Easy bounds for $n!$

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Theorem 1.1

$$n^n \leq e^n n! \leq (n+1)^{n+1}$$

We can derive this inequality from the simpler

$$x \leq \lceil x \rceil \leq x + 1 \tag{$*$}$$

using the idea of a product integral.

Define ${}_\pi\!\int f$ to be $\exp \int \log f$ for any positive function $f : \mathbb{R} \to \mathbb{R}$. We can think of ${}_\pi\!\int$ as a way of *multiplying* the values of $f(x)$ over a range in contrast to the standard $\int$, which *adds* these values.

For any positive sequence $(a_k)_1^n$, we can see that

$${}_\pi\!\int_0^n a_{\lceil x \rceil} dx = {}_\pi\!\int_0^1 a_1 \cdot {}_\pi\!\int_1^2 a_2 \cdots {}_\pi\!\int_{n-1}^n a_n = \prod_1^n a_k,$$

so that ${}_\pi\!\int_0^n \lceil x \rceil dx = n!$ It is clear from the definition of ${}_\pi\!\int$ that

$${}_\pi\!\int_0^y x \; dx = \left(\frac{y}{e}\right)^y.$$

We may extend one version of the fundamental theorem of calculus to its product analog: if ${}_\pi\partial F = f$, then

$${}_\pi\!\int_a^b f = F(b) \div F(a),$$

where we define the product derivative ${}_\pi\partial F = \exp \partial \log F$. If $a = g(c)$ and $b = g(d)$, then

$${}_\pi\!\int_a^b f(x) dx = {}_\pi\!\int_c^d [f \circ g(y)]^{g'(y)} \, dy$$

represents the substitution $x = g(y)$. Now we can see that

$${}_\pi\!\int_0^y (x+1) \; dx = {}_\pi\!\int_1^{y+1} x \; dx = \left(\frac{y+1}{e}\right)^{y+1} \div \left(\frac{1}{e}\right)^1 = \frac{(y+1)^{y+1}}{e^y}$$

To prove the theorem, just take ${}_\pi\!\int_0^n \cdot \, dx$ of $(*)$ and multiply through by e^n. $\square$