THE COMPLEXITY OF A CLASS OF INFINITE GRAPHS

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Given a class of graphs $\mathcal{G}$, we say that $\mathcal{G}$ has a *universal* element $G_0 \in \mathcal{G}$ if any other graph $G \in \mathcal{G}$ is isomorphic to a (not necessarily induced) subgraph of G_0.

In a recent paper (Komjáth-Pach,1989) we have extended this definition, as follows. Let $c(\mathcal{G})$, the *complexity* of a class of graphs $\mathcal{G}$, be defined as the least cardinality of a subset $\mathcal{G}_0 \subseteq \mathcal{G}$ with the property that any element of $\mathcal{G}$ is isomorphic to a subgraph of some $G_0 \in \mathcal{G}_0$. Obviously, $\mathcal{G}$ has a universal element if and only if $c(\mathcal{G}) = 1$.

In all interesting examples $\mathcal{G}$ is *closed under containment*, i.e., $G \in \mathcal{G}$ implies that $G' \in \mathcal{G}$ for any $G' \subseteq G$. This condition is satisfied e.g. for all classes of graphs which can be obtained in the following way. Given a cardinal κ and a family $\mathcal{H}$ of so-called *forbidden subgraphs*, let $\mathcal{G}_\kappa(\mathcal{H})$ be defined as the class of all graphs with at most κ vertices containing no subgraph isomorphic to any element of $\mathcal{H}$.

Let $\mathcal{G}_k$ denote the class of all countable graphs containing no k vertex-disjoint cycles. That is, using the above notation, $\mathcal{G}_k = \mathcal{G}_\omega(\mathcal{H}_k)$, where $\mathcal{H}_k$ stands for the family of all (finite) graphs consisting of k vertex-disjoint cycles. In particular, $\mathcal{G}_1$ is the class of all countable forests. Since $\mathcal{G}_k$ has continuum many elements, its complexity is at most 2^ω.

Theorem 1. *Let $1 < k < \omega$, and let $\mathcal{G}_k$ be the class of all countable graphs containing no k vertex-disjoint cycles. Then $c(\mathcal{G}_k) = \omega$.*

Proof. First we show that $c(\mathcal{G}_k) \leq \omega$.

Let G be a fixed countable graph without k vertex-disjoint cycles. Fix a finite subset $K \subseteq V(G)$ with the property that any cycle of G contains at least one element of K. (Note that the vertex set of any maximal system of cycles in G obviously satisfies this condition. Moreover, by a result of Erdös and Pósa, K can always be chosen so as to have fewer than $ck \log k$ elements.) We will refer to K, as the *kernel* of G. The vertices of G outside the kernel are called *external*. The external vertices induce a forest in G, and they can be classified according to which elements of K they are connected to. We color two external points with the same *color*, if and only if their sets of neighbors in K are the same. Thus, we obtain a coloring function ϕ: $(V(G) - K) \to \Gamma$, where Γ is the set of colors and $|\Gamma| = 2^{|K|}$.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a set of at most $|K|$ vertex-disjoint paths in G-K, and let v_i and v_i' denote the endpoints of P_i. (We do not exclude the possibility that $v_i = v_i'$, i.e., P_i consists of a single vertex.) The *type of* P_i (with respect to K) is defined by the colors of its endpoints:

$$\text{type}(P_i) = \begin{cases} [\phi(v_i)] \text{ if } v_i = v_i' \ , \\ [\phi(v_i), \phi(v_i')] \text{ if } v_i \neq v_i' \ ; \end{cases}$$

$$\text{Type}(\mathcal{P}) = [\text{type}(P_i) \colon 1 \leq i \leq m] \ .$$

(We write $[\cdot]$ instead of $\{\cdot\}$ to indicate that some of the elements may be repeated, i.e., they form a *multiset.*) Furthermore, let $\mathcal{T}(G)$ be defined as the set of all Type($\mathcal{P}$), where $\mathcal{P}$ is a system of at most $|K|$ vertex-disjoint paths in G-K. Clearly, $\mathcal{T}(G)$ is closed under containment, i.e., $T \in \mathcal{G}(G)$ implies that every submultiset $T' \subseteq T$ also belongs to $\mathcal{T}(G)$.

