) + [r(\lambda)^{ij}, r(\lambda)^{ij}].
\]

The second equality follows from the CDYBE and the antisymmetry on \(r(\lambda)\). Then

\[
[h^i_\nu + h^i_{\overline{\nu}}, r(\lambda)]\partial_\nu - (\partial_\nu r(\lambda))^{ij}\vec{X}_\nu = [(h^i_\nu + h^i_{\overline{\nu}}, r(\lambda)^{ij}) - \partial_\nu r^{ij}(\lambda, \lambda, [h^i_\nu, h^i_{\overline{\nu}}])]\partial_\nu = 0
\]

by the \(\mathfrak{h}\)-invariance of \(r(\lambda)\). Applying \(x^i y^j z^k \rightarrow x^i(yz)^j\) to the CDYBE identity

\[
[r(\lambda)^{ij}, r(\lambda)^{jk}] + [r(\lambda)^{ij}, r(\lambda)^{jk}] + [h^i_\nu, \partial_\nu, r(\lambda)^{jk}] + h^i_{\overline{\nu}}(\partial_\nu r(r(\lambda)^{jk})) + h^i_{\overline{\nu}}(\partial_\nu r(r(\lambda)^{ij})) = 0,
\]

we get

\[
(1/2) \sum_{\alpha, \beta} e_i e_{\beta}(\lambda)[a_{\alpha}, b_{\beta}]^i + [\alpha(\lambda)^{ij}, \lambda(\lambda)^{ij}] + [h^i_\nu, \partial_\nu, r(\lambda)^{ij}] + [h^i_{\overline{\nu}}, \partial_\nu, r(\lambda)^{ij}] = 0.
\]

Since \(r(\lambda)\) is antisymmetric, the sum \((1/2) \sum_{\alpha, \beta} \ldots\) is symmetric in \((i, j)\); antisymmetrizing in \((i, j)\), we get

\[
[h^i_\nu + h^i_{\overline{\nu}}, r(\lambda)]\partial_\nu - (\partial_\nu r(\lambda))^{ij}\vec{X}_\nu = [(h^i_\nu + h^i_{\overline{\nu}}, r(\lambda)^{ij}) - \partial_\nu r^{ij}(\lambda, \lambda, [h^i_\nu, h^i_{\overline{\nu}}])\partial_\nu = 0
\]

All this implies that \([\rho_{\mathfrak{g}, \mathfrak{h}}(\vec{y}_i), \rho_{\mathfrak{g}, \mathfrak{h}}(\vec{y}_j)] = 0\).

Let us check that \([\vec{x}_i, t_{jk}] = 0\) is preserved \((i, j, k\) distinct). We have \([\rho_{\mathfrak{g}, \mathfrak{h}}(\vec{x}_i), \rho_{\mathfrak{g}, \mathfrak{h}}(t_{jk})] = [(\lambda^{(ij)}], t_{\mathfrak{g}, k}^j] = 0\).

Let us prove that \([\vec{y}_i, t_{jk}] = 0\) is preserved \((i, j, k\) distinct). We have \([\rho_{\mathfrak{g}, \mathfrak{h}}(\vec{y}_i), \rho_{\mathfrak{g}, \mathfrak{h}}(t_{jk})] = [\vec{x}_i, t_{\mathfrak{g}, k}] = 0\) because \(t_{\mathfrak{g}}\) is \(\mathfrak{g}\)-invariant. \(\square\)

**Proposition 6.4.** If \(V_1, \ldots, V_n\) are \(\mathfrak{g}\)-modules, then \((S(\mathfrak{h})[1/P] \otimes (\otimes_i V_i))^\mathfrak{h}\) is a \(\mathfrak{t}_{1, n} \rtimes \mathfrak{d}\)-module. The \(\mathfrak{t}_{1, n}\)-module structure is induced by the morphism \(\mathfrak{t}_{1, n} \rightarrow \mathcal{H}(\mathfrak{g}, \mathfrak{h})\) of Proposition 6.3, so

\[
\rho_{(V_1)}(\vec{x}_i)(f(\lambda) \otimes (\otimes_i v_i)) = (\lambda^{(ij)})(f(\lambda) \otimes (\otimes_i v_i)),
\]

\[
\rho_{(V_1)}(\vec{y}_i)(f(\lambda) \otimes (\otimes_i v_i)) = (-h^i_\nu, \partial_\nu + \sum_j r(\lambda)^{ij})(f(\lambda) \otimes (\otimes_i v_i)),
\]

\[
\rho_{(V_1)}(\vec{t}_{ij})(f(\lambda) \otimes (\otimes_i v_i)) = t^j_{\mathfrak{g}}(f(\lambda) \otimes (\otimes_i v_i)),
\]

and the \(\mathfrak{d}\)-module structure is given by

\[
\rho_{(V_1)}(\delta_{2m})(f(\lambda) \otimes (\otimes_i v_i)) = \frac{1}{2} \sum_i ((\text{ad } \lambda^{(ij)})^m(\epsilon_\alpha) \cdot \epsilon_\alpha)(f(\lambda) \otimes (\otimes_i v_i)),
\]

\[
\rho_{(V_1)}(\Delta_0)(f(\lambda) \otimes (\otimes_i v_i))
\]

\[
= \left( -\frac{1}{2} \partial_\nu + \frac{1}{2} (\mu(r(\lambda)), h_\nu)\partial_\nu + \frac{1}{2} \psi(\lambda)^{11} - \frac{1}{2} (\text{ad } \lambda^{(ij)})^{-1}(\mu(r(\lambda)))_{m}^n \right)(f(\lambda) \otimes (\otimes_i v_i)),
\]

\[
\rho_{(V_1)}(\mathfrak{d})(f(\lambda) \otimes (\otimes_i v_i)) = \frac{1}{2} ((\lambda, h_\nu)\partial_\nu + \partial_\nu (\lambda, h_\nu) + (\mu(r(\lambda)), \lambda^{(ij)})(f(\lambda) \otimes (\otimes_i v_i)),
\]

\[
\rho_{(V_1)}(\mathfrak{x})(f(\lambda) \otimes (\otimes_i v_i)) = (1/2)(\lambda^{(ij)}, \lambda^{(ij)})(f(\lambda) \otimes (\otimes_i v_i)).
\]

Here \(x_n\) is the projection of \(x \in \mathfrak{g}\) on \(n\) along \(\mathfrak{h}\).
To summarize, we have a diagram

\[
\begin{array}{ccc}
\tilde{\mathbf{t}}_{1,n} & \rightarrow & \mathcal{H}_n(\mathfrak{g}, \mathfrak{h}) \\
\subset & \cong (1) & \rightarrow \\
\mathbf{t}_{1,n} & \times & \mathfrak{d}
\end{array}
\]

As before, the restriction morphism \( (S(\mathfrak{g}) \otimes (\otimes_i V_j))^g \rightarrow (S(\mathfrak{h})[1/P] \otimes (\otimes_i V_j))^h \) extends to a \( \mathbf{t}_{1,n} \times \mathfrak{d} \)-modules morphism.

The action of \( \mathbf{t}_{1,n} \times \mathfrak{d} \) factors through a morphism \( \tilde{\rho}_{\mathfrak{g}, \mathfrak{h}} : \tilde{\mathbf{t}}_{1,n} \times \mathfrak{d} \rightarrow \mathcal{H}_n(\mathfrak{g}, \mathfrak{h}) \) extending \( \rho_{\mathfrak{g}, \mathfrak{h}} : \mathbf{t}_{1,n} \rightarrow \mathcal{H}_n(\mathfrak{g}, \mathfrak{h}) \) (denoted by (1) in the diagram).

**Proof.** Let \( \lambda \in \mathfrak{h}^*_\text{reg} \). Then if \( V \) is a \( \mathfrak{g} \)-module, we have \( (\hat{\mathcal{O}}_{\mathfrak{g}^*, \lambda} \otimes V)^g = (\hat{\mathcal{O}}_{\mathfrak{b}^*, \lambda} \otimes V)^h \) (where \( \hat{\mathcal{O}}_{X,x} \) is the completed local ring of a variety \( X \) at the point \( x \)). We then have a morphism \( \tilde{\mathbf{t}}_{1,n} \times \mathfrak{d} \rightarrow \mathcal{H}_n(\mathfrak{g}) \rightarrow \text{End}(\mathcal{H}_n(\mathfrak{g}, \mathfrak{h})^g) \) for any \( \lambda \in \mathfrak{g}^* \), so when \( \lambda \in \mathfrak{h}^*_\text{reg} \) we get a morphism \( \tilde{\mathbf{t}}_{1,n} \times \mathfrak{d} \rightarrow \text{End}(\mathcal{H}_n(\mathfrak{g}, \mathfrak{h}))^h \).

Let show that the images of the generators of \( \tilde{\mathbf{t}}_{1,n} \times \mathfrak{d} \) under this morphism are given by the above formulas.

Since the actions of \( \tilde{x}_i, \tilde{t}_i \) and \( X \) on \( (\hat{\mathcal{O}}_{\mathfrak{g}^*, \lambda} \otimes (\otimes_i V_j))^g \) are given by multiplication by elements of \( (\hat{\mathcal{O}}_{\mathfrak{b}^*, \lambda} \otimes U(\mathfrak{g}^{\otimes n}))^g \), their actions on \( (\hat{\mathcal{O}}_{\mathfrak{b}^*, \lambda} \otimes (\otimes_i V_j))^h \) are given by multiplication by restrictions of these elements to \( \mathfrak{h}^* \).

Let us compute the action of \( \tilde{y}_i \). Let \( \tilde{f}(\lambda) \in (\hat{\mathcal{O}}_{\mathfrak{b}^*, \lambda} \otimes (\otimes_i V_j))^h \) and \( \tilde{F}(\lambda) \in (\hat{\mathcal{O}}_{\mathfrak{b}^*, \lambda} \otimes (\otimes_i V_j))^g \) be its equivariant extension to a formal map \( \mathfrak{g}^* \rightarrow \otimes_i V_j \). Then for \( x \in \mathfrak{n} \), we have \( (\partial_{x^\lambda} + \sum_i (ad x^\lambda)^{-1}(x^1))(\tilde{F}(\lambda))|_{\mathfrak{b}^*} = 0 \) (the map \( x \mapsto x^\lambda \) is the inverse of \( \mathfrak{g}^* \rightarrow \mathfrak{g}, \lambda \mapsto \lambda^\vee \)). Then \( \rho_{(V)}(\tilde{y}_i)(\tilde{f}(\lambda)) = \left( -h_{ij}^i \partial_{\partial_{x^\lambda}} + \sum_j e_j^\beta (ad x^\lambda)^{-1}(e_{\beta}) \right) \tilde{f}(\lambda) = (-h_{ij}^i \partial_{\partial_{x^\lambda}} + \sum_j r(\lambda)) (\tilde{f}(\lambda)). \)

Let us now compute the action of \( \Delta_0 \). Let \( \lambda_0 \in \mathfrak{h}^* \) be such that \( \lambda_0^\vee \in U \) and \( \lambda \in \mathfrak{g}^* \) be close to \( \lambda_0 \). We set \( \delta \lambda := \lambda - \lambda_0 \). We then have \( \lambda = e^{ad x}(\lambda_0 + \delta \lambda) \), where \( x \in \mathfrak{n} \) and \( h \in \mathfrak{h} \) are close to \( 0 \). We have the expansions

\[
h = (\delta \lambda)^\vee + \frac{1}{2}[(ad \lambda_0^\vee)^{-1}(\delta \lambda)^\vee, (\delta \lambda)^\vee],
\]

\[
x = -(ad \lambda_0^\vee)^{-1}(\delta \lambda)^\vee + [(ad \lambda_0^\vee)^{-1}(\delta \lambda)^\vee, (\delta \lambda)^\vee] + \frac{1}{2}[(ad \lambda_0^\vee)^{-1}((\delta \lambda)^\vee), (\delta \lambda)^\vee]
\]

up to terms of order \( > 2 \); here the indices \( u_0 \) and \( u_h \) mean the projections of \( u \in \mathfrak{g} \) to \( \mathfrak{n} \) and \( \mathfrak{h} \). If now \( \tilde{f}(\lambda) : \mathfrak{g}^* \supset V(\lambda_0, \mathfrak{h}^*) \rightarrow \otimes_i V_j \) is an \( \mathfrak{h} \)-equivariant function defined at the vicinity of \( \lambda_0 \) and \( \tilde{F}(\lambda) : \mathfrak{g}^* \supset V(\lambda_0, \mathfrak{g}^*) \rightarrow \otimes_i V_j \) its \( \mathfrak{g} \)-equivariant extension to a neighborhood of \( \lambda_0 \) in \( \mathfrak{g}^* \), then \( \tilde{F}(\lambda) = (e^{ad x})(\lambda_0 + \delta \lambda) \), which implies the expansion

\[
\tilde{F}(\lambda) = \tilde{f}(\lambda_0) + \left( (\delta \lambda)^\vee \right. + \frac{1}{2}([(ad \lambda_0^\vee)^{-1}(e_{\beta}), e_{\beta'}], h_{\beta'}) (\delta \lambda)^\vee (\delta \lambda)_{\beta} (\delta \lambda)_{\beta'} \partial_{\partial_{x^\lambda}} \tilde{f}(\lambda_0) + \left. \frac{1}{2} (\delta \lambda)^\vee (\delta \lambda)_{\beta} \partial_{\partial_{x^\lambda}}^2 \tilde{f}(\lambda_0) \right) + \left( - (ad \lambda_0^\vee)^{-1}(e_{\beta}) \right. (\delta \lambda)^\vee (\delta \lambda)_{\beta} + \left. (ad \lambda_0^\vee)^{-1}(e_{\beta}) \right) \tilde{f}(\lambda_0)
\]

up to terms of order \( > 2 \).

