THE GÖDEL SOLUTION TO THE EINSTEIN FIELD EQUATIONS

AL MOMIN

1. Introduction

In 1949, Kurt Gödel introduced a new solution to the Einstein equations with two significant properties that no prior solution possessed. First, it modelled a rotating universe (in a sense we will make precise below). This demonstrated for the first time that "Mach's principle" is not completely incorporated in general relativity. It is a little surprising because Mach's principle was one of the motivating factors that led Einstein to general relativity[1]. Second, it permitted the existence of closed, time-like curves. In fact, such a curve passes through every point in this space-time. Thus, it is possible for an observer to travel into the past. We shall see many reasons to believe that the Gödel solution does not describes our universe. However, it demonstrates the kind of phenomena that we cannot easily dismiss in general relativity.

2. Preliminaries: Rotation and Time

Before presenting the Gödel solution and proving properties about it, we should first clarify some concepts that may be unfamiliar in curved space-time. For example, the condition in Newtonian Mechanics for a vector field to be irrotational is that the curl of the field is zero. In 4-dimensional curved space-time, this requires a generalization. As for time, we must generalize our notions of time in Minkowskii space-time to curved space-times. Our claim that closed time-like curves exist indicates that this will be a little more subtle than in Minkowskii space-time. We will see how to describe locally what it means to be moving forward in time, as well as what we mean by an absolute time coordinate for a space-time.
First, we discuss the rotation of a vector field in space-time. As mentioned above, in Newtonian mechanics we have a well defined notion of the rotation of a field given by its curl. We would like to have such a notion in our curved space-time as well. The corresponding rotation of the vector field v^μ in space-time is the following vector field ω,

$$\begin{aligned}
\omega^\mu &:= \frac{1}{12\sqrt{-g}}\epsilon^{\mu ijk}a_{ijk} \\
a_{ijk} &:= (v_i, v_j, v_k)\cdot((\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}))\times(v_i, v_j, v_k)
\end{aligned}$$

[1]e.g. see Wald, pg. 9, pg. 71

To see why we should think of this as the rotation of the vector field v, let us suppose we are in a geodesic normal coordinate system (i.e. $\frac{\partial}{\partial x^\sigma} g_{\mu\nu} = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$ = Lorentz metric). In these coordinates, geodesics correspond to straight lines through the origin, so in this sense this is the coordinate system that most closely maps the manifold to a flat space-time. Suppose further that matter is at rest at the origin in this frame (i.e. the velocity vector for matter is (1,0,0,0) in these coordinates). Then if we calculate ω, we obtain (the other components are given by similar expressions)

$$\omega^1 = \frac{1}{2}(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}) = \frac{1}{2}(\frac{\partial}{\partial x^2}\frac{v^3}{v^4} - \frac{\partial}{\partial x^3}\frac{v^2}{v^4})$$
$$\omega^0 = 0$$

In the first three components, this is the angular velocity in Newtonian physics, and the fourth component is zero. Apparently, ω is the only vector derived from v such that that coincides with the classical notion in these coordinates, so it is natural to define this as rotation in space-time.

We claimed earlier that rotation has relevance to Mach's Principle. While there may not be a precise statement of Mach's principle[2], it seems to imply that motion, such as what it means to be inertial or non-rotating, is not determined not by space but by the objects that occupy it. For example, if we accept the principle, then we take it that matter in space determines the curvature of space-time[3], which led Einstein to the field equations for gravity. The Gödel universe is homogeneous with constant matter distribution. So roughly speaking, no point of space-time is any different from any other. If we accept Mach's principle, it would seem that there could not be any preferred motion of any kind, since there is nothing special in the matter distribution to distinguish any preferred directions. However, the Gödel universe does exhibit a preferred motion in its rotation. Therefore, Mach's principle is not completely incorporated in the theory of general relativity.

The other aspect that will attract our attention is time. We shall see that the Gödel solution is time orientable, but pathological in the sense that there is no global notion of time, and that in fact there exist time-like closed loops, i.e. possible paths which are always travelling "forward" in time but self intersect.