For each member $T \in \mathcal{T}(G)$, fix a system $\mathcal{P}^T$ of vertex-disjoint paths with Type$(\mathcal{P}^T) = T$. Put

$$L = \bigcup_{T \in \mathcal{T}(G)} V(\mathcal{P}^T) \ .$$

Evidently, L is a finite set and $L \cap K = \emptyset$. Since G-K is a forest, there are only finitely many external vertices lying on some path connecting two elements of L. Let $\overline{L}$ denote the set obtained from L by adding all of these vertices.

Now $G - (K \cup \overline{L})$ falls into countably many connected components (trees) G_i $(i = 1, 2, \ldots)$. Every G_i has at most one point adjacent to some element of $\overline{L}$. If such a point exists, then it is called the *root of* G_i. By the definition of $\overline{L}$, the root of G_i has only one neighbor in $\overline{L}$.

Next we describe a procedure that will enable us to add new vertices to G, without creating k vertex-disjoint cycles.

Let G_1 be a fixed component of $G - (K \cup \overline{L})$, and pick a color $\gamma \in \Gamma$ which occurs among the vertices of G_1 at least twice. Thus, one can find two distinct points

$u_1, u_2 \in V(G_1)$ which are adjacent to the same elements of K. Let G' denote the graph obtained from G by adding a new vertex u of color γ and connecting it to any point $w \in V(G_1)$. That is,

$$\begin{aligned} V(G') &= V(G) \cup \{u\} \ , \\ E(G') &= E(G) \cup \{uv\colon v \in K \ , \ u_1 v \in E(G)\} \cup \{uw\} \ . \end{aligned}$$

Obviously, any cycle of G' passes through at least one element of K. In other words, K is also a kernel of G', hence it can be used to define $\mathcal{T}(G')$.

Lemma 1. $\mathcal{T}(G') = \mathcal{T}(G)$.

We have to prove only that $\mathcal{T}(G') \subseteq \mathcal{T}(G)$. Assume, in order to obtain a contradiction, that there is a system $\mathcal{P} = \{P_i\colon 1 \le i \le m\}$ of at most $|K|$ vertex-disjoint paths in $G' - K$ such that

$$T = \ \text{Type}(\mathcal{P}) = [\text{type}(P_i)\colon 1 \le i \le m] \in \mathcal{T}(G') - \mathcal{T}(G) \ .$$

Suppose, without loss of generality, that $\mathcal{P}$ is a *minimal* system satisfying this condition, i.e.,

$$T_j = [\text{type}(P_i)\colon 1 \le i \le m \ , \ i \ne j] \in \mathcal{T}(G)$$

for every j $(1 \le j \le m)$.

Clearly, one of the paths P_i (say, P_1) must contain the new vertex u, otherwise $T = \text{Type}(\mathcal{P}) \in \mathcal{T}(G)$. Moreover, u must be an endpoint of P_1, because the degree of u in $G' - K$ is 1. Let v_i and v_i' denote the (not necessarily distinct) endpoints of P_i. Thus, we can assume that $u = v_1$.

Let $G_1' \subseteq G'$ denote the tree obtained from G_1 by adding the vertex u and the edge uw.

Observe that no path P_j can be entirely contained in G_1'. To see this, recall that there is a system $\mathcal{P}^{T_j}$ of vertex-disjoint paths in L with $\text{Type}(\mathcal{P}^{T_j}) = T_j$. So, if P_j were in G_1' for some $j \ne 1$, then $\mathcal{P}^{T_j} \cup \{P_j\}$ would form a system of vertex-disjoint paths in G-K, whose type is T. If $P_1 \subseteq G_1'$, then consider the uniquely determined paths P_{11} and $P_{12} \subseteq G_1$ connecting v_1' to u_1 and u_2, respectively. At least one of them (say, P_{11}) is of the same type as P_1. Hence, $\mathcal{P}^{T_1} \cup \{P_{11}\}$ is a system of vertex-disjoint paths in G-K, whose type is T. In both cases we can conclude that $T \in \mathcal{T}(G)$, contradiction.

Thus, we can assume that v_1' is not in G_1'. This implies that G_1 has a root r, and P_1 must pass through r. Let P_{11} denote the (unique) path connecting v_1' and u_1 in G-K. Clearly, P_{11} also passes through r and type(P_{11}) = type(P_1). Notice that P_{11} is disjoint from any P_j $(2 \leq j \leq m)$, otherwise P_j would lie entirely in G_1, contradicting our previous observation. Hence, $\{P_{11}, P_2, P_3, \ldots, P_m\}$ is a system of vertex-disjoint paths in G-K, whose type is T, which is again a contradiction. This completes the proof of Lemma 1.

