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3D Gyrocompass*

In an ordinary gyrocompass, the axis of the gyroscope is constrained to lie in a locally horizontal plane. Here, we allow the axis to point in any direction.

Let a gyroscope be mounted in such a way that its center of mass rotates with the earth but its axis can point in any direction. In the absence of friction, the axis of the gyroscope will then point in a fixed direction with respect to an inertial frame, and therefore the axis of the gyroscope will appear when viewed from the earth to rotate about an axis parallel to that of the earth. Here, however, we assume there is a frictional force applied to the axis of the gyroscope that opposes motion of the axis itself relative to the earth (without opposing the spin of the gyroscope). The result will be that the gyroscope aligns with the axis of the earth, and we investigate the stability of this alignment.

*Thanks to Shafer Smith for suggesting this kind of gyrocompass.

We assume that the gyroscope has an axis of symmetry. Let $\underline{a}(t)$ be a unit vector parallel to this axis. Let $\underline{\omega}(t)$ be the angular velocity of the gyroscope. We can always write

$$(1) \qquad \underline{\omega}(t) = \omega_0(t)\,\underline{a}(t) + \underline{\omega}_1(t)$$

where

$$(2) \qquad \underline{\omega}_1(t) \cdot \underline{a}(t) = 0$$

Note that $\omega_0(t)$ is a scalar but $\underline{\omega}_1(t)$ is a vector.

Since $\underline{a}(t)$ moves rigidly along with the gyroscope, it satisfies

$$(3) \qquad \frac{d\underline{a}}{dt} = \underline{\omega}(t) \times \underline{a}(t) = \underline{\omega}_1(t) \times \underline{a}(t)$$

Because of axial symmetry, the angular momentum of the gyroscope is of the form

$$\underline{L}(t) = I_0\, \omega_0(t)\, \underline{a}(t) + I_1\, \underline{\omega}_1(t) \tag{4}$$

where I_0 is the moment of inertia of the gyroscope about its axis of symmetry, and I_1 is the moment of inertia about any line through the center of mass that is perpendicular to the axis of symmetry.

Let $\Omega\, \underline{z}$ be the angular velocity of the earth, where $\underline{z}$ is a constant unit vector that defines the z direction of our inertial frame. We assume that the gyroscope experiences a frictional torque of the form

$$\begin{aligned} \underline{\tau} &= -\beta\, \underline{a}(t) \times \left(\frac{d\underline{a}}{dt} - \Omega\left(\underline{z} \times \underline{a}(t)\right) \right) \\ &= -\beta\, \underline{a}(t) \times \left(\left(\underline{\omega}_1(t) - \Omega\, \underline{z} \right) \times \underline{a}(t) \right) \end{aligned} \tag{5}$$

The explanation of this form of the frictional torque is as follows. Consider a material point on the axis of the gyroscope. With the center of the earth as origin and in an inertial frame, the position of such a point is given by

(6) $$\underline{x}(t) = \underline{x}_{cm}(t) + r\,\underline{a}(t)$$

where r is a constant. The velocity of this point is

(7) $$\frac{d\underline{x}}{dt} = \frac{d\underline{x}_{cm}}{dt} + r\frac{d\underline{a}}{dt}$$

$$= \Omega\,\underline{z} \times \underline{x}_{cm}(t) + r\frac{d\underline{a}}{dt}$$

On the other hand, a point that is rigidly attached to the earth but happens to be at position $\underline{x}(t)$ at time t has velocity

(8) $$\underline{v}_{ref} = \Omega\,\underline{z} \times \underline{x}(t)$$

$$= \Omega\,\underline{z} \times \underline{x}_{cm}(t) + r\,\Omega\,\underline{z} \times \underline{a}(t)$$

Let a force $\underline{f}(t)$ that opposes the motion of $\underline{X}(t)$ relative to the earth be applied to the point $\underline{X}(t)$. Such a force is given by

$$(9) \qquad \underline{f}(t) = -\beta_0 \left(\frac{d\underline{X}}{dt} - \underline{v}_{ref}(t) \right)$$

$$= -\beta_0 r \left(\frac{d\underline{a}}{dt} - \Omega \underline{z} \times \underline{a}(t) \right)$$

and since this is applied at the point $\underline{X}(t)$, it produces a torque about the center of mass equal to

$$(10) \qquad \underline{\tau} = r \underline{a}(t) \times \underline{f}(t)$$

$$= -\beta_0 r^2 \underline{a}(t) \times \left(\frac{d\underline{a}}{dt} - \underline{\Omega} \underline{z} \times \underline{a}(t) \right)$$

Thus, with $\beta = \beta_0 r^2$, we have equation (5). More generally, we could have derived (5) by considering a force like that given by (9) but distributed along the axis of the gyroscope instead of being concentrated at a single point.