In Minkowskii space-time, we have a well defined notion of a light cone. Furthermore, we can separate the light cone into two halves, one of which we call the future and the other the past. With this structure we can relate two time-like related points A and B by saying A is to the past of B if B is in the future light cone of A. This defines the past and future relation between time-like related points in this space-time, with no danger of ambiguity.

In general curved space-time (i.e. a 4-manifold with Lorentzian metric), the notion of a light cone does not carry over in the same global sense. Instead, we associate to each point on the manifold the light cone in the tangent space , which is isomorphic to Minkowskii space-time as a pseudo-inner product space[4]. We can assign half of the light cone to be future and the other half to be past. If a "continuous" choice

[2]see Wald, pg. 9

[3]see Wald again, p. 71

[4]Thus, the light cone in the tangent space is the set of vectors with non-negative norm

of future and past can be made, then we will we have a definite notion of future and past, at least for time-like vectors. We specify what we mean by continuous by saying that a space-time is **time orientable** if we can classify all time-like and null vectors in the space-time into classes + and - in such a way that: (a) if ζ is a + vector (resp. - vector) then -ζ is a - vector (resp. + vector), (b) if a sequence of vectors of the same sign converges to a non-zero vector, then that vector also has the same sign. Given a curve γ, we will say that it is **time-like** if its tangent vector is everywhere a + vector. This structure gives a curved space-time the time direction we seek, so that at least we can say what it means for a particle to be moving "forward in time".
However, in the Minkowskii space-time it is possible to take any two time-like related points and label one as past and the other as future. In general space-time, this may not be possible, even if it is time orientable. The natural definition to try would be to say that an event A is to the past of event B if B can be reached from A by a time-like curve. The problem with this is that it may also be possible to reach A from B via a time-like curve, in which case we would also have B to the past of A. Of course, the relation we seek should be irreflexive, so this is unacceptable if this is possible.
As for a global time coordinate, what we mean is a mapping from the space-time into the real numbers that increases along every time-like curve. This is the most natural notion of time that we have. For example, in Minkowskii space time the projection onto the x_0 axis defines a global time coordinate. It is clear that the existence of closed time-like curves precludes the existence of such a global time in the Gödel universe, for if $\gamma(0) = \gamma(1)$ is such a curve, then the time at $\gamma(1)$ is greater than at $\gamma(0)$, which is impossible since they are the same point in the manifold.

3. The Gödel solution

Gödel's solution to the Einstein equations is the manifold (M,$g_a b$), where $M = \mathbb{R}^4$, and the metric g is given by (in the canonical coordinates for $\mathbb{R}^4$, which we will call rectangular coordinates):

$$ds^2 = a^2(dx_0^2 - dx_1^2 + \frac{e^{2x_1}}{2}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0dx_2)$$

where a > 0 is a constant. The matrix and inverse for $g_{\mu\nu}$ is

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 & e^{x_1} & 0 \\ 0 & -1 & 0 & 0 \\ e^{x_1} & 0 & \frac{1}{2}e^{2x_1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} -1 & 0 & 2e^{-x_1} & 0 \\ 0 & -1 & 0 & 0 \\ 2e^{-x_1} & 0 & -2e^{-2x_1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The following geometrically inspired change of coordinates [5] will prove useful later

[5]This involves a representation of this manifold in the hyperbolic quaternions. The resulting calculation of the metric in these coordinates is quite complicated and therefore we omit it. Apparently it is easier to derive it directly from its geometrical interpretation. See Gödel (1949) for a brief explanation.

on:

$$e^{x_1} = cosh(2r) + cos(\phi)sinh(2r)$$
$$x_2 e^{x_1} = \sqrt{2}sin(\phi)sinh(2r)$$
$$tan(\frac{\phi}{2} + \frac{x_0 - 2t}{2\sqrt{2}}) = tan(\frac{\phi}{2})e^{-2r}$$
$$y = 2x_3$$

for $r \geq 0$, $0 \leq \phi \leq 2\pi$. The 2π periodicity of ϕ and association of r with distance suggests that we call these cylindrical coordinates for M. The metric in this coordinate system becomes[6]

$$ds^2 = 4a^2(dt^2 - dr^2 - dy^2 + (sinh^4 r - sinh^2 r)d\phi^2 + 2\sqrt{2}sinh^2 r d\phi dt)$$

The matter in the Gödel solution is dust with constant density and 4-velocity u^μ, so its stress-energy-momentum tensor is $T_{\mu\nu} = \rho u^\mu u^\nu$. Finally, the solution given has negative cosmological constant, which corresponds to a positive pressure.
We will now see that this does in fact satisfy the field equations.