Then

\[
(\partial_{\partial_{x^\lambda}}^2 \tilde{F})(\lambda_0) = (\partial_{\partial_{x^\lambda}}^2 \tilde{f})(\lambda_0) + \left( - (ad \lambda_0^\vee)^{-1}(e_{\beta}) \right. (\delta \lambda)^\vee (\delta \lambda)_{\beta} + \left. (ad \lambda_0^\vee)^{-1}(e_{\beta}) \right) \tilde{f}(\lambda_0)
\]

which implies the formula for the action of \( \Delta_0 \).
Then $(S(b)[1/P] \otimes (\otimes_i V_i))^b \subset \prod_{\lambda \in \mathfrak{g}^{\text{reg}}_{\text{reg}}} (\mathcal{O}_{\mathfrak{h}^*} \otimes (\otimes_i V_i))^b$ is preserved by the action of the generators of $\mathfrak{g}_{1,n} \times \mathfrak{d}$-module, hence it is a sub-$\mathfrak{g}_{1,n} \times \mathfrak{d}$-module, with action given by the above formulas.

6.4. Realization of the universal KZB system. The realization of the flat connection $d - \sum_i K_i(z|\tau) d z_i - \Delta(z|\tau) d \tau$ on $(\mathfrak{g} \times \mathbb{C}^n) - \text{Diag}_n$ is a flat connection on the trivial bundle with fiber $(\mathcal{O}_{\mathfrak{h}^{\text{reg}}} \otimes (\otimes_i V_i))^b$.

We now compute this realization, under the assumption that $\mathfrak{h} \subset \mathfrak{g}$ is a maximal abelian subalgebra. In this case, two simplifications occur:

(a) $[(\text{ad} \lambda|^\vee)^{-1}(\epsilon_\beta), \epsilon_\beta|_n] = 0$ since $[(\text{ad} \lambda^{-1}(\epsilon_\beta), \epsilon_\beta)]$ commutes with any element in $\mathfrak{h}$, so that it belongs to $\mathfrak{h}$.

The image of $K_i(z|\tau)$ is then the operator

$$K_i^{(V)}(z|\tau) = h_i' \partial_\nu - \sum_j r(\lambda)^{ij} + \sum_{j \neq i} k(z_{ij}, (\text{ad} \lambda|\tau)_n^i) t_n^{ij}$$

$$= h_i' \partial_\nu - r(\lambda)^{ii} + \sum_{j \neq i} \frac{\theta(z_{ij} + (\text{ad} \lambda|\tau)_n^i)}{\theta(z_{ij}|\tau)} t_n^{ij} + \sum_{j \neq i} \frac{\theta'}{\theta}(z_{ij}|\tau) t_n^{ij}$$

The image of $2\pi i \Delta(z|\tau)$ is the operator

$$2\pi i \Delta^{(V)}(z|\tau) = \frac{1}{2} \partial_\nu^2 + \frac{1}{2} ([(\text{ad} \lambda|\tau)_n^i(\epsilon_\beta), \epsilon_\beta], h_\nu) \partial_\nu - g(0, 0|\tau) \sum_i \frac{1}{2} t_n^{ii}$$

$$+ \sum_{i,j} \frac{1}{2} \text{tr}[(g(z_{ij}) + (\text{ad} \lambda|\tau)_n^i(\epsilon_\beta)) t_n^{ij} + \sum_i \frac{1}{2} g(z_{ij}, 0|\tau) h_i^j h_\nu^j$$

and the connection is now

$$\nabla^{(V)} = d - \sum_i K_i^{(V)}(z|\tau) - \Delta^{(V)}(z|\tau).$$

Recall that $P(\lambda) = \text{det}((\text{ad} \lambda|\tau)_n).$ We compute the conjugation $P^{1/2} \nabla^{(V)} P^{-1/2}$, where $P^{\pm 1/2}$ is the operator of multiplication by (inverse branches of) $P^{\pm 1/2}$ on $(\mathcal{O}_{\mathfrak{h}^{\text{reg}}} \otimes (\otimes_i V_i))^b$.

Lemma 6.5. $\partial_\nu \log P(\lambda) = -\langle h_\nu, \mu(\lambda) \rangle$, $P^{1/2} h_\nu^i \partial_\nu - r(\lambda)^{ii} P^{-1/2} = h_i' \partial_\nu, P^{1/2} \partial_\nu^2 + ([(\text{ad} \lambda|\tau)_n^i(\epsilon_\beta), \epsilon_\beta], h_\nu) \partial_\nu P^{-1/2} = \partial_\nu^2 + \partial_\nu \langle h_\nu, \frac{1}{2} \mu(\lambda) \rangle - \langle h_\nu, \frac{1}{2} \mu(\lambda) \rangle^2$.

Proof. $\partial_\nu \log P(\lambda) = (d/dt)|_{t=0} \text{det}[(\text{ad} \lambda + h_\nu|\tau)_n(\text{ad} \lambda|\tau)_n^{-1}] = \text{tr}[(\text{ad} h_\nu, \omega(\text{ad} \lambda|\tau)_n^{-1}] = \langle \epsilon_\beta, (\text{ad} h_\nu) \circ (\text{ad} \lambda|\tau)_n^{-1}(\epsilon_\beta) \rangle = \langle (\text{ad} \lambda|\tau)_n^{-1}(\epsilon_\beta), \epsilon_\beta \rangle = -\langle h_\nu, \mu(\lambda) \rangle$. The next equality follows from $\mu(\lambda)^i = 2 \lambda \lambda^i$. The last equality is a direct consequence.

We then get:

Proposition 6.6. $P^{1/2} \nabla^{(V)} P^{-1/2} = d - \sum_i \tilde{K}_i(z|\tau) d z_i - \tilde{\Delta}(z|\tau) d \tau$, where

$$\tilde{K}_i(z|\tau) = h_i' \partial_\nu + \sum_{j \neq i} \frac{\theta(z_{ij} + (\text{ad} \lambda|\tau)_n^i)}{\theta(z_{ij}|\tau)} t_n^{ij} + \sum_{j \neq i} \frac{\theta'}{\theta}(z_{ij}|\tau) t_n^{ij}$$

$$2\pi i \tilde{\Delta}(z|\tau) = \frac{1}{2} \partial_\nu^2 + \partial_\nu \langle h_\nu, \frac{1}{2} \mu(\lambda) \rangle - \langle h_\nu, \frac{1}{2} \mu(\lambda) \rangle^2 - g(0, 0|\tau) \sum_i \frac{1}{2} t_n^{ii}$$

$$+ \sum_{i,j} \frac{1}{2} \text{tr}[(g(z_{ij}) - (\text{ad} \lambda|\tau)_n^{-2}(\epsilon_\beta)) t_n^{ij} + \sum_i \frac{1}{2} g(z_{ij}, 0|\tau) h_i^j h_\nu^j,$$

where

$$g(z, 0|\tau) = \frac{1}{2} \frac{\theta'}{\theta}(z|\tau) - 2\pi i \frac{\partial_\eta}{\eta}(\tau)$$
and
\[ g(z, \alpha|\tau) - \alpha^{-2} = \frac{1}{2} \theta(z + \alpha|\tau) \left( \frac{\theta'(z + \alpha|\tau) - \theta'(\alpha|\tau)}{\theta(\alpha|\tau)} \right) \]

The term in \( \sum_i (1/2)t_i^{t_i} \) is central and can be absorbed by a suitable further conjugation. Rescaling \( t_i \) to \( \kappa^{-1} \), where \( \kappa \in \mathbb{C}^\times \), \( \tilde{K}_t(z|\tau) \) and \( \tilde{\Delta}(z|\tau) \) get multiplied by \( \kappa \). Moreover, we have:

**Lemma 6.7.** When \( g \) is simple and \( h \subset g \) is the Cartan subalgebra, \( \partial_x \{ (h_{\nu}, \frac{1}{2} \mu(r(\lambda))) \} = \langle h_{\nu}, \frac{1}{2} \mu(r(\lambda)) \rangle^2 \).

**Proof.** Let \( D(\lambda) := \prod_{\alpha \in \Delta^+} (\alpha, \lambda) \), where \( \Delta^+ \) is the set of positive roots of \( g \). Then \( D(\lambda) \) is \( W \)-antiinvariant, where \( W \) is the Weyl group. Therefore \( \partial_\nu^2 D(\lambda) \) is also \( W \)-antiinvariant, so it is divisible (as a polynomial on \( h^* \)) by all the \( (\alpha, \lambda) \), where \( \alpha \in \Delta^+ \), so it is divisible by \( D(\lambda) \); since \( \partial_\nu^2 D(\lambda) \) has degree strictly lower than \( D(\lambda) \), we get \( \partial_\nu^2 D(\lambda) = 0 \).

Now if \( (e_\alpha, f_\alpha, h_\alpha) \) is a basis of the \( \mathfrak{sl}_2 \)-triple associated with \( \alpha \), we have \( r(\lambda) = \sum_{\alpha \in \Delta^+} - (e_\alpha \otimes f_\alpha - f_\alpha \otimes e_\alpha)/(\alpha, \lambda) \), so \( \frac{1}{2} \mu(r(\lambda)) = - \sum_{\alpha \in \Delta^+} h_\alpha/(\alpha, \lambda) \). Therefore \( \frac{1}{2} \mu(r(\lambda)) = - \partial_\nu \log D(\lambda) h_{\nu} \).

Then \( \partial_\nu^2 D(\lambda) = 0 \) implies that \( \partial_\nu^2 \log D + (\partial_\nu \log D)^2 = 0 \), which implies the lemma.

The resulting flat connection then coincides with that of [Be1, FW].

7. The universal KZB connection and representations of Cherednik algebras

7.1. The rational Cherednik algebra of type \( A_{n-1} \). Let \( k \) be a complex number, and \( n \geq 1 \) an integer. The rational Cherednik algebra \( H_n(k) \) of type \( A_{n-1} \) is the quotient of the algebra \( \mathbb{C}[S_n] \times \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n] \) by the relations
\[
\sum_i x_i = 0, \quad \sum_i y_i = 0, \quad [x_i, x_j] = 0 = [y_i, y_j],
\]
\[
[x_i, y_j] = \frac{1}{n} - ks_{ij}, \quad i \neq j,
\]
where \( s_{ij} \in S_n \) is the permutation of \( i \) and \( j \) (see e.g. [EG]).

Let \( e := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \mathbb{C}[S_n] \) be the Young symmetrizer. The spherical subalgebra \( B_n(k) \) (often called the spherical Cherednik algebra) is defined to be the algebra \( e H_n(k) e \).

We define an important element
\[
h := \frac{1}{2} \sum_i (x_i y_i + y_i x_i).
\]

We recall that category \( \mathcal{O} \) is the category of \( H_n(k) \)-modules which are locally nilpotent under the action of the operators \( x_i \), and decompose into a direct sum of finite dimensional generalized eigenspaces of \( h \). Similarly, one defines category \( \mathcal{O} \) over \( B_n(k) \) to be the category of \( B_n(k) \)-modules which are locally nilpotent under the action of \( \mathbb{C}[y_1,\ldots,y_n]^{S_n} \) and decompose into a direct sum of finite dimensional generalized eigenspaces of \( h \).

7.2. The homomorphism from \( \tilde{t}_{1,n} \) to the rational Cherednik algebra.

**Proposition 7.1.** For each \( k, a, b \in \mathbb{C} \), we have a homomorphism of Lie algebras \( \xi_{a,b} : \tilde{t}_{1,n} \to H_n(k) \), defined by the formula
\[
\begin{align*}
\tilde{x}_i & \mapsto ax_i, \\
\tilde{y}_i & \mapsto by_i, \\
\tilde{t}_{ij} & \mapsto ab \left( \frac{1}{n} - ks_{ij} \right).
\end{align*}
\]

**Proof.** Straightforward.

---

\(^5\) The generators \( x_\alpha, \partial_\alpha \) of Section 6.1 will be henceforth renamed \( q_\alpha, p_\alpha \).
Remark 7.2. Obviously, $a, b$ can be rescaled independently, by rescaling the generators $\bar{x}_i$ and $\bar{y}_i$ of the source algebra $\mathfrak{t}_{1,n}$. On the other hand, if we are only allowed to apply automorphisms of the target algebra $H_n(k)$, then $a, b$ can only be rescaled in such a way that the product $ab$ is preserved.