Lemma 2. *G' has no k vertex-disjoint cycles.*

Assume, for contradiction, that there is a system $\{C_i\colon 1 \leq i \leq k\}$ of k vertex-disjoint cycles in G'. Since every cycle must visit K, the pieces of the C_i lying outside K form a system $\mathcal{P}'$ of at most $|K|$ vertex-disjoint pahts in G'-K. By Lemma 1, there exists a system $\mathcal{P}$ of vertex-disjoint paths in G-K such that Type$(\mathcal{P})$ = Type$(\mathcal{P}')$. For every cycle C_i, replace each piece lying outside K by the corresponding path in $\mathcal{P}$. Thus, we obtain k vertex-disjoint cycles in G, the desired contradiction establishing Lemma 2.

By the repeated application of the above procedure, we can add countably many new vertices to G, to obtain a graph G^* satisfying the conditions summarized in the following statement.

Lemma 3. *Let G be a countable graph without k vertex-disjoint cycles, and let $K, \overline{L} \subseteq V(G)$ be finite sets, as defined above.*

Then there exists a countable graph G^* with the following properties.

(i) *G^* contains G as an induced subgraph;*

(ii) *G^* has no k vertex-disjoint cycles;*

(iii) *every cycle of G^* meets K.*

Furthermore, let $\phi^*\colon (V(G^*) - K) \to \Gamma$ be a coloring assigning the same color to two vertices if and only if they are connected to the same elements of K. Let G_i^* $(i = 1, 2, \ldots)$ denote the connected components of $G^* - (K \cup \overline{L})$.

(iv) *Each component G_i^* is connected to $\overline{L}$ by at most one edge;*

(v) *if* $\gamma \in \Gamma$ *is any color assigned to at least two points of* G_i^*, *then every vertex of* G_i^* *has infinitely many neighbors of color* γ $(i = 1, 2, \ldots)$.

Let $\mathcal{G}_k^*$ be the family of all countable graphs that can be obtained as G^* for some $G \subset \mathcal{G}_k$. Obviously, $\mathcal{G}_k^* \subseteq \mathcal{G}_k$ and every element of $\mathcal{G}_k$ can be embedded into some element of $\mathcal{G}_k^*$ as an induced subgraph. On the other hand, $\mathcal{G}_k^*$ is clearly a *countable* family of graphs. To see this, we have to note only that

(a) there are only countably many different graphs that can be obtained as the restriction of some $G^* \in \mathcal{G}_k^*$ to the corresponding subset $K \cup \overline{L}$ (because K and $\overline{L}$ are finite);

(b) there are only countably many different colored graphs (trees) that can be obtained as G_i^* for some $G^* \in \mathcal{G}_k^*$ (because G_i^* is either finite or it is a tree whose every vertex has degree ω, and those points whose color does not appear anywhere else in G_i^* can be situated in this tree in countably many different ways);

(c) given K and $\overline{L}$, there are only countably many different ways that a colored tree G_i^* can be connected to these sets (because of Lemma 3 (iv)).

Hence, $c(\mathcal{G}_k) \leq \omega$.

Next we show that $c(\mathcal{G}_k) \geq \omega$. Let K_4 denote the complete graph on four vertices, and let G_0 be the graph obtained from the union of $k-1$ vertex-disjoint copies of K_4 by adding a vertex connected to every other point. Let $\mathcal{G}_0$ be the family of all *subdivisions* of G_0, i.e., the set of all graphs arising from G_0 by replacing its edges with independent paths. Clearly, $\mathcal{G}_0$ is a countable subfamily of $\mathcal{G}_k$. On the other hand, it is easy to check that, if G is a graph containing two subgraphs isomorphic to distinct elements of $\mathcal{G}_0$, then $G \notin \mathcal{G}_k$. Thus, $c(\mathcal{G}_k) \geq |\mathcal{G}_0| = \omega$, completing the proof.

The analogous result for countable graphs containing no k edge-disjoint cycles can be established by a similar argument.

Theorem. *Let* $1 < k < \omega$, *and let* $\mathcal{G}'_k$ *be the class of all countable graphs containing no* k *edge-disjoint cycles. Then* $c(\mathcal{G}'_k) = \omega$.

To see that $c(\mathcal{G}_k') \geq \omega$, we can repeat the above argument with the only difference that now G_0 has to be defined as the graph obtained from the union of $k-1$ vertex-disjoint *triangles* by adding a vertex connected to every other point. The minor modifications in the other part of the proof are left to the reader.

References

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