Theorem 3.1. *The manifold (M,g) solves the field equations for dust with constant density* $\rho = \frac{1}{8\pi G a^2}$*, (where G is Newton's gravitational constant) and cosmological constant* $\Lambda = \frac{-1}{2a^2}$*.*

Proof: First note that the metric given is in fact Lorentzian, which we see by completing the square:

$$ds^2 = a^2((dx_0 + \frac{1}{2}e^{x_1}dx_2)^2 - dx_1^2 - \frac{1}{2}e^{2x_1}dx_2^2 - dx_3^2)$$

The Ricci curvature tensor in a coordinate chart is given by the formula[7]:

$$R_{\mu\nu} = \frac{\partial}{\partial x^\sigma}\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\frac{\partial^2 log g}{\partial x^\mu \partial x^\nu} + \frac{1}{2}\Gamma^\sigma_{\mu\nu}\frac{\partial log g}{\partial x^\sigma} - \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu}$$

The Christoffel symbols are obtained by:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma})$$

This calculation is easy because many of the Christoffel symbols are zero. The 9 non-zero Christoffel symbols are (remember they are symmetric in the lower indices)

$$\Gamma^0_{01} = 1$$
$$\Gamma^0_{12} = \Gamma^1_{02} = \frac{1}{2}e^{x_1}$$
$$\Gamma^1_{22} = \frac{1}{2}e^{2x_1}$$
$$\Gamma^2_{01} = -e^{-x_1}$$

[6]The calculation of the metric from the change of variables formula is difficult. Apparently, it is easier to compute directly from the geometric interpretation mentioned in the prior footnote.

[7]cf. Wald pg. 48

For all Christoffel symbols, $\frac{\partial}{\partial x^\mu} = 0$ unless $\mu = 1$. Furthermore, $g = detg_{\mu\nu} = \frac{a^8}{2} e^{2x_1}$, so $log(g)$ is linear in x^1. This allows us to simplify the expression for the Ricci tensor to

$$R_{\mu\nu} = \frac{\partial}{\partial x^1}\Gamma^1_{\mu\nu} + \Gamma^1_{\mu\nu} - \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu}$$

The non-zero coefficients are

$$R_{00} = 1$$
$$R_{02} = R_{20} = e^{x_1}$$
$$R_{22} = e^{2x_1}$$

Let the vector u be $u^\mu = (1/a, 0, 0, 0)$, so that $u_\mu = (a, 0, ae^{x_1}, 0)$, and therefore we have $R_{\mu\nu} = \frac{1}{a^2} u_\mu u_\nu$. For the Ricci scalar, $R = R^\mu{}_\mu = \frac{1}{a^2} u^\mu u_\nu = \frac{1}{a^2}$. So the Einstein tensor is

$$\begin{aligned} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{1}{a^2} u_\mu u_\nu - \frac{1}{2}\frac{1}{a^2} g_{\mu\nu} \\ &= 8\pi G \rho u_\mu u_\nu + \Lambda g_{\mu\nu} \end{aligned}$$

This is Einstein's equation for dust with density ρ and cosmological constant Λ.
□

The particles in this universe have velocity vector $u^\nu = (\frac{1}{a}, 0, 0, 0)$, so it is clear that they travel along the x_0-lines in our rectangular coordinate system with constant speed. We will call these lines the **world lines of matter**.

4. Main Properties of the Solution and Proofs

We are now ready to prove the main geometrical properties of the Gödel solution.