This shows that any representation $V$ of the rational Cherednik algebra $H_n(k)$ yields a family of realizations for $\mathfrak{t}_{1,n}$ parametrized by $a, b \in \mathbb{C}$, and gives rise to a family of flat connections $\nabla_{a,b}$ over the configuration space $\mathcal{C}(E_\tau, n)$.

7.3. Monodromy representations of double affine Hecke algebras. Let $\mathcal{H}_n(q, t)$ be Cherednik’s double affine Hecke algebra of type $A_{n-1}$. By definition, $\mathcal{H}_n(q, t)$ is the quotient of the group algebra of the orbifold fundamental group $\mathcal{B}_{1,n}$ of $\mathcal{C}(E_\tau, n)/S_n$ by the additional relations

$$(T - q^{-1}t)(T + q^{-1}t^{-1}) = 0,$$

where $T$ is any element of $\mathcal{B}_{1,n}$ homotopic (as a free loop) to a small loop around the divisor of diagonals in the counterclockwise direction.

Let $V$ be a representation of $H_n(k)$, and let $\nabla_{a,b}(V)$ be the universal connection $\nabla_{a,b}$ evaluated in $V$. In some cases, for example if $a, b$ are formal, or if $V$ is finite dimensional, we can consider the monodromy of this connection, which obviously gives a representation of $\mathcal{H}_n(q, t)$ on $V$, with

$q = e^{-2\pi i a b/n}, \quad t = e^{-2\pi i k a b}.$

In particular, taking $a = b$, $V = H_n(k)$, this monodromy representation defines an homomorphism $\theta_a : \mathcal{H}_n(q, t) \to H_n(k)[[a]]$, where

$q = e^{-2\pi i a^2/n}, \quad t = e^{-2\pi i k a^2}.$

It is easy to check that this homomorphism becomes an isomorphism upon inverting $a$. The existence of such an isomorphism was pointed out by Cherednik (see [Ch2], end of Section 6, and the end of [Ch1]), but his proof is different.

Example 7.3. Let $k = r/n$, where $r$ is an integer relatively prime to $n$. In this case, it is known (see e.g. [BEG1]) that the algebra $H_n(k)$ admits an irreducible finite dimensional representation $Y(r,n)$ of dimension $r^{n-1}$. By virtue of the above construction, the space $Y(r,n)$ carries an action of $\mathcal{H}_n(q, t)$ with any nonzero $q, t$ such that $q^r = t$. This finite dimensional representation of $\mathcal{H}_n(q, t)$ is irreducible for generic $q$, and is called a perfect representation; it was first constructed in [E], p. 500, and later in [Ch2], Theorem 6.5, in a greater generality.

7.4. The modular extension of $\xi_{a,b}$. Assume that $a, b \neq 0$.

Proposition 7.4. The homomorphism $\xi_{a,b}$ can be extended to the algebra $U(\mathfrak{t}_{1,n} \rtimes \mathfrak{a}) \rtimes S_n$ by the formulas

$$\xi_{a,b}(s_{ij}) = s_{ij},$$

$$\xi_{a,b}(d) = h = \frac{1}{2} \sum_i (x_i y_i + y_i x_i), \quad \xi_{a,b}(X) = -\frac{1}{2} a b^{-1} \sum_i x_i^2,$$

$$\xi_{a,b}(\Delta_0) = \frac{1}{2} b a^{-1} \sum_i y_i^2, \quad \xi_{a,b}(\delta_{2m}) = -\frac{1}{2} a^{2m-1} b^{-1} \sum_{i<j} (x_i - x_j)^{2m}.$$

Proof. Direct computation.}

Thus, the flat connections $\nabla_{a,b}$ extend to flat connections on $\mathcal{M}_{1,[n]}$.

This shows that the monodromy representation of the connection $\nabla_{a,b}(V)$, when it can be defined, is a representation of the double affine Hecke algebra $\mathcal{H}_n(q, t)$ with a compatible action of the extended modular group $\text{SL}_2(\mathbb{Z})$. In particular, this is the case if $V = Y(r,n)$. 


Remark is then that of Proposition 6.2, and the composition of the two last maps is the map Proposition 8.1. This implies the statement.

8. Explicit Realizations of Certain Highest Weight Representations of the Rational Cherednik Algebra of Type $A_{n-1}$

8.1. The representation $V_N$. Let $N$ be a divisor of $n$, and $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$, $G = SL_N(\mathbb{C})$. Let $V_N = (\mathbb{C}[\mathfrak{g}] \otimes (\mathcal{C}^N)^{\otimes n})^g$ (the divisor condition is needed for this space to be nonzero). It turns out that $V_N$ has a natural structure of a representation of $H_n(k)$ for $k = n/N$.

**Proposition 8.1.** We have a homomorphism $\zeta_N : H_n(N/n) \to \text{End}(V_N)$, defined by the formulas

$$\zeta_N(s_{ij}) = s_{ij}, \quad \zeta_N(x_i) = X_i, \quad \zeta_N(y_i) = Y_i, \quad (i = 1, \ldots, n)$$

where for $f \in V_N, A \in \mathfrak{g}$ we have

$$(X_i f)(A) = A_i f(A),$$

$$(Y_i f)(A) = \frac{N}{n} \sum_p (b_p)_i \frac{\partial f}{\partial b_p}(A),$$

where $\{b_p\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the trace form.

**Proof.** Straightforward verification. \qed

The relationship of the representation $V_N$ to other results in this paper is described by the following proposition.

**Proposition 8.2.** The connection $\nabla_{a,1}(V_N)$ corresponding to the representation $V_N$ is the usual KZB connection for the $n$-point correlation functions on the elliptic curve for the Lie algebra $\mathfrak{sl}_N$ and $n$ copies of the vector representation $\mathbb{C}^N$, at level $K = -\frac{a}{\mathfrak{g} N} - N$.

**Proof.** We have a sequence of maps

$$U(\mathfrak{t}_{1,n} \times \mathfrak{g}) \times S_n \to H_n(N/n) \to H_n(\mathfrak{g}) \times S_n \to \text{End}(V_N),$$

where the first map is $\xi_{a,b}$, the second map sends $s_{ij}$ to $s_{ij}, x_i$ to the class of $\sum_a q_a \otimes e^i_a$, and $y_i$ to the class of $\sum_a p_a \otimes e^i_a$ (recall that the $x_a, \partial_a$ of Section 6.1 have been renamed $q_a, p_a$), and the last map is explained in Section 6.1. The composition of the two first maps is then that of Proposition 6.2, and the composition of the two last maps is the map $\zeta_N$ of Proposition 8.1. This implies the statement. \qed

**Remark 8.3.** Suppose that $K$ is a nonnegative integer, i.e. $a = -\frac{n}{(K+1)n}$, where $k \in \mathbb{Z}_+$. Then the connection $\nabla_{a,1}$ on the infinite dimensional vector bundle with fiber $V_N$ preserves a finite dimensional subbundle of conformal blocks for the WZW model at level $K$. Th subbundle gives rise to a finite dimensional monodromy representation $V^{\mathcal{C}}_N$ of the Cherednik algebra $H_n(q,t)$ with

$$q = e^{\frac{2\pi i}{(K+1)n}}, \quad t = q^N$$

(so both parameters are roots of unity). The dimension of $V^{\mathcal{C}}_N$ is given by the Verlinde formula, and it carries a compatible action of $SL_2(\mathbb{Z})$ to the action of the Cherednik algebra. Representations of this type were studied by Cherednik in [Ch2].
8.2. The spherical part of $V_N$. Note that
\begin{equation}
((\sum_{i=1}^{n} X_i^p)f)(A) = \frac{n}{N}(\text{tr } A^p)f(A),
\end{equation}
\begin{equation}
((\sum_{i=1}^{n} Y_i^p)f)(A) = \left(\frac{N}{n}\right)^{p-1}(\text{tr } \partial_A^p)f(A)
\end{equation}

Consider the space $U_N = eV_N = (\mathbb{C}[\mathfrak{g}] \otimes S^n \mathbb{C}^N)^{g}$ as a module over the spherical subalgebra $B_n(k)$. It is known (see e.g. [BEG2]) that the spherical subalgebra is generated by the elements $(\sum x_i^p)e$ and $(\sum y_i^p)e$. Thus formulas (43,44) determine the action of $B_n(k)$ on $U_N$.

We note that by restriction to the set $\mathfrak{h}$ of diagonal matrices $\text{diag}(\lambda_1, \ldots, \lambda_N)$, and dividing by $\Delta^{n/N}$, where $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$, one identifies $U_N$ with $\mathbb{C}[\mathfrak{h}]^{S^N}$. Moreover, it follows from [EG] that formulas (43,44) can be viewed as defining an action of another spherical Cherednik algebra, namely $B_N(1/k)$, on $\mathbb{C}[\mathfrak{h}]^{S^N}$. Moreover, this representation is the symmetric part $W$ of the standard polynomial representation of $H_N(1/k)$, which is faithful and irreducible since $1/k = n/N$ is an integer ([GGOR]). In other words, we have the following proposition.

**Proposition 8.4.** There exists a surjective homomorphism $\phi : B_n(N/n) \to B_N(n/N)$, such that $\phi^*W = U_N$. In particular, $U_N$ is an irreducible representation of $B_n(N/n)$.

Proposition 8.4 can be generalized as follows. Let $0 \leq p \leq n/N$ be an integer. Consider the partition $\mu(p) = (n - p(N - 1), p, \ldots, p)$ of $n$. The representation of $\mathfrak{g}$ attached to $\mu(p)$ is $S^{n-pN}\mathbb{C}^N$.

Let $e(p)$ be a primitive idempotent of the representation of $S_n$ attached to $\mu(p)$. Let $U_N^p = e(p)V_N = (\mathbb{C}[\mathfrak{g}] \otimes S^{n-pN}\mathbb{C}^N)^{g}$. Then the algebra $e(p)H_n(N/n)e(p)$ acts on $U_N^p$, and the above situation of $U_N$ is the special case $p = 0$.

**Proposition 8.5.** There exists a surjective homomorphism $\phi_p : e(p)H_n(N/n)e(p) \to B_N(n/N - p)$, such that $\phi_p^*W = U_N^p$. In particular, $U_N^p$ is an irreducible representation of $B_n(N/n - p)$.

**Proof.** Similar to the proof of Proposition 8.4. □

**Example 8.6.** $p = 1, n = N$. In this case $e(p) = e_\omega = \frac{1}{n!} \sum_{\sigma \in S_n} e(\sigma)\sigma$, the antisymmetrizer, and the map $\phi_p$ is the shift isomorphism $e_\omega H_N(1)e_\omega \to eH_N(0)e_\omega$.

8.3. Coincidence of the two $\mathfrak{sl}_2$ actions. As before, let $\{b_p\}$ be an orthonormal basis of $\mathfrak{g}$ (under some invariant inner product). Consider the $\mathfrak{sl}_2$-triple
\begin{equation}
H = \sum b_p \frac{\partial}{\partial b_p} + \frac{\dim \mathfrak{g}}{2}
\end{equation}
(the shifted Euler field),
\begin{equation}
F = \frac{1}{2} \sum b_p^2, \quad E = \frac{1}{2} \Delta_{\mathfrak{g}}
\end{equation}
where $\Delta_{\mathfrak{g}}$ is the Laplace operator on $\mathfrak{g}$. Recall also (see e.g. [BEG2]) that the rational Cherednik algebra contains the $\mathfrak{sl}_2$-triple $h = \frac{1}{2} \sum (x_iy_i + y_ix_i), \quad e = \frac{1}{2} \sum y_i^2, \quad f = \frac{1}{2} \sum x_i^2$.

The following proposition shows that the actions of these two $\mathfrak{sl}_2$ algebras on $V_N$ essentially coincide.

**Proposition 8.7.** On $V_N$, one has
\begin{equation}
h = H, \quad e = \frac{N}{n}E, \quad f = \frac{n}{N}F.
\end{equation}

**Proof.** The last two equations follow from formulas (43,44), and the first one follows from the last two by taking commutators. □
8.4. The irreducibility of $V_N$. Let $\Delta(n,N)$ be the representation of the symmetric group $S_n$ corresponding to the rectangular Young diagram with $N$ rows (and correspondingly $n/N$ columns), i.e. to the partition $(\frac{n}{N}, \frac{n}{N}, \ldots)$; e.g., $\Delta(n,1)$ is the trivial representation.

For a representation $\pi$ of $S_n$, let $L(\pi)$ denote the irreducible lowest weight representation of $H_n(k)$ with lowest weight $\pi$.

**Theorem 8.8.** The representation $V_N$ is isomorphic to $L(\Delta(n,N))$.

**Proof.** The representation $V_N$ is graded by the degree of polynomials, and in degree zero we have $V_N[0] = ((\mathbb{C}^N)^\otimes n)^0 = \Delta(n,N)$ by the Weyl duality.

