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<u>Modeling the interaction of a structure with the ground.</u>

Let the surface of the ground be

$$H(\underline{X}) = 0,$$

with $H(\underline{X}) > 0$ above the ground
and $H(\underline{X}) < 0$ below the ground.

Let $\underline{X}(t)$ be any point of a structure that interacts with the ground, and let $\underline{U}(t)$ be the velocity of that point.

We allow the point $\underline{X}(t)$ to go below the surface, but when it does so, it experiences a force pushing it out and also a tangential force that models sliding friction.

The normal direction to the ground, pointing upwards, is approximately given by

$$\frac{\nabla H}{\|\nabla H\|}$$

and the distance below the ground when $H(\underline{X}) < 0$ is approximately given by

$$\frac{-H(\underline{X})}{\|\nabla H(\underline{X})\|}$$

See APPENDIX for justification of these statements.

Also, the tangential velocity is approximately given by

$$\underline{U}^{(tan)} = \underline{U} - \frac{\underline{U} \cdot \nabla H(\underline{X})}{\|\nabla H(\underline{X})\|} \frac{\nabla H(\underline{X})}{\|\nabla H(\underline{X})\|}$$

Any point of the structure for which $H(\underline{X}) < 0$ therefore feels a force given by

$$\underline{F} = S\,\frac{-H(\underline{X})}{\|\nabla H(\underline{X})\|}\,\frac{\nabla H(\underline{X})}{\|\nabla H(\underline{X})\|}$$

$$- \mu\, S\,\frac{-H(\underline{X})}{\|\nabla H(\underline{X})\|}\,\frac{\underline{U}^{(tan)}}{\|\underline{U}^{tan}\|}$$

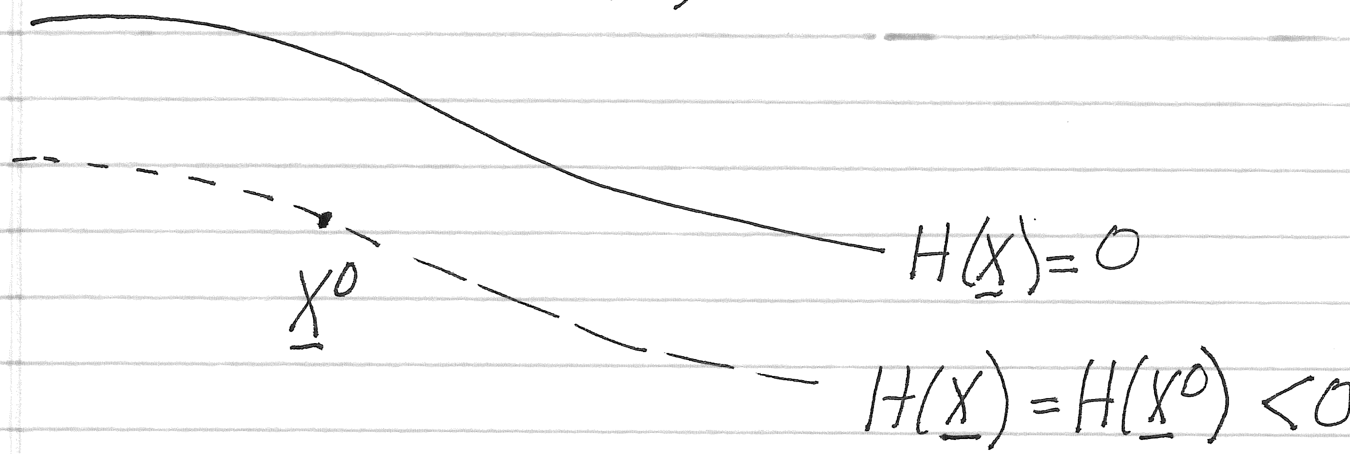
$$= S\,\frac{-H(\underline{X})}{\|\nabla H(\underline{X})\|}\left(\frac{\nabla H(\underline{X})}{\|\nabla H(\underline{X})\|} - \mu\,\frac{\underline{U}^{(tan)}}{\|\underline{U}^{tan}\|}\right)$$

Here μ is the coefficient of sliding friction.

Recall that sliding friction is proportional to the normal force and in the direction opposing the tangential motion.

APPENDIX

$H(\underline{X}) > 0$



$$H(\underline{X}) = H(\underline{X}^0) + (\nabla H)^0 \cdot (\underline{X} - \underline{X}^0) + \cdots$$

The surface $H(\underline{X}) = H(\underline{X}^0)$ is therefore given by

$$0 = (\nabla H)^0 \cdot (\underline{X} - \underline{X}^0) + \cdots$$

If we drop $\cdots$, we get the equation of the tangent plane to this surface, namely

$$0 = (\nabla H)^0 \cdot (\underline{X} - \underline{X}^0)$$

This shows that $(\nabla H)^0$ is normal to the tangent plane to the surface $H(\underline{X}) = H(\underline{X}^0)$ at the point $\underline{X} = \underline{X}^0$, since it is orthogonal to $\underline{X} - \underline{X}^0$ for every $\underline{X}$ in the tangent plane. By definition, being normal to the tangent plane is the same thing as being normal to the surface.

Now we seek an approximate formula for the distance between a point $\underline{X}^0$ and the closest point on the surface $H(\underline{X})=0$.

Instead of using the exact function $H(\underline{X})$ we use its linear approximation in the neighborhood of $\underline{X}^0$. Thus, we seek $\underline{X}$ that minimizes

$$\|\underline{X}-\underline{X}^0\|^2$$

subject to the constraint

$$0 = H(\underline{X}^0) + (\nabla H)^0 \cdot (\underline{X}-\underline{X}^0)$$

let

$$\underline{N}^0 = \frac{(\nabla H)^0}{\|(\nabla H)^0\|}$$

Then

$$0 = \frac{H(\underline{X}^0)}{\|(\nabla H)^0\|} + \underline{N}^0 \cdot (\underline{X}-\underline{X}^0)$$

We can always write

$$(\underline{X}-\underline{X}^0) = ((\underline{X}-\underline{X}^0)\cdot\underline{N}^0)\underline{N}^0 + \underline{Z}$$

and then $\underline{Z}\cdot\underline{N}^0 = 0$, so

$$\|\underline{X}-\underline{X}^0\|^2 = ((\underline{X}-\underline{X}^0)\cdot\underline{N}^0)^2 + \|\underline{Z}\|^2$$

$$= \left(\frac{H(\underline{X}^0)}{\|(\nabla H)^0\|}\right)^2 + \|\underline{Z}\|^2$$

Clearly the choice of $\underline{Z}$ that minimizes this expression is $\underline{Z}=0$, and then we have

$$\|\underline{X}-\underline{X}^0\| = \frac{|H(\underline{X}^0)|}{\|(\nabla H)^0\|}$$

(Here $\underline{X}$ is the point that solves the minimization problem.)

If $H(\underline{X}^0)<0$, which is the case of interest

$$\|\underline{X}-\underline{X}^0\| = \frac{-H(\underline{X}^0)}{\|(\nabla H)^0\|}$$