The equation of motion is now

(11) $$\frac{d\underline{L}}{dt} = \underline{\tau}$$

with $\underline{L}$ given by (4) and with $\underline{\tau}$ given by equation (5). If we apply $\underline{a}(t)\cdot$ to both sides of (11), we get

(12) $$I_0 \frac{d\omega_0}{dt} = 0$$

since $\underline{a}(t)\cdot\frac{d\underline{a}}{dt} = 0$ because $\underline{a}(t)$ is a unit vector, and since $\underline{a}(t)\cdot\underline{\omega}_1(t) = 0$ by definition. Thus ω_0 is constant, and from now on we will write ω_0 instead of $\omega_0(t)$.

Next, we apply $\times\,\underline{a}(t)$ on the right to both sides of (11). This gives

$$(13) \quad I_0 \omega_0 \left(\frac{d\underline{a}}{dt} \times \underline{a}(t) \right) + I_1 \left(\frac{d\underline{\omega}_1}{dt} \times \underline{a}(t) \right)$$

$$= -\beta \left(\underline{a}(t) \times \left(\frac{d\underline{a}}{dt} - \Omega(\underline{z} \times \underline{a}(t)) \right) \right) \times \underline{a}(t)$$

$$= -\beta \left(\frac{d\underline{a}}{dt} - \Omega(\underline{z} \times \underline{a}(t)) \right)$$

To get the last line of (13), we made use of the fact that $\frac{d\underline{a}}{dt}$ and $\underline{z} \times \underline{a}(t)$ are both orthogonal to $\underline{a}(t)$.

We can make (13) into an equation of motion for $\underline{a}(t)$ by eliminating $d\underline{\omega}_1/dt$. To do this, we start by differentiating in (3) with respect to t to obtain

$$(14) \quad \frac{d^2\underline{a}}{dt^2} = \frac{d\underline{\omega}_1}{dt} \times \underline{a}(t) + \underline{\omega}_1(t) \times \frac{d\underline{a}}{dt}$$

$$= \frac{d\underline{\omega}_1}{dt} \times \underline{a}(t) + \underline{\omega}_1(t) \times \left(\underline{\omega}_1(t) \times \underline{a}(t) \right)$$

$$= \frac{d\underline{\omega}_1}{dt} \times \underline{a}(t) + \|\underline{\omega}_1(t)\|^2 \left(-\underline{a}(t) \right)$$

$$= \frac{d\underline{\omega}_1}{dt} \times \underline{a}(t) - \left\|\frac{d\underline{a}}{dt}\right\|^2 \underline{a}(t)$$

In the various steps of (14), we repeatedly used (3) together with $\underline{\omega}_1(t) \cdot \underline{a}(t) = 0$.

Substituting (14) into (13) we get an equation of motion for $\underline{a}(t)$:

(15) $$I_0 \omega_0 \left(\frac{d\underline{a}}{dt} \times \underline{a}(t)\right) + I_1 \left(\frac{d^2\underline{a}}{dt^2} + \left\|\frac{d\underline{a}}{dt}\right\|^2 \underline{a}(t)\right)$$

$$= -\beta\left(\frac{d\underline{a}}{dt} - \Omega\underline{z} \times \underline{a}(t)\right)$$

Since this is a second-order equation, it requires initial data

(16) $$\underline{a}(0), \quad \frac{d\underline{a}}{dt}(0)$$

And since $\underline{a}(t)$ is supposed to be a unit vector, these must satisfy

(17) $$\|\underline{a}(0)\| = 1, \quad \underline{a}(0) \cdot \frac{d\underline{a}}{dt}(0) = 0$$