Let us show that the module $V_N$ is semisimple. It is sufficient to show that $V_N$ is a unitary representation, i.e. admits a positive definite contravariant Hermitian form. Such a form can be defined by the formula

$$(f,g) = \langle f(\partial A), g(A) \rangle|_{A=0},$$

where $\langle -,- \rangle$ is the Hermitian form on $(\mathbb{C}^N)^\otimes n$ obtained by tensoring the standard forms on the factors. This form is obviously positive definite, and satisfies the contravariance properties:

$$(Y_if,g) = \frac{N}{n} (f,X_ig), \quad (f,Y_jg) = \frac{N}{n} (X_if,g).$$

The existence of the form $\langle -,- \rangle$ implies the semisimplicity of $V_N$. In particular, we have a natural inclusion $L(\Delta(n,N)) \subset V_N$.

Next, formula (43) implies that $V_N$ is a torsion-free module over $R := \mathbb{C}[x_1, \ldots, x_N]^{S_N} = \mathbb{C}[\sum_{i=1}^{N} x_i^p, 2 \leq p \leq N]$. Since $V_N$ is semisimple, this implies that $V_N/L(\Delta(n,N))$ is torsion-free as well.

On the other hand, we will now show that the quotient $V_N/L(\Delta(n,N))$ is a torsion module over $R$. This will imply that the quotient is zero, as desired.

Let $v_1, \ldots, v_N$ be the standard basis of $\mathbb{C}^N$, and for each sequence $J = (j_1, \ldots, j_N), \ j_i \in \{1, \ldots, N\}$, let $v_J := v_{j_1} \otimes \cdots \otimes v_{j_N}$. Let us say that a sequence $J$ is balanced if it contains each of its members exactly $n/N$ times. Let $B$ be the set of balanced sequences. The set $B$ has commuting left and right actions $S_N$ and $S_n$, $\sigma \cdot (j_1, \ldots, j_N) = (\sigma(j_1), \ldots, \sigma(j_N))$.

Let $J_0 = (1\ldots1,2\ldots2, \ldots, N\ldots N)$, then any $J \in B$ has the form $J = J_0 \ast \tau$ for some $\tau \in S_n$.

Let $f \in V_N$. Then $f$ is a function $\mathfrak{h} \rightarrow ((\mathbb{C}^N)^\otimes n)^\mathfrak{h}$, equivariant under the action of $S_N$ (here $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra, so $\mathfrak{h} = \{(\lambda_1, \ldots, \lambda_N) | \sum_1 \lambda_i = 0\}$), so

$$f(\lambda) = \sum_{J \in B} f_J(\lambda)v_J,$$

(47)

where $\lambda = (\lambda_1, \ldots, \lambda_N)$, and $f_J$ are scalar functions (the summation is over $B$ since $f(\lambda)$ must have zero weight). By the $S_N$-invariance, we have $f_{\sigma J}(\sigma(\lambda)) = f_J(\lambda)$. We then decompose $f(\lambda) = \sum_{\sigma \in S_N} \sum_{J \in B} f_\sigma(\lambda) v_J$, where $f_\sigma(\lambda) = \sum_{J \in B} f_J(\lambda)v_J$.

For each $o \in S_N \setminus B$, we construct a nonzero $\phi_o \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\phi_o \cdot f_0(\lambda) \in L(\Delta(n,N))$. Then $\phi := \prod_{o \in S_N \setminus B} \prod_{\sigma \in S_N} \sigma(\phi_o) \in R$ is nonzero and such that $\phi \cdot f(\lambda) \in L(\Delta(n,N))$.

We first construct $\phi_o$ when $o = o_0$, the class of $J_0$. By $S_N$-invariance, $f_{o_0}(\lambda)$ has the form

$$f_{o_0}(\lambda) = \sum_{\sigma \in S_N} g(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})v_{\sigma(1)}^{\otimes n/N} \otimes \cdots \otimes v_{\sigma(N)}^{\otimes n/N},$$

where $g(\lambda, \ldots, \lambda_N) \in \mathbb{C}[\lambda_1, \ldots, \lambda_N]$.

For $\phi_{o_0} \in \mathbb{C}[x_1, \ldots, x_N]$, we have

$$\phi_{o_0} \cdot f_{o_0}(\lambda) = \sum_{\sigma \in S_N} (\phi_{o_0}g)(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})v_{\sigma(1)}^{\otimes n/N} \otimes \cdots \otimes v_{\sigma(N)}^{\otimes n/N}.$$

(48)

On the other hand, let $v \in \Delta(n,N)$; expand $v = \sum_{J \in B} c_J v_J$. One checks that $v$ can be chosen such that $c_{J_0} \neq 0$ (one starts with a nonzero vector $v'$ and $J' \in B$ such that the
coordinate of \(v'\) along \(J'\) is nonzero, and then acts on \(v'\) by an element of \(S_n\) bringing \(J'\) to \(J_0\). Then since \(v\) is \(g\)-invariant (and therefore \(S_N\)-invariant), we have

\[
c_{\sigma(1)}\cdots c_{\sigma(1)}c_{\sigma(N)}c_{\sigma(N)} = c_{J_0}
\]

for any \(\sigma \in S_N\).

If \(Q \in \mathbb{C}[x_1, \ldots, x_n]\), then

\[
(Q \cdot v)(\lambda) = \sum_{(j_1, \ldots, j_n) \in B} c_{j_1 \cdots j_n} Q(\lambda_j_1, \ldots, \lambda_j_n) v_{j_1} \otimes \cdots \otimes v_{j_n} \in L(\Delta(n, N)).
\]

Set \(Q_0(\lambda_1, \ldots, \lambda_n) := \prod_{1 \leq a < b \leq n, j_a \neq j_b} (\lambda_a - \lambda_b)\), where \(J_0 = (1, \ldots, n; N = (j_1^0, \ldots, j_n^0)\),

\[
q_0(\lambda_1, \ldots, \lambda_N) := Q_0(\lambda_1, \ldots, \lambda_N, 0), \quad \text{so} \quad q_0(\lambda_1, \ldots, \lambda_N) = (\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j))^{(n/N)^2}.
\]

Then (48) and (50) coincide, as: (a) for \(J \notin o_0\), \(Q_0(\lambda_1, \ldots, \lambda_n) = 0\) so the coefficient of \(v_J\) in both expressions is zero, (b) the coefficients of \(v_{J_0}\) in both expressions coincide, (c) for \(J \in o_0\), the coefficients of \(v_J\) coincide because of (b) and of (49). The functions \(\phi_o\) are constructed in the same way for a general \(o \in S_N \setminus B\). This ends the proof of the theorem.

\[\square\]

**Remark 8.9.** Theorem 8.8 is a special case of a much more general (but much less elementary) Theorem 9.8, which is proved below.

8.5. **The character formula for** \(V_N\). For each partition \(\mu\) of \(n\), let \(V(\mu)\) be the representation of \(g\), and \(\pi(\mu)\) the representation of \(S_n\) corresponding to \(\mu\).

Let \(P_\mu(q)\) be the \(q\)-analogue of the weight multiplicity of the zero weight in \(V(\mu)\). Namely, we have a filtration \(F^i\) on \(V(\mu)[0]\) such that \(F^0\) is the space of vectors in \(V(\mu)[0]\) killed by the \(i\)-th power of the principal nilpotent element \(\sum e_i\) of \(g\). Then \(P_\mu(q) = \sum_{j \geq 0} \dim(F^j/F^{j-1})q^j\).

The coefficients of \(P_\mu(q)\) are called the generalized exponents of \(V(\mu)\) (see [K, He, Lu1] for more details).

We have \(V_N = \bigoplus_\mu \pi(\mu) \otimes (\mathbb{C}[g] \otimes V(\mu))^g\). This together with Theorem 8.8 implies the following.

**Corollary 8.10.** The character of \(L(\Delta(n, N))\) is given by the formula

\[
\text{Tr}(L(\Delta(n, N)))(w) = q^{(N^2 - 1)/2} \sum_\mu \chi_\pi(\mu)(w) P_\mu(q) / (1 - q) \cdots (1 - q^N)
\]

where \(w \in S_n\), and \(\chi_\pi(\mu)\) is the character of \(\pi(\mu)\). Here the summation is over partitions \(\mu\) of \(n\) with at most \(N\) parts.

**Proof.** The formula follows, using Proposition 8.7, from Kostant’s result ([K]) that \((\mathbb{C}[g] \otimes V(\mu))^g\) is a free module over \(\mathbb{C}[g]^g\), and the fact that the Hilbert polynomial of the space of generators for this module is the \(q\)-weight multiplicity of the zero weight, \(P_\mu(q)\) ([K, Lu1, He]).

\[\square\]

**Remark 8.11.** It would be interesting to compare this formula with the character formula of [Ro] for the same module.

9. **Equivariant \(D\)-modules and representations of the rational Cherednik algebra**

9.1. **The category of equivariant \(D\)-modules on the nilpotent cone.** The theory of equivariant \(D\)-modules on the nilpotent cone arose from Harish-Chandra’s work on invariant distributions on nilpotent orbits of real groups, and was developed further in many papers,
see e.g. [HK, LS, L, Mi] and references therein. Let us recall some of the basics of this theory.

Let $G$ be a simply connected simple algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone of $\mathfrak{g}$. We denote by $\mathcal{D}(\mathfrak{g})$ the category of finitely generated $\mathfrak{g}$-modules on $\mathfrak{g}$, by $\mathcal{D}_G(\mathfrak{g})$ the subcategory of $G$-equivariant $\mathfrak{g}$-modules, and by $\mathcal{D}_G(\mathcal{N})$ the category of $G$-equivariant $\mathfrak{g}$-modules which are set-theoretically supported on $\mathcal{N}$ (here we do not make a distinction between a $\mathfrak{g}$-module on an affine space and the space of its global sections). Since $G$ acts on $\mathcal{N}$ with finitely many orbits, it is well known that any object in $\mathcal{D}_G(\mathcal{N})$ is regular and holonomic.

Moreover, the category $\mathcal{D}_G(\mathcal{N})$ has finitely many simple objects, and every object of this category has finite length (so this category is equivalent to the category of modules over a finite dimensional algebra).

9.2. Simple objects in $\mathcal{D}_G(\mathcal{N})$. Recall (see e.g. [Mi] and references) that irreducible objects in the category $\mathcal{D}_G(\mathcal{N})$ are parametrized by pairs $(O, \chi)$, where $O$ is a nilpotent orbit of $G$ in $\mathfrak{g}$, and $\chi$ is an irreducible representation of the fundamental group $\pi_1(O)$, which is clearly isomorphic to the component group $A(O)$ of the centralizer $G_x$ of a point $x \in O$. Namely, $\chi$ defines a local system $L_\chi$ on $O$, and the simple object $M(O, \chi) \in \mathcal{D}_G(\mathcal{N})$ is the direct image of the Goresky-Macpherson extension of $L_\chi$ to the closure $\overline{O}$ of $O$, under the inclusion of $\overline{O}$ into $\mathfrak{g}$.

9.3. Semisimplicity of $\mathcal{D}_G(\mathcal{N})$. The proof of the following theorem was explained to us by G. Lusztig.

**Theorem 9.1.** The category $\mathcal{D}_G(\mathcal{N})$ is semisimple.

**Proof.** We may replace the category $\mathcal{D}_G(\mathcal{N})$ by the category of $G$-equivariant perverse sheaves (of complex vector spaces) on $\mathfrak{g}$ supported on $\mathcal{N}$, $\text{Perv}_G(\mathcal{N})$, as these two categories are known to be equivalent. We must show that $\text{Ext}^1(P, Q) = 0$ for every two simple objects $P, Q \in \text{Perv}_G(\mathcal{N})$.

Let $P', Q'$ be the Fourier transforms of $P, Q$. Then $P', Q'$ are character sheaves on $\mathfrak{g}$, and it suffices to show that $\text{Ext}^1(P', Q') = 0$.

Recall that to each character sheaf $S$ one can naturally attach a conjugacy class of pairs $(L, \theta)$, where $L$ is a Levi subgroup of $G$, and $\theta$ is a cuspidal local system on a nilpotent orbit for $L$. It is shown by arguments parallel to those in [Lu3] (which treats the more difficult case of character sheaves on the group) that if $(L_i, \theta_i)$ corresponds to $(S_i, i = 1, 2)$, and $(L_1, \theta_1)$ is not conjugate to $(L_2, \theta_2)$ then $\text{Ext}^* (S_1, S_2) = 0$. Thus it is sufficient to assume that the pair $(L, \theta)$ attached to $P'$ and $Q'$ is the same.

Using standard properties of constructible sheaves (in particular, Poincaré duality), we have

$$\text{Ext}^1(P', Q') = H^1(\mathfrak{g}, \text{Hom}(P', Q')) =$$

$$H^2_\text{c} \dim \mathfrak{g}^{-1}(\mathfrak{g}, \text{Hom}(P', Q'))^* = H^2_\text{c} \dim \mathfrak{g}^{-1}(\mathfrak{g}, (Q')^* \otimes P'^*),$$

where $*$ for sheaves denotes the Verdier duality functor.