Theorem 4.1. *The Gödel solution has the following properties.*

1. *M is homogeneous. That is, for any two events A and B in M, there is a transformation of M carrying A into B. Furthermore, M has rotational symmetry, meaning for every point A of M, there exists a one parameter group of transformations of M carrying A into itself.*
2. *M is rotating everywhere with velocity $2\sqrt{(\pi G\rho)}$, where ρ is the matter density, and G is Newton's Gravitational constant.*
3. *M is time-orientable. However, M has closed time-like curves, and furthermore, any two points of this space-time can be connected by a time-like loop. Therefore, no global time coordinate exists for M.*

Proof:

(1) Consider the following transformations of M:

$I)$	$II)$	$III)$	$IV)$
$x_0 = x_0 + a$	$x_0 = x_0$	$x_0 = x_0$	$x_0 = x_0$
$x_1 = x_1$	$x_1 = x_1 + b$	$x_1 = x_1$	$x_1 = x_1$
$x_2 = x_2$	$x_2 = e^{-b} x_2$	$x_2 = x_2 + c$	$x_2 = x_2$
$x_3 = x_3$	$x_3 = x_3$	$x_3 = x_3$	$x_3 = x_3 + d$

One can directly verify that each of these is an isometry of M. These transformations allow one to map any point of M into any other without changing metric properties of M. This shows that M is homogeneous. This

is an important property, one reason being that in any proof we may assume at the outset that any given point lies at the origin (or any other given point).
To exhibit rotational symmetry, consider M in the (r,ϕ,y,t) coordinates defined in section 3. By homogeneity we may assume that a given point P lies at the origin, so $r = 0$. Since the metric is independent of ϕ, the group of transformations for real parameter c sending (r,ϕ,y,t) $\mapsto$ (r,ϕ + c, y, t) is a one-parameter group of isometries of M fixing P. It is natural to call these transformations rotations because of the 2π periodicity.

(2) The velocity vector of matter in this space-time is the constant unit vector $u^\mu = (1/a,0,0,0)$. The rotation vector ω (defined in section 1) for matter is $\omega^\mu = (0,0,0,\sqrt{2}/a^2)$. Therefore, matter rotates everywhere at an angular velocity $\sqrt{g_{\mu\nu}\omega^\mu\omega^\nu} = a/\sqrt{2} = 2\sqrt{\pi G\rho}$.

(3) The vector field u^ν defines a time orientation for M. Specifically, if ξ is a time-like vector, we define it to be a + vector if $g_{\mu\nu}u^\mu\xi^\nu > 0$, and a - vector if $g_{\mu\nu}u^\mu\xi^\nu < 0$. This mapping is linear, so it satisfies the two properties for time-orientability in section 2. Therefore M is time orientable.
To produce a closed time-like curve, consider the closed curve given in cylindrical coordinates by $\gamma(s)$ = (R,sα, 0 , 0). This curve has velocity vector in these coordinates v^μ = (0, α, 0, 0), which has norm $g_{\mu\nu}v^\mu v^\nu = (sin^4(R) - sin^2(R))\alpha^2$. If we take $R > \log(1 + \sqrt{2})$, this number is positive and therefore γ is a closed time-like curve. This immediately implies that no global time coordinate exists.
With this curve, we are able to construct more closed time-like curves. In fact, we will be able to put a closed time-like curve through any two points in M. Working in the cylindrical coordinates, we note that the world lines of matter correspond to the t lines[8]. If we perturb the curve γ slightly, it will remain a time-like curve for our solution. Suppose we are given two points P and Q on a world line of matter, with t coordinates $t_1 < t_2$ respectively. Thus Q is "to the future" of P. We may assume without loss of generality that P lies at the point (R,0,0,t_1), by homogeneity. Take the curve $\gamma(s) = (R, \alpha s, 0, t_2 + \frac{\alpha 2\pi(t_1-t_2)(s}{n})$, where n is a sufficiently large integer so that this curve remains time-like. This $\gamma(s)$ is a time-like curve from Q (at $s = 0$) to P (at $s = 2\pi n$). Therefore there exists a closed tim-like loop through P and Q.
For the more general case, let P and Q be any two points in our space-time. There exists a time-like curve connecting a point on the world matter of P to a point on the world matter of Q[9]. So suppose we have time-like curves from P to R and from S to P, where R and S lie on the world matter of Q. By the previous construction, we have a time-like curve from R to Q and from Q to S. Therefore, we can make a time-like curve from P to R to Q to