If $\underline{a}(t)$ is independent of t, then (15) reduces to

(18) $$\underline{z} \times \underline{a} = 0$$

Since $\underline{a}$ and $\underline{z}$ are both unit vectors, this implies

(19) $$\underline{a} = \pm \underline{z}$$

We now investigate the stability of these solutions. Without loss of generality, we may consider $\underline{a} = +\underline{z}$, since (15) is invariant under the change of variable

(20) $$\underline{a} \rightarrow -\underline{a} \quad , \quad \omega_0 \rightarrow -\omega_0$$

Accordingly, we look for a solution of (15) of the form

(21) $$\underline{a}(t) = \underline{z} + \varepsilon A_1(t)\underline{x} + \varepsilon A_2(t)\underline{y}$$

where $(\underline{x}, \underline{y}, \underline{z})$ is a right-handed orthonormal triad and ε is a small parameter.

Note that, according to (21),

(22) $$\|\underline{a}(t)\|^2 = 1 + O(\varepsilon^2)$$

Now we substitute (21) into (15) and keep only those terms that are $O(\varepsilon)$. The result, after dividing by ε, is

(23) $$I_0 \omega_0 \left(-\frac{dA_1}{dt}\underline{y} + \frac{dA_2}{dt}\underline{x} \right) + I_1 \left(\frac{d^2 A_1}{dt^2}\underline{x} + \frac{d^2 A_2}{dt^2}\underline{y} \right) = -\beta \left(\frac{dA_1}{dt}\underline{x} + \frac{dA_2}{dt}\underline{y} - \Omega \left(A_1 \underline{y} - A_2 \underline{x} \right) \right)$$

Dividing by I_1 and separating the $\underline{x}$ and $\underline{y}$ components of this equation, we get

$$\frac{d^2A_1}{dt^2} + \frac{\beta}{I_1}\frac{dA_1}{dt} + \frac{I_0}{I_1}\omega_0\frac{dA_2}{dt} + \frac{\beta}{I_1}\Omega A_2 = 0 \tag{24}$$

$$\frac{d^2A_2}{dt^2} + \frac{\beta}{I_1}\frac{dA_2}{dt} - \frac{I_0}{I_1}\omega_0\frac{dA_1}{dt} - \frac{\beta}{I_1}\Omega A_1 = 0 \tag{25}$$

To simplify the notation, let

$$\gamma = \frac{\beta}{I_1}, \qquad \delta = \frac{I_0}{I_1} \tag{26}$$

To study the stability of the linear system (24-25), we look for a solution of the form

$$\begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} = \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix} e^{\lambda t} \tag{27}$$

This gives the homogeneous linear system

$$\begin{pmatrix} (\lambda^2 + \gamma\lambda) & (\delta\omega_0\lambda + \gamma\Omega) \\ -(\delta\omega_0\lambda + \gamma\Omega) & (\lambda^2 + \gamma\lambda) \end{pmatrix} \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix} = 0 \tag{28}$$

Equation (28) has nontrivial solutions if and only if

(29) $$(\lambda^2+\gamma\lambda)^2 + (\delta\omega_0\lambda+\gamma\Omega)^2 = 0$$

which is equivalent to

(30) $$\lambda^2+\gamma\lambda \pm i(\delta\omega_0\lambda+\gamma\Omega) = 0$$

If λ solves (30) with the choice $(+i)$, then $\bar{\lambda}$ solves (30) with the choice $(-i)$. Since λ and $\bar{\lambda}$ have the same real part, and since stability is determined by the real part of λ, we need only consider one of these two cases. Making the choice $(-i)$ in (30), we get the quadratic equation

(31) $$\lambda^2 + (\gamma - i\omega_0\delta)\lambda - i\gamma\Omega = 0$$

In which γ and δ are positive, but ω_0 and Ω can have either sign.

Instead of solving (31) explicitly, we think of each of the two solutions of (31) as a function of γ, and we determine the behaviors of these functions for small γ.