Recall that to each character sheaf one can attach an irreducible representation of a certain Weyl group, via the generalized Springer correspondence. Let $R$ be the direct sum of all character sheaves corresponding to a given pair $(L, \theta)$ with multiplicities given by the dimensions of the corresponding representations. Then it is sufficient to show that $\text{H}^2_\text{c} \dim \mathfrak{g}^{-1}(\mathfrak{g}, (R')^* \otimes R') = 0$. 
This fact is essentially proved in [Lu2]. Namely, it follows from the computations of [Lu2] that $H^2(\mathfrak{g}, (R^\ast)^{\ast} \otimes R^\ast)$ is the cohomology with compact support of a certain generalized Steinberg variety with twisted coefficients, and it is shown that this cohomology is concentrated in even degrees. $^6$ The theorem is proved.  

9.4. Monodromicity. We will need the following lemma.

**Lemma 9.2.** Let $Q \in D_G(N)$. Then for any finite dimensional representation $U$ of $\mathfrak{g}$, the action of the shifted Euler operator $H$ defined by (45) on $(Q \otimes U)^g$ is locally finite (so $Q$ is a monodromic $D$-module), and has finite dimensional generalized eigenspaces. Moreover, the eigenvalues of $H$ on $(Q \otimes U)^g$ are bounded from above. In particular, $(Q \otimes U)^g$ belongs to category $\mathcal{O}$ for the $\mathfrak{sl}_2$-algebra spanned by $H$ and the elements $E, F$ given by (46).

**Proof.** Since $Q$ has finite length, it is sufficient to assume that $Q$ is irreducible. We may further assume that $Q$ is generated by an irreducible $G$-submodule $V$, annihilated by multiplication by any invariant polynomial on $\mathfrak{g}$ of positive degree. Indeed, let $V_0$ be an irreducible $G$-submodule of $Q$, let $J_{V_0} := \{ f \in \mathbb{C}[\mathfrak{g}]^g | fV_0 = 0 \}$ and for any $v \in V_0$, let $J_v := \{ f \in \mathbb{C}[\mathfrak{g}]^g | f v = 0 \}$. Then if $v \in V_0$ is nonzero, $J_v = J_{V_0}$ as $Gv = V_0$. Moreover, the support condition implies that $J_v \subset m^k$ for some $k \geq 0$, where $m = \mathbb{C}[\mathfrak{g}]^g$. So $J_{V_0} \subset m^k$ and is an ideal of $\mathbb{C}[\mathfrak{g}]^g$. Let $f \in \mathbb{C}[\mathfrak{g}]^g$ be such that $f \notin J_{V_0}$ and $f m \subset J_{V_0}$; we set $V := fV_0$.

Then $Q$ is a quotient of the $D$-module $\hat{Q} \otimes V$ by a $G$-stable submodule, where

$$\hat{Q} := D(\mathfrak{g})/(D(\mathfrak{g}) \mathrm{ad}(\mathrm{Ann}(V)) + D(\mathfrak{g}) I),$$

$\mathrm{Ann}(V)$ is the annihilator of $V$ in $U(\mathfrak{g})$, and $I$ is the ideal in $\mathbb{C}[\mathfrak{g}]$ generated by invariant polynomials on $\mathfrak{g}$ of positive degree. Thus, it suffices to show that the lemma holds for the module $\hat{Q}$ (which is only weakly $G$-equivariant, i.e. the group action and the Lie algebra action coming from differential operators do not agree, in general).

The algebra $D(\mathfrak{g})$ has a grading in which $\deg(\mathfrak{g}^\ast) = -1, \deg(\mathfrak{g}) = 1$. This grading descends to a grading on $\hat{Q}$. We will show that for each $U$, this grading on $(\hat{Q} \otimes U)^g$ has finite dimensional pieces, and is bounded from above. This implies the lemma, since the Euler operator preserves the grading.

Consider the associated graded module $\hat{Q}_0$ of $\hat{Q}$ under the Bernstein filtration. This is a bigraded module over $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]$ (where we identify $\mathfrak{g}$ and $\mathfrak{g}^\ast$ using the trace form). We have to show that the homogeneous subspaces of $(\hat{Q}_0 \otimes U)^g$ under the grading defined by $\deg(\mathfrak{g} \oplus 0) = -1, \deg(0 \oplus \mathfrak{g}) = 1$ are finite dimensional.

The associated graded of the ideal $\mathrm{Ann}(V) \subset U(\mathfrak{g})$ is such that $\mathbb{C}[\mathfrak{g}]^+_k \subset \mathrm{grAnn}(V) \subset \mathbb{C}[\mathfrak{g}]^+_+$ for some $k \geq 1$, therefore

$$\hat{Q}_0 = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]/J,$$

where $J$ is a (not necessarily radical) ideal whose zero set is the variety $Z$ of pairs $(u, v) \in N \times \mathfrak{g}$ such that $[u, v] = 0$. Let

$$Q' = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]/\sqrt{J}.$$  

Because of the Hilbert basis theorem, it suffices to prove that the homogeneous subspaces of $(Q'_0 \otimes U)^g$ are finite dimensional, and the degree is bounded above. But $Q'_0$ is the algebra of regular functions on $Z$. By the result of [J], one has $\mathbb{C}[Z]^g = \mathbb{C}[\mathfrak{g}]^g$, the algebra of invariant polynomials of $Y$. But it follows from the Hilbert’s theorem on invariants that every isotypic component of $\mathbb{C}[Z]$ is a finitely generated module over $\mathbb{C}[\mathfrak{g}]^g$. This implies the result.  

$^6$More precisely, in the arguments of [Lu2] the vanishing of odd cohomology is proved for $G$-equivariant cohomology with compact supports, and in the non-equivariant case one should use parallel arguments, rather than exactly the same arguments.
9.5. Characters. Lemma 9.2 allows one to define the character of an object \( M \in \mathcal{D}_G(N) \). Namely, let \( \mu = (\mu_1, \ldots, \mu_N) \) be a dominant integral weight for \( \mathfrak{g} \), and \( V(\mu) \) the irreducible representation of \( \mathfrak{g} \) with highest weight \( \mu \). Let \( K_M(\mu) = (M \otimes V(\mu))^g \). Then the character of \( M \) is defined by the formula

\[
\text{Ch}_M(t,g) = \text{Tr}_{M}(gt^{-\mathcal{H}}) = \sum \mu \text{Tr}_{K_M(\mu)}(t^{-\mathcal{H}})\chi_\mu(g), \quad g \in G,
\]

where \( \chi_\mu \) denotes the character of \( \mu \). It can be viewed as a linear functional from \( \mathbb{C}[G]^G \) to \( \mathbb{F} := \oplus_{\beta \in \mathcal{U}^0 \mathbb{C}[t]} \), via the integration pairing.

In other words, the multiplicity spaces \( K_M(\mu) \) are representations from category \( \mathcal{O} \) of the Lie algebra \( \mathfrak{sl}_2 \) spanned by \( E, F, H \), and the character of \( M \) carries the information about the characters of these representations.

The problem of computing characters of simple objects in \( \mathcal{D}_G(N) \) is interesting and, to our knowledge, open. Below we will show how these characters for \( G = SL_N(\mathbb{C}) \) can be expressed via characters of irreducible representations of the rational Cherednik algebra.

**Example 9.3.** Recall (see e.g. [Mi]) that an object \( M \in \mathcal{D}_G(N) \) is cuspidal iff \( \mathcal{F}(M) \in \mathcal{D}_G(N) \), where \( \mathcal{F} \) is the Fourier transform (Lusztig’s criterion). If follows that in the case of cuspidal objects \( M \), the spaces \( K_M(\mu) \) are also in the category \( \mathcal{O} \) for the opposite Borel subalgebra of \( \mathfrak{sl}_2 \), hence are finite dimensional representations of \( \mathfrak{sl}_2 \), and, in particular, their dimensions are of interest.

9.6. The functors \( F_n, F_n^* \). The representation \( V_N \) is a special case of representations of the rational Cherednik algebra which can be constructed via a functor similar to the one defined in [GG1]. Namely, the construction of \( V_N \) can be generalized as follows.

Let \( n \) and \( N \) be positive integers (we no longer assume that \( N \) is a divisor of \( n \)), and \( k = N/n \). We again consider the special case \( G = SL_N(\mathbb{C}) \), \( \mathfrak{g} = \mathfrak{sl}_N(\mathbb{C}) \). Then we have a functor \( F_n : D(\mathfrak{g}) \to H_n(k)\)-mod defined by the formula

\[
F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^g,
\]

where \( g \) acts on \( M \) by adjoint vector fields. The action of \( H_n(k) \) on \( F_n(M) \) is defined by the same formulas as in Proposition 8.1, and Proposition 8.7 remains valid.

Note that \( F_n(M) = F_n(M_{\text{fin}}) \), where \( M_{\text{fin}} \) is the set of \( g \)-finite vectors in \( M \). Clearly \( M_{\text{fin}} \) is a \( G \)-equivariant \( D \)-module. Thus, it is sufficient to consider the restriction of \( F_n \) to the subcategory \( \mathcal{D}_G(\mathfrak{g}) \), which we will do from now on.

In general, \( F_n(M) \) does not belong to category \( \mathcal{O} \). However, we have the following lemma.

**Lemma 9.4.** If the Fourier transform \( \mathcal{F}(M) \) of \( M \) is set-theoretically supported on the nilpotent cone \( \mathcal{N} \) of \( \mathfrak{g} \), then \( F_n(M) \) belongs to the category \( \mathcal{O} \).

**Proof.** Since \( \mathcal{F}(M) \) is supported on \( \mathcal{N} \), invariant polynomials on \( \mathfrak{g} \) act locally nilpotently on \( \mathcal{F}(M) \). Hence invariant differential operators on \( \mathfrak{g} \) with constant coefficients act locally nilpotently on \( M \). Thus, it follows from formula (44) that the algebra \( \mathbb{C}[y_1,\ldots,y_n]^{S_n} \) acts locally nilpotently on \( F_n(M) \). Also, by Lemma 9.2, the operator \( h \) acts with finite dimensional generalized eigenspaces on \( F_n(M) \). This implies the statement. \( \square \)

Thus we obtain an exact functor \( F_n^* = F_n \circ \mathcal{F} : \mathcal{D}_G(N) \to \mathcal{O}(H_n(k)) \).

9.7. The symmetric part of \( F_n \). Consider the symmetric part \( eF_n(M) \) of \( F_n(M) \). We have \( eF_n(M) = (M \otimes S^n\mathbb{C}^N)^g \), and we have an action of the spherical subalgebra \( B_n(k) \) on \( eF_n(M) \), given by formulas (43,44).

This allows us to relate the functor \( F_n \) with the functor defined in [GG1]. Namely, recall from [GG1] that for any \( c \in \mathbb{Z} \), one may define the category \( \mathcal{D}_c(\mathfrak{g} \times \mathbb{P}^{N-1}) \) of coherent \( D \)-modules on \( \mathfrak{g} \times \mathbb{P}^{N-1} \) which are twisted by the \( c \)-th power of the tautological line bundle on
the second factor (this makes sense for all complex $c$ even though the $c$-th power is defined only for integer $c$). Then the paper [GG1]\(^7\) defines a functor

$\mathbb{H} : D_c(g \times \mathbb{F}^{N-1}) \to \mathbb{B}_N(c/N)$ -mod,

given by $\delta(M) = M^g$.

**Proposition 9.5.** (i) If $n$ is divisible by $N$ then one has a functorial isomorphism $eF_n(M) \simeq \phi^\tau \mathbb{H}(M \otimes S^n \mathbb{C}^N)$, where $S^n \mathbb{C}^N$ is regarded as a twisted $D$-module on $\mathbb{F}^{N-1}$ (with $c = n$).

(ii) For any $n$, the actions of $B_n(N/n)$ and $B_N(n/N)$ on the space $eF_n(M) = \mathbb{H}(M \otimes S^n \mathbb{C}^N)$ have the same image in the algebra of endomorphisms of this space.

**Proof.** This follows from the definition of $\mathbb{H}$ and formulas (43,44). \(\square\)

**Corollary 9.6.** The functor $eF_n^*$ on the category $D_G(N)$ maps irreducible objects into irreducible ones.

**Proof.** This follows from Proposition 9.5, (ii) and Proposition 7.4.3 of [GG1], which states that the functor $\mathbb{H}$ maps irreducible objects to irreducible ones. \(\square\)

Formulas 43,44 can also be used to study the support of $F_n^*(M)$ for $M \in D_G(N)$, as a $\mathbb{C}[x_1, \ldots, x_n]$-module. Namely, we have the following proposition.

**Proposition 9.7.** Let $q = \text{GCD}(n, N)$ be the greatest common divisor of $n$ and $N$. Then the support $S$ of $F_n^*(M)$ is contained in the union of the $S_n$-translates of the subspace $E_q$ of $\mathbb{C}^n$ defined by the equations $\sum_{i=1}^n x_i = 0$ and $x_i = x_j$ if $\frac{n}{q}(l - 1) + 1 \leq i, j \leq \frac{nl}{q}$ for some $1 \leq l \leq q$.

**Proof.** It follows from equation (44) that for any $(x_1, \ldots, x_n) \in S$ there exists a point $(z_1, \ldots, z_N) \in \mathbb{C}^N$ such that one has

$$\frac{1}{n} \sum_{i=1}^n x_i^p = \frac{1}{N} \sum_{j=1}^N z_j^p$$

for all positive integer $p$. In particular, writing generating functions, we find that

$$\frac{N}{n} \sum_{i=1}^n \frac{1}{1 - tx_i} = \frac{n}{N} \sum_{j=1}^N \frac{1}{1 - tz_j}.$$

In particular, every fraction occurs on both sides at least $\text{LCM}(n, N)$ times, and hence the numbers $x_i$ fall into $n/q$-tuples of equal numbers (and the numbers $z_j$ into $N/q$-tuples of equal numbers). The proposition is proved. \(\square\)

9.8. **Irreducible equivariant $D$-modules on the nilpotent cone for $G = \text{SL}_N(\mathbb{C})$.** Nilpotent orbits for $\text{SL}_N(\mathbb{C})$ are labelled by Young diagrams, or partitions. Namely, if $x \in \mathfrak{sl}_N(\mathbb{C})$ is a nilpotent element, then we let $\mu$ be the sizes of its Jordan blocks enumerated in the decreasing order. The partition $\mu = (\mu_1, \ldots, \mu_m)$ and the corresponding Young diagram whose rows have lengths $\mu_i$ are attached to $x$. If $O$ is the orbit of $x$ then we will denote $\mu$ by $\mu(O)$. For instance, if $O = \{0\}$ then $\mu = (1^N)$ and if $O$ is the open orbit then $\mu = (N)$.

It is known (and easy to show) that the group $A(O)$ is naturally isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where $d$ is the greatest common divisor of the $\mu_i$. Namely, let $Z = \mathbb{Z}/N\mathbb{Z}$ be the center of $G$ (we identify $\mathbb{Z}/N\mathbb{Z}$ with $Z$ by $p \to e^{2\pi ip/N}\text{Id}$). Then we have a natural surjective homomorphism $\theta : Z \to A(O)$ induced by the inclusion $Z \to G_x$, $x \in O$. This homomorphism sends $d$ to 0, and thus $A(O)$ gets identified with $\mathbb{Z}/d\mathbb{Z}$.

Thus, any character $\chi : A(O) \to \mathbb{C}^*$ is defined by the formula $\chi(p) = e^{-2\pi ip s/d}$, where $0 \leq s < d$. We will denote this character by $\chi_s$.

\(^7\)There seems to be a misprint in [GG1]: in the definition of $\mathbb{H}$, $c$ should be replaced by $c/N$. 
9.9. **The action of $F_n^*$ on irreducible objects.** Obviously, the center $Z$ of $G$ acts on $F_n^*(M)$ by $z \mapsto z^{-sN/d}$. Thus, a necessary condition for $F_n^*(M(O, \chi_s))$ to be nonzero is

$$n = N \left( p + \frac{s}{d} \right),$$

where $p$ is a nonnegative integer.

Our main result in this section is the following theorem.

**Theorem 9.8.** The functor $F_n^*$ maps irreducible objects into irreducible ones or zero. Specifically, if condition (51) holds, then we have

$$F_n^*(M(O, \chi_s)) = L(\pi(n\mu(O)/N)),$$

the irreducible representation of $\mathcal{H}_n(k)$ whose lowest weight is the representation of $S_n$ corresponding to the partition $n\mu(O)/N$.

**Remark 9.9.** Here if $\mu$ is a partition and $c \in \mathbb{Q}$ is a rational number, then we denote by $c\mu$ the partition whose parts are $c\mu_i$, provided that these numbers are all integers. In our case, this integrality condition holds since all parts of $\mu(O)$ are divisible by $d$. 

**Corollary 9.10.** Let $\lambda$ be a partition of $n$ into at most $N$ parts. Let $M = M(O_\lambda, \chi_s)$, and assume that condition (51) is satisfied. Then

$$(M \otimes V(\lambda))^d = \text{Hom}_{S_n}(\pi(\lambda), L(\pi(n\mu/O))$$

as graded vector spaces.

This corollary allows us to express the characters of the irreducible $D$-modules $M(O, \chi)$ in terms of characters of certain special lowest weight irreducible representations of $H_n(k)$. We note that characters of lowest weight irreducible representations of rational Cherednik algebras of type $A$ have been computed by Rouquier, [Ro].

**Remark 9.11.** Note that Theorem 8.8 is the special case of Theorem 9.8 for $O = \{0\}$.

9.10. **Proof of Theorem 9.8.** Our proof of Theorem 9.8 is based on the following result of [GS].

**Theorem 9.12.** Let $k > 0$. Then the functor $V \mapsto eV$ is an equivalence of categories between $H_n(k)$-modules and $B_n(k)$-modules.

**Remark 9.13.** We note that Theorem 9.12 is proved in [GS] under the technical assumption $k \notin \mathbb{Z} + 1/2$. It was noticed by V. Ginzburg that this assumption is really unnecessary. Indeed, the only place where this assumption is used is in the proof of Lemma 3.5. Namely, it is used in the proof of this lemma that Hom between Verma modules over $H_n(k)$ is isomorphic to Hom between the corresponding dual Specht modules, which is known, from [GGOR], only for $k \notin \mathbb{Z} + 1/2$. However, it is sufficient for the proof of Lemma 3.5 of [GS] to know just that the first Hom injects into the second one, which is known for all positive $k$ thanks to a lemma by Opdam and Rouquier (Lemma 2.10 of [BEG2]).

Theorem 9.12 implies the first statement of the theorem, i.e. that if (51) holds then $F_n^*(M(O_\mu, \chi_s))$ is irreducible. Indeed, it follows from Corollary 9.6 that $eF_n^*(M(O_\mu, \chi_s))$ is irreducible over $B_n(k)$. Thus, it remains to find the lowest weight of $F_n^*(M(O_\mu, \chi_s))$.

Let $\mu = (\mu_1, ..., \mu_N)$ be a partition of $N$ ($\mu_i \geq 0$). Let $O_\mu$ be the nilpotent orbit of $\mathfrak{g}$ corresponding to the partition $\mu$. Denote by $d$ the greatest common divisor of $\mu_i$, and by $m$ a divisor of $d$. Define the following function $f$ on $O_\mu$ with values in $\otimes_{i=1}^{N} S^{\mu_i} \mathbb{C}^N$:

$$f(X, \xi_1, ..., \xi_N) = \bigwedge_{i=1}^{N} \bigwedge_{j=0}^{\mu_i-1} \xi_j X^j,$$

$\xi_i \in (\mathbb{C}^N)^*$ (here $X^j \in M_N(\mathbb{C})$ is the $j$th power of $X$, so $\xi_j X^j \in \mathbb{C}^N$).
Lemma 9.14. (i) For any $X \in O_\mu$, $f(X, \ldots)^{1/m}$ is a polynomial in $\xi_1, \ldots, \xi_N$. Thus, $f^{1/m}$ is a regular function on the universal cover $\tilde{O}_\mu$ of $O_\mu$ with values in $\otimes_{i=1}^N S^{\mu_i/m} \mathbb{C}^N$.

(ii) For any $X \in O_\mu$, the function $f(X, \ldots)^{1/m}$ generates a copy of the representation $V(\mu/m)$ inside $\otimes_{i=1}^N S^{\mu_i/m} \mathbb{C}^N$.

(iii) Specifically, let the standard basis $u_1, \ldots, u_N$ of $(\mathbb{C}^N)^*$ be filled into the squares of the Young diagram of $\mu$ (filling the first column top to bottom, then the second one, etc.), and let $X$ be the matrix $J$ acting by the horizontal shift to the right on this basis. Then $f(J, \ldots)^{1/m}$ is a highest weight vector of the representation $V(\mu/m)$.

Proof. It is sufficient to prove (iii). Let $\mu^* = (\mu_1^*, \ldots, \mu_N^*)$ be the conjugate partition. Let $p_j$ be the number of times the part $j$ occurs in this partition. Clearly, $p_j$ is divisible by $m$. By looking at the matrix whose determinant is $f$, we see that we have, up to sign:

$$f(J, \xi_1, \ldots, \xi_N) = \prod_j \Delta_j(\xi_1, \ldots, \xi_N)^p_j,$$

where $\Delta_j$ is the left upper $j$-by-$j$ minor of the matrix $(\xi_1, \ldots, \xi_N)$. Thus $f^{1/m} = \prod_j \Delta_j^{p_j/m}$ is clearly a highest weight vector of weight $\sum_j p_j \varpi_j/m$, where $\varpi_j$ are the fundamental weights. But $\sum p_j \varpi_j = \mu$, so we are done. \qed

Corollary 9.15. The function $f$ gives rise to a $G$-equivariant regular map $f : \tilde{O}_\mu \to V(\mu/d)$, whose image is the orbit of the highest weight vector. In particular, we have a $G$-equivariant inclusion of commutative algebras

$$f^* : \oplus_{\ell \geq 0} V(\ell \mu/d)^* \to \mathbb{C}[\tilde{O}_\mu].$$

Now let $0 \leq s \leq d - 1$, and denote by $\mathbb{C}[\tilde{O}_\mu]_s$ the subspace of $\mathbb{C}[\tilde{O}_\mu]$, on which central elements $z \in G$ act by $z \to z^{-s}$. Then we have an inclusion

$$f^* : \oplus_{\ell \geq 0} V((\ell + s) \mu/d)^* \to \mathbb{C}[\tilde{O}_\mu]_s.$$

Now recall that by construction, $\mathbb{C}[\tilde{O}_\mu]_s$ sits inside $M = M(O_\mu, \chi_s)$ as a $\mathbb{C}[O_\mu]$-submodule. In particular, the operators $X_i$, corresponding to $\mu$ and regard $V(\lambda) \otimes \pi(\lambda)$, for any partition $\lambda$ of $n$, as a subspace of $(\mathbb{C}^N)^{\otimes n}$ using the Weyl duality. Then for any $u \in \pi(n \mu/N)$, we can define the element $a(u) \in F^*_n(M)$ by $a(u) = f^* \otimes u$, where $f^*_n \in \mathbb{C}[\tilde{O}_\mu]_s \otimes V(n \mu/N)$ is the homogeneous part of $f^*$ of degree $n$.

Lemma 9.16. $a(u)$ is annihilated by the elements $y_i$ of $H_n(k)$.

Proof. We need to show that the operators $X_i$ (or, equivalently, the elements $x_i \in H_n(k)$) annihilate $a(u) \in F^*_n(M)$. Since $a(u)$ is $G$-invariant, it is sufficient to prove the statement at the point $X = J$. This boils down to showing that for any $j$ not exceeding the number of parts of $\mu$ (i.e., $j \leq \mu_1^*$), the application of $J$ in any component annihilates the element $\Delta_j(\xi_1, \ldots, \xi_N) \in N \mathbb{C}^N \subset (\mathbb{C}^N)^{\otimes j}$. This is clear, since the first $\mu_1^*$ columns of $J$ are zero. \qed

This implies that the lowest weight of $F^*_n(M(O_\mu, \chi_s))$ is $\pi(n \mu/N)$, as desired. The theorem is proved.

Remark 9.17. Here is another, short proof of Theorem 9.8 for $n = N$. We have

$$e_- F^*_n(M(O, 1)) = F(M(O, 1))^G.$$

According to [L, LS],

$$F(M(O, 1))^G = (\mathbb{C}[h] \otimes \pi(\mu(O)))^{S_N}$$

as a module over $D(h)^W = e_- H_N(1)e_-$. Thus, $e_- F^*_n(M(O, 1)) = e_- L(\pi(\mu(O)))$ as $e_- H_N(1)e_-$-modules. But the functor $V \to e_- V$ is an equivalence of categories $H_N(1)$-mod $\to e_- H_N(1)e_-$-mod (see [BEG2]). Thus, $F^*_n(M(O, 1)) = L(\pi(\mu(O)))$ as $H_N(1)$-modules, as desired.
9.11. The support of \( L(\pi(n\mu/N)) \).

**Corollary 9.18.** Let \( \mu \) be a partition of \( N \) such that \( n\mu_i/N \) are integers. Then the support of the representation \( L(\pi(n\mu/N)) \) of \( H_n(N/n) \) as a module over \( \mathbb{C}[x_1, \ldots, x_n] \) is contained in the union of \( S_n \)-translates of \( E_q \), \( q = \text{GCD}(n, N) \).

**Proof.** This follows from Theorem 9.8 and Proposition 9.7. \( \Box \)

We note that in the case when \( \mu = (N) \), Corollary 9.18 follows from Theorem 3.2 from [CE].