When $\gamma=0$, the two solutions of (31) are

(32) $$\lambda_1(0)=0 \quad , \quad \lambda_2(0)=i\omega_0\delta$$

Differentiating (31) with respect to γ, we get, for both solutions

(33) $$2\lambda(\gamma)\lambda'(\gamma)+(\gamma-i\omega_0\delta)\lambda'(\gamma)+\lambda(\gamma)-i\Omega=0$$

or

(34) $$\lambda'(\gamma)=\frac{i\Omega-\lambda(\gamma)}{2\lambda(\gamma)+(\gamma-i\omega_0\delta)}$$

Evaluating this at $\gamma=0$ and making use of (32) gives

(35) $$\lambda_1'(0)=\frac{i\Omega}{-i\omega_0\delta}=-\frac{\Omega}{\omega_0\delta}$$

(36) $$\lambda_2'(0)=\frac{i\Omega-i\omega_0\delta}{2i\omega_0\delta-i\omega_0\delta}=\frac{i(\Omega-\omega_0\delta)}{i\omega_0\delta}=\frac{\Omega}{\omega_0\delta}-1$$

Therefore, the Taylor series for the two functions $\lambda(\gamma)$ are

(37) $$\lambda_1(\gamma) = -\frac{\Omega}{\omega_0 \delta}\gamma + O(\gamma^2)$$

(38) $$\lambda_2(\gamma) = i\omega_0\delta - \left(1 - \frac{\Omega}{\omega_0\delta}\right)\gamma + O(\gamma^2)$$

Thus, the solution $\underline{a} = \underline{z}$ is asymptotically stable for sufficiently small but positive γ if and only if

(39) $$0 < \frac{\Omega}{\omega_0 \delta} < 1$$

which is equivalent to

(40) $$0 < \frac{I_1 \Omega}{I_0 \omega_0} < 1$$

The first inequality of (39) or (40) says that Ω and ω_0 must have the same sign for the solution to be stable. This means that the spin of the gyroscope and the spin of the earth must be parallel (not anti-parallel).

If we had considered $\underline{a} = -\underline{z}$ instead of $\underline{a} = \underline{z}$, we would have obtained the same stability conditions as above except that ω_0 would have been replaced by $-\omega_0$, see (20). Thus, a necessary condition for stability would then have been that ω_0 and Ω have opposite sign, but since $\underline{a}$ and $\underline{z}$ would then point in opposite directions, this would again mean that the spin of the gyroscope and the spin of the earth have to be <u>parallel</u> (not anti-parallel) for stability.

The second inequality of (39) or (40) says that the spin of the gyroscope has to be sufficiently fast in relation to the spin of the earth for the time-independent solution $\underline{a} = \underline{z}$ to be stable. If

(41) $$I_0 |\omega_0| < I_2 |\Omega|$$

then neither of the time-independent solutions $\underline{a} = \pm \underline{z}$ is stable. We leave it to the reader to figure out what happens in this case. Note, however, that (41) is unlikely to hold for any practical gyroscope.

In the foregoing, we determined whether the solution $\underline{a} = \underline{Z}$ is stable or not only for γ sufficiently small, but we can easily extend the validity of these results by asking whether either of the two functions $\lambda(\gamma)$ can cross the imaginary axis at some positive value of γ. To study this, set $\lambda = i\mu$ in equation (31), where μ is real. Separating the real and imaginary parts, we then get the two equations

$$-\mu^2 + \omega_0 \delta \mu = 0 \tag{42}$$

$$\gamma(\mu - \Omega) = 0 \tag{43}$$

Since $\gamma > 0$, $(43) \Rightarrow \mu = \Omega$
and since $\Omega \neq 0$, $\mu \neq 0$, and then $(42) \Rightarrow \mu = \omega_0 \delta$. Thus, a crossing of the imaginary axis for $\gamma > 0$ can only occur in the very special case that $\omega_0 \delta = \Omega$. Excluding this special case, no such crossing can occur, so the results we found for small γ are actually applicable to all $\gamma > 0$.

In contrast to the ordinary gyrocompass, the 3D unconstrained gyrocompass has the following nice features.

- Its function is exactly the same at all lattitudes

- Instead of pointing north in a locally horizontal plane, it points at the north star, i.e., parallel to the axis of rotation of the earth. In this way it actually indicates the lattitude by its angle of inclination with respect to a locally horizontal plane.