9.12. The cuspidal case. An interesting special case of Theorem 9.8 is the cuspidal case. In this case \( N \) and \( n \) are relatively prime, \( d = N \) (i.e., \( O \) is the open orbit), and \( s \) is relatively prime to \( N \).

Here is a short proof of Theorem 9.8 in the cuspidal case.

Since the Fourier transform of \( M(O, \chi_s) \) in the cuspidal case is supported on the nilpotent cone, \( F^*(M(O, \chi_s)) \) belongs not only to the category \( O \) generated by lowest weight modules, but also to the “dual” category \( O_- \) generated by highest weight modules over \( H_n(k) \). Thus, by the results of [BEG1], \( F^*(M(O, \chi_s)) \) is a multiple of the unique finite dimensional irreducible \( H_n(k) \)-module \( L(\mathbb{C}) = Y(N, n) \), of dimension \( N^{n-1} \). But this multiple must be a single copy by Corollary 9.6, so the theorem is proved.

Theorem 9.8 implies the following formula for the characters of the cuspidal \( D \)-modules \( M(O, \chi_s) \).

Let \( \mu \) be a dominant integral weight for \( \mathfrak{g} \), such that the center \( Z \) of \( G \) acts on \( V(\mu) \) via \( z \to z^\mu = z^n \). Let \( \rho \) be the half-sum of positive roots of \( \mathfrak{g} \). Let \( K_s(\mu) = (M(O, \chi_s) \otimes V(\mu))^g \) be the isotypic components of \( M(O, \chi_s) \).

**Theorem 9.19.** We have

\[
\text{Tr}_{K_s(\mu)}(q^{2H}) = \frac{q - q^{-1}}{q^N - q^{-N}} \varphi_\mu(q),
\]

where

\[
\varphi_\mu(q) := \prod_{1 \leq p < r \leq N} \frac{q^{\mu_r - \mu_p + r - p} - q^{\mu_p - \mu_r + p - r}}{q^{r - p} - q^{p - r}} = \chi_V(\mu)(q^{2\rho}),
\]

where \( \chi_V(\mu) \) is the character of \( V(\mu) \). In particular,

\[
\dim K_s(\mu) = \frac{1}{N} \prod_{1 \leq p < r \leq N} \frac{\mu_r - \mu_p + r - p}{r - p} = \frac{1}{N} \dim V(\mu).
\]

**Proof.** We extend the representation \( V(\mu) \) to \( \text{GL}_N(\mathbb{C}) \) by setting \( z \to z^n \) for all scalar matrices \( z \), so that its \( \text{GL}_N(\mathbb{C}) \)-highest weight is

\[
\hat{\mu} := (\mu_1 + n/N, \ldots, \mu_N + n/N).
\]

Note that we automatically have \( \mu_i + n/N \in \mathbb{Z} \). Assume that \( n \) is so big that \( \hat{\mu} \) is a partition of \( n \) (i.e., \( \mu_i + n/N \geq 0 \)).

It follows from the results of [BEG1] that the character of the irreducible representation \( L(\mathbb{C}) \) of the rational Cherednik algebra \( H_n(k) \), \( k = N/n \), is given by the formula

\[
\text{Tr}_{L(\mathbb{C})}(gq^{2h}) = \frac{q - q^{-1}}{q^N - q^{-N}} \frac{\det(q^{-N} - q^Ng)}{\det(q^{-1} - qg)}, \quad g \in S_n,
\]

where the determinants are taken in \( \mathbb{C}^n \).

Let us equip \( \mathbb{C}^N \) with the structure of an irreducible representation of \( \mathfrak{sl}_2 \) with basis \( e, f, h \).

Let \( g \in S_n \). Then

\[
\text{Tr}_{\text{Hom}_S(\pi(\hat{\mu}), (\mathbb{C}^N)^{\otimes n})}(q^h) = \text{Tr}_{V(\mu)}(q^{2\rho}) = \varphi_\mu(q),
\]

as desired.\( \Box \)
by the Weyl character formula. On the other hand, it is easy to show that
\[
\text{Tr}_{(|\mathbb{C}^N|)^n}(gg^h) = \frac{\det(q^{-N} - q^Ng)}{\det(q^{-1} - qg)}.
\]
Thus,
\[
\text{Tr}_{\text{Hom}_{\mathbb{C}^n}(\pi(\mu),L(\mathbb{C}))}(q^{2h}) = \frac{q - q^{-1}}{q^N - q^{-N}} \text{Tr}_{\text{Hom}_{\mathbb{C}^n}(\pi(\mu),(|\mathbb{C}^N|)^n)}(q^h)
\]
\[
= \frac{q - q^{-1}}{q^N - q^{-N}} \varphi_\mu(q).
\]
By Theorem 9.8 and Weyl duality, this implies that
\[
\text{Tr}_{[(O(\lambda_s)\otimes V(\mu))^s]}(q^{2H}) = \frac{q - q^{-1}}{q^N - q^{-N}} \varphi_\mu(q),
\]
as desired. \qed

Example 9.20. Let \(N = 2, s = 1\). In this case Theorem 9.19 gives us the following decomposition of \(M(O, \lambda_s)\):
\[
M(O, \lambda_s) = \bigoplus_{j \geq 1} N_j \otimes V_{2j-1},
\]
where \(V_j\) is the irreducible representation of \(\mathfrak{sl}_2\) of dimension \(j + 1\), and the spaces \(N_j\) satisfy the equation
\[
\text{Tr}_{|N_j^s}(q^{2H}) = \frac{q^{2j} - q^{-2j}}{q^2 - q^{-2}}.
\]
This shows that \(N_j = V_{j-1}\) as a representation of the \(\mathfrak{sl}_2\)-subalgebra spanned by \(E, F, H\), which commutes with \(g\).

9.13. The case of general orbits. Let \(W = S_N\) the Weyl group of \(G, \lambda \in \mathfrak{h}/W\), and \(N_\lambda\) be the closure in \(g\) of the adjoint orbit of a regular element of \(g\) whose semisimple part is \(\lambda\). Denote by \(\mathcal{D}_G(N_\lambda)\) the category of \(G\)-equivariant \(D\)-modules on \(G\) which are concentrated on \(N_\lambda\). We also let \(\mathcal{O}_\lambda\) be the category of finitely generated \(H_n(k)\)-modules in which the subalgebra \(\mathbb{C}[y_1, \ldots, y_n]^{S_n}\) acts through the character \(\lambda\). Then one can show, similarly to the above, that the functor \(F^*_n\) restricts to a functor \(F^*_{n,\lambda} : \mathcal{D}_G(N_\lambda) \to \mathcal{O}_\lambda\). The functor considered above is \(F^*_{n,0}\). We plan to study the functor \(F^*_{n,\lambda}\) for general \(\lambda\) in a future work.

9.14. The trigonometric case. Our results about rational Cherednik algebras can be extended to the trigonometric case. For this purpose, \(D\)-modules on the Lie algebra \(g\) should be replaced with \(D\)-modules on the group \(G\). Let us describe this generalization.

First, letus introduce some notation. As above, we let \(G = \text{SL}_N(\mathbb{C})\). For \(b \in g\), let \(L_b\) be the right invariant vector field on \(G\) equal to \(b\) at the identity element; that is, \(L_b\) generates the group of left translations by \(e^b\). As before, we let \(k = N/n\).

Now let \(M\) be a \(D\)-module on \(G\). Similarly to the above, we define \(F_n(M)\) to be the space
\[
F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^G,
\]
where \(G\) acts on itself by conjugation.

Consider the operators \(X_i, Y_i, i = 1, \ldots, n\), on \(F_n(M)\), defined by the formulas similar to the rational case:
\[
X_i = \sum_{j,l} A_{jl} \otimes (E_{ij})_l, \quad Y_i = \frac{N}{n} \sum_p L_p \otimes (h_p)_i,
\]
where \(A_{jl}\) is the \(jl\)-th matrix element of \(A \in G\) regarded as the multiplication operator in \(M\) by a regular function on \(G\).
Proposition 9.21. The operators $X_i, Y_i$ satisfy the following relations:
\[
\prod_i X_i = 1, \quad \sum_i Y_i + k \sum_{i<j} s_{ij} = 0,
\]
\[
s_{ij} X_i = X_j s_{ij}, \quad s_{ij} Y_i = Y_j s_{ij}, \quad [s_{ij}, X_i] = [s_{ij}, Y_i] = 0,
\]
\[
[X_i, X_j] = 0, \quad [Y_i, Y_j] = k s_{ij} (Y_i - Y_j),
\]
\[
[Y_i, X_j] = \left( k s_{ij} - \frac{1}{n} \right) X_j,
\]
where $i, j, l$ denote distinct indices.

Proof. Straightforward computation. \qed

Corollary 9.22. The operators $\tilde{Y}_i = Y_i + k \sum_{j<i} s_{ij}$ pairwise commute.

The relations of Proposition 9.21 are nothing but the defining relations of the degenerate double affine Hecke algebra of type $A_{n-1}$, which we will denote $H_n^\text{tr}(k)$ (where “tr” stands for trigonometric, to illustrate the fact that this algebra is a trigonometric deformation of the rational Cherednik algebra $H_n(k)$). Thus we have defined an exact functor $F_n : D(G) \to H_n^\text{tr}(k)\text{-mod}$. As before, it is sufficient to consider the restriction of this functor to the category of equivariant finitely generated $D$-modules, $D_G(G)$.

This allows us to generalize much of our story for rational Cherednik algebras to the trigonometric case. In particular, let $U$ be the unipotent variety on $G$, and $D_G(U)$ be the category of finitely generated $G$-equivariant $D$-modules on $G$ concentrated on $U$. If we restrict the functor $F_n$ to this category, we get a situation identical to that in the rational case. Indeed, one can show that for any $M$ in this category, $F_n(M)$ belongs to the category $O_U$ of finitely generated modules over $H_n^\text{tr}(k)$ which are locally unipotent with respect to the action of $X_i$. The latter category is equivalent to the category $O_-$ over the rational Cherednik algebra $H_n(k)$, because the completion of $H_n^\text{tr}(k)$ with respect to the ideal generated by $X_i - 1$ is isomorphic to the completion of $H_n(k)$ with respect to the ideal generated by $X_i$. On the other hand, the exponential map identifies the categories $D_G(U)$ and $D_G(N)$. It is clear that after we make these two identifications, the functor $F_n$ becomes the functor $F_n$ in the rational case that we considered above.

On the other hand, because of the absence of Fourier transform on the group (as opposed to Lie algebra), the trigonometric story is richer than the rational one. Namely, we can consider another subcategory of $D_G(G)$, the category of character sheaves. By definition, a character sheaf on $G$ is an object $M$ in $D_G(G)$ which is locally finite with respect to the action of the algebra of biinvariant differential operators, $U(\mathfrak{g})^G$. This category is denoted by $\text{Char}(G)$. It is known that one has a decomposition
\[
\text{Char}(G) = \oplus_{\lambda \in T^\vee/W} \text{Char}_{\lambda}(G),
\]
where $T^\vee$ is dual torus, and $\text{Char}_{\lambda}(G)$ the category of those $M \in D_G(G)$ for which the generalized eigenvalues of $U(\mathfrak{g})^G$ (which we identify with $U(\mathfrak{h})^W$ via the Harish-Chandra homomorphism) project to $\lambda$ under the natural projection $\mathfrak{h}^* \to T^\vee$.

On the other hand, one can define the category $\text{Rep}_{Y - \text{fin}}(H_n^\text{tr}(k))$ of modules over $H_n^\text{tr}(k)$ on which the commuting elements $\tilde{Y}_i$ act in a locally finite manner. We have a similar decomposition
\[
\text{Rep}_{Y - \text{fin}}(H_n^\text{tr}(k)) = \oplus_{\lambda \in T^\vee/W} \text{Rep}_{Y - \text{fin}}(H_n^\text{tr}(k))_{\lambda},
\]
where $\text{Rep}_{Y - \text{fin}}(H_n^\text{tr}(k))_{\lambda}$ is the subcategory of all objects where the generalized eigenvalues of $\tilde{Y}_i$ project to $\lambda \in T^\vee/W$. Then one can show, similarly to the rational case, that the functor $F_n$ gives rise to the functors
\[
F_{n, \lambda} : \text{Char}_{\lambda}(G) \to \text{Rep}_{Y - \text{fin}}(H_n^\text{tr}(k))_{\lambda}
\]
for each $\lambda \in T^\vee/W$. The most interesting case is $\lambda = 0$ (unipotent character sheaves). We plan to study these functors in subsequent works.
9.15. **Relation with the Arakawa-Suzuki functor.** Note that the elements $Y_i$ and $s_{ij}$ generate the degenerate affine Hecke algebra $\mathcal{H}_n$ of Drinfeld and Lusztig (of type $A_{n-1}$). To define the action of this algebra on $F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^\mathfrak{g}$ by the formula of Proposition 9.21, we only need the action of the operators $L_b, b \in \mathfrak{g}$ in $M$. So $M$ can be taken to be an arbitrary $\mathfrak{g}$-bimodule which is locally finite with respect to the diagonal action of $\mathfrak{g}$ (in this case, $\sum_i Y_i + \sum_i s_{ij}$ is a central element which does not necessarily act by zero, so we get a representation of a central extension $\tilde{\mathcal{H}}_n$ of $\mathcal{H}_n$). In particular, we have an exact functor $F_n : \text{HC}(\mathfrak{g}) \to \tilde{\mathcal{H}}_n$-mod from the category of Harish-Chandra bimodules over $\mathfrak{g}$ to the category of finite dimensional representations of the degenerate affine Hecke algebra $\mathcal{H}_n$. This functor was essentially considered in [AS] (where it was applied to the Harish-Chandra modules of the form $M = \text{Hom}_{\mathfrak{g}-\text{finite}}(M_1, M_2)$, where $M_1$ and $M_2$ are modules from category $\mathcal{O}$ over $\mathfrak{g}$). We note that the paper [AST] describes the extension of this construction to affine Lie algebras, which yields representations of degenerate double affine Hecke algebras.

9.16. **Directions of further study.** In conclusion we would like to discuss (in a fairly speculative manner) several directions of further study and generalizations (we note that these generalizations can be combined with each other).

1. The $q$-case: the group $G$ is replaced with the corresponding quantum group, $D$-modules with $q$-$D$-modules, and degenerate double affine Hecke algebras with the usual double affine Hecke algebras (defined by Cherednik). It is especially interesting to consider this generalization if $q$ is a root of unity.

2. The quiver case. This generalization was suggested by Ginzburg, and will be studied in his subsequent work with the third author. In this case, one has a finite subgroup $\Gamma \subset \text{SL}_2(\mathbb{C})$, and one should consider equivariant $D$-modules on the representation space of the affine quiver attached to $\Gamma$ (with some orientation). Then there should exist an analog of the functor $F_n$, which takes values in the category of representations of an appropriate symplectic reflection algebra for the wreath product $S_n \rtimes \Gamma^n$, [EG] (or, equivalently, the Gan-Ginzburg algebra, [GG2]). This generalization should be especially nice in the case when $\Gamma$ is a cyclic group, when the symplectic reflection algebra is a Cherednik algebra for a complex reflection group, and one has the notion of category $\mathcal{O}$ for it.

3. The symmetric space case. This is the trigonometric version of the previous generalization for $\Gamma = \mathbb{Z}/2$. In this generalization one considers (monodromic) equivariant $D$-modules on the symmetric space $\text{GL}_{p+q}(\mathbb{C})/((\text{GL}_p \times \text{GL}_q)(\mathbb{C})$ (see [Gin]), and one expects a functor from this category to the category of representations of an appropriate degenerate affine Hecke algebra of type $C^e C_n$. This functor should be related, similarly to the previous subsection, to an analog of the Arakawa-Suzuki functor, which would attach to a Harish-Chandra module for the pair $\text{GL}_{p+q}(\mathbb{C}), \text{GL}_p(\mathbb{C}) \times \text{GL}_q(\mathbb{C})$, a finite dimensional representation of the degenerate double affine Hecke algebra of type $BC_n$.

**APPENDIX A**

Let $\mathcal{O}$ be the ring $\mathbb{C}[[u_1, \ldots, u_n], \ell_1, \ldots, \ell_n]$. Define commuting derivations $D_i$ of $\mathcal{O}$ by $D_i(u_j) = \delta_{ij} u_i, D_i(\ell_j) = \delta_{ij}$ (we will later think of $\ell_i$ and $D_i$ as $\log u_i$ and $u_i \frac{\partial}{\partial u_i}$).

We set $\mathcal{O}_+ := \mathfrak{m}[\ell_1, \ldots, \ell_n], \text{ where } \mathfrak{m} = \text{Ker}(\mathbb{C}[[u_1, \ldots, u_n]] \to \mathbb{C}) \text{ is the augmentation ideal. Let } A = \oplus_{k \geq 0} A_k \text{ be a graded ring with finite dimensional homogeneous components.}

**Proposition A.1.** Let $X_i(u_1, \ldots, u_n) \in \hat{\otimes}_{k \geq 0} (A_k \otimes \mathcal{O}_+)$ be such that $D_i(X_j) = D_j(X_i)$. Then there exists a unique $F(u_1, \ldots, u_n) \in \hat{\otimes}_{k \geq 0} (A_k \otimes \mathcal{O}_+)$ such that $D_i(F) = X_i$ for $i = 1, \ldots, n$.

Let us say that $f \in \mathcal{O}$ has radius of convergence $R > 0$ if $f = \sum_{k_1, \ldots, k_n \geq 0} f_{k_1, \ldots, k_n}(u_1, \ldots, u_n) \ell_1^{k_1} \ldots \ell_n^{k_n}$, where each $f_{k_1, \ldots, k_n}(u_1, \ldots, u_n)$ converges for $|u_1|, \ldots, |u_n| \leq R$. Then if $X_1, \ldots, X_n$ have radius of convergence $R$, so does $F$.

**Proof.** For each $i$, $D_i$ restricts to an endomorphism of $\mathcal{O}_+$; one checks that $\cap_{n=1}^\infty \text{Ker}(D_i : \mathcal{O}_+ \to \mathcal{O}_+) = 0$ which implies the uniqueness. To prove the existence, we work by induction.
One proves that $D_n : O_+ \to O_+$ is surjective, and its kernel is $m_{n-1}[\ell_1, \ldots, \ell_{n-1}]$, where $m_{n-1} = \text{Ker}[\mathcal{C}[u_1, \ldots, u_{n-1}]]$. Let $G$ be a solution of $D_n(G) = X_n$, then the system $D_i(F') = X_i - D_i(G)$ $(i = 1, \ldots, n)$ is compatible, which implies $D_n(X'_n) = 0$, where $X'_n = X_n - D_i(G)$, so $X'_n \in \oplus_{k>0} (O_k \otimes O^{(n-1)}_+)$. We have $O^{(n-1)}_+$ is the analogue of $O_+$ at order $n - 1$. Hence the system $D_i(F') = X_i - D_i(G)$ $(i = 1, \ldots, n - 1)$ is compatible and we may apply to it the result at order $n - 1$ to obtain a solution $F'$. Then a solution of $D_i(F) = X_i$ is $F' + G$.

Let $D : u\mathbb{C}[u] \to u\mathbb{C}[u]$ be the map $u \frac{\partial}{\partial u}$ and let $I := D^{-1}$. The map $D_i : u\mathbb{C}[u][\ell] \to u\mathbb{C}[u][\ell]$ is bijective and its inverse is given by $D^{-1}_i (F(u) \ell^n) = \sum_{k=0} (-1)^k a(a - 1) \cdots (a - k + 1) (F^{k+1})(u)^{a-k}$. We have $O_+ = O^{(n-1)}_+ \oplus u \mathbb{C}[u] [\ell_n] \oplus \mathfrak{m}^{(n-1)} \otimes \mathbb{C}[\ell_n]$ (where $O^{(n-1)}_+$, $\mathfrak{m}^{(n-1)}$ are the analogues of $O$, $\mathfrak{m}$ at order $n - 1$, $\otimes$ is the completed tensor product). The endomorphism $D_n$ preserves this decomposition and a section of $D_n$ is given by $(id \otimes D^{-1}_n) + (id \otimes J)$, where $J \in \text{End}(\mathbb{C}[\ell])$ is a section of $\partial/\partial \ell$.

It follows from the fact that $I$ preserves the radius of convergence of a series that the same holds for the section of $D_n$ defined above. One then follows the above construction of a solution $X$ of $D_i(X) = X_i$ and uses the fact that $D_i$ also preserves the radius of convergence to show by induction that $X$ has radius $R$ if the $X_i$ do.

Proposition A.2. Let $X_i(u_1, \ldots, \ell_n) \in \oplus_{k>0} (A_k \otimes O_+)$ be such that $D_i(X_j) - D_j(X_i) = [X_i, X_j]$. Then there exists a unique $F(u_1, \ldots, \ell_n) \in 1 + \oplus_{k>0} (A_k \otimes O_+)$ such that $D_i(F) = X_iF$ for $i = 1, \ldots, n$. If the $X_i$ have radius $R$, then so does $F$.

Proof. Let us prove the uniqueness. If $F', F''$ are two solutions, then $F^{-1} F''$ is a constant (as $\mathfrak{g}^{\text{an}}_0 = \text{Ker}(D : O \to O) = 0$), and it also belongs to $1 + \oplus_{k>0} (A_k \otimes O_+)$, which implies that $F = F''$. To prove the existence, one sets $F = f_1 + f_2 + \ldots$, $X_i = x_i^{(1)} + \ldots$, where $f_k, x_k^{(i)} \in A_k \otimes O_+$ and solves by induction the system $D_i(f_k) = x_i^{(1)} f_{k-1} + \ldots + x_i^{(k)}$ using Proposition A.1.

Proposition A.3. Let $C_i(u_1, \ldots, u_n) \in \oplus_{k>0} (A_k \otimes [u_1, \ldots, u_n])$ $(i = 1, \ldots, n)$ be such that $u_i D_i(C_j) = [C_i, C_j]$ for any $i, j$. Assume that the series $C_i$ have radius $R$.

Then there exists a unique solution of the system $u_i D_i(X) = C_i X$, analytic in the domain $\{u||u| \leq R, u \notin \mathbb{R}^-\}^n$, such that the ratio $(u_1^{C_1} \cdots u_n^{C_n})^{-1} X(u_1, \ldots, u_n)$ (we set $C_0 := C_i(0, \ldots, 0)$) has the form $1 + \sum_{k>0} \sum_{a_1, \ldots, a_n} r_k^{a_1, \ldots, a_n} (u_1, \ldots, u_n)$ (the second sum is finite for any $k$), $r_k^{a_1, \ldots, a_n}$ has degree $k$, $a_i \geq 0$, $i \in \{1, \ldots, n\}$, and leads (after substitution $u_1 = 0, \ldots, u_n = 0$) to an analytic function $X(u_1, \ldots, u_n) = O(u_1^{C_1} \cdots u_n^{C_n})$. The same is then true of the ratio $X(u_1, \ldots, u_n) (u_1^{C_1} \cdots u_n^{C_n})^{-1}$; we write $X(u_1, \ldots, u_n) \simeq u_1^{C_1} \cdots u_n^{C_n}$.

Proof. Let us show the existence of $X$. The compatibility condition implies that $[C_0, C_0] = 0$. If we set $Y(u_1, \ldots, u_n) := (u_1^{C_1} \cdots u_n^{C_n})^{-1} X(u_1, \ldots, u_n)$, then $X$ is a solution iff $Y$ is a solution of $u_i D_i(Y) = \exp(-\sum_j (1 \log u_j) C_j^0) (C_i - C_0^0) \cdot Y$.

Let us set $X_i(u_1, \ldots, \ell_n) := \exp(-\sum_j (1 \log u_j) C_j^0) (C_i(u_1, \ldots, u_n) - C_i(0, \ldots, 0))$, then $X_i(u_1, \ldots, \ell_n) \in \oplus_{k>0} (A_k \otimes O_+)$. We then apply Proposition A.2 and find a solution $Y \in 1 + \oplus_{k>0} (A_k \otimes O_+)$ of $D_i(Y) = X_iY$. Let $Y_k$ be the component of $Y$ of degree $k$. Since $Y$ has radius $R$, the replacement $\ell_i = \log u_i$ in $Y_k$ for $u_i \in \{u||u| \leq R, u \notin \mathbb{R}^-\}^n$ gives an analytic function on $\{u||u| \leq R, u \notin \mathbb{R}^-\}^n$. Moreover, $O_+ = \sum_{a_1, \ldots, a_n} u_1^{a_1} \cdots u_n^{a_n} y_{a_1, \ldots, a_n} (u_1, \ldots, u_n)$, which gives a decomposition $Y_k = \sum_{a_1, \ldots, a_n} u_1^{a_1} \cdots u_n^{a_n} y_{a_1, \ldots, a_n} (u_1, \ldots, u_n)$ and leads (after substitution $\ell_i = \log u_i$) to the above estimates.

The ratio $X(u_1, \ldots, u_n) (u_1^{C_1} \cdots u_n^{C_n})^{-1}$ is then $1 + \exp(\sum_j (1 \log u_j) (Y(u_1, \ldots, u_n) - 1)$; the term of degree $k$ has finitely many contributions to which we apply the above estimates.
Let us prove the uniqueness of \( X \). Any other solution has the form \( X = X(1 + c_k + \ldots) \) where \( c_j \in \mathbb{A}_k \), and \( c_k \neq 0 \). Then the degree \( k \) term is transformed by the addition of \( c_k \), which cannot be split as a sum of terms in the various \( O(u_i (\log u_1)^{a_i}\ldots(\log u_n)^{a_n}) \).

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