[8] We can see this as the coordinates (x_1,x_2,x_3) determine the coordinates (r,ϕ,y) completely in the change of variable expression, so the t-lines must correspond to the x_0-lines

[9] For example, if we take P to lie at the origin, and Q at the point (x_0, x_1, x_2, x_3), then we can follow the curves [0,1] $\mapsto M$, s $\mapsto (sT, sx_1, 0, 0)$; s $\mapsto (T + sT', x_1, sx_2, 0)$; s $\mapsto (T + T' + sT'', x_1, x_2, sx_3)$, where T, T' and T" are taken large enough so that these curves are time-like

S and back to P by concatenation. It follows that there exists a time-like loop connecting any two events.

□

5. Further Properties of the Solution

We have seen that M has a number of interesting properties. The existence of time-like loops connecting every point of space-time means that an observer with a powerful enough rocket would be able to travel to any event in the space-time. This has obvious philosophical implications. We have already discussed how rotation relates to Mach's principle.

Another property of the solution is that there is no red-shift observed for distant objects. Suppose that a particle travelling along a world line of matter emits two light signals at times t and $t+\Delta t$ (i.e. the x_0 component in the rectangular coordinate system), and that the first signal arrives at a distant particle travelling along a world line of matter at time s. By the transformation I), we can send $t+\Delta t$ to t. The light signal will then follow the same path as the first light signal and arrive at time s. Then if we invert the transformation, we should get the original path of the light signal. This means that the second light signal arrives at time $s+\Delta t$. Therefore, no red shift is observed, so the universe is neither expanding nor contracting.

M has the further property that there does not exist everywhere space-like surface[10] intersecting every world line in one point. Suppose we had such a surface Σ. Then a closed time-like curve can only pass through Σ once, since Σ divides M in half and any time a time-like curve passes through Σ it passes from one half into the other (in the future direction, corresponding to increasing x_0). Since continuous deformations change the number of crossings through Σ by an even number, we discover that this curve cannot be contracted to a point. But M is homeomorphic to $\mathbb{R}^4$ and therefore it is simply connected, a contradiction. Another proof of this fact is possible by constructing a global time coordinate from Σ and the x_0 coordinate using transformation I).

Imagine trying to solve an initial value P.D.E. problem in the whole space-time. Since the space is time-orientable, we might expect to specify the initial conditions of some problem in space and solve in time, at least for small times. However, this property shows that we cannot expect this. Any surface intersecting every world line of matter would contain a portion of some time-like curve, hence the initial data would have to include information about the evolution of the solution. This is a highly un-physical property.

Another interesting fact about this space-time discovered by Gödel is that even if we weaken our idea of a global time coordinate on a space-time, M still does not admit such a coordinate. By this, we mean that if τ is a partition of M into 3-spaces that intersect each world line of matter, then there is an isometry of M sending the positive direction of time into itself but not τ into itself. We think of these cross-sections of our space-time as our coordinate for time. This is more general because we drop the condition that the direction of time must agree for all

[10] This means that each vector in the tangent space of the surface is space-like

observers on the same slice of time, since we do not assign a real value to each slice in τ and therefore do not distinguish future and past, but only present. To prove this, take any element U in τ. Let M_b be the subset of M for which $x_3 = b$, for each $b \epsilon \mathbb{R}$. Let V be the intersection of M_0 and U, which is a two-dimensional surface because U intersects each x_0 line in one point, so we can express each point in V uniquely as a function of x_1 and x_2. V is not orthogonal to each x_0 line, else the three surface formed by V and the x_3 lines would be an everywhere space-like surface intersecting each world-line of matter, which we have proved impossible. Take a line on which it is not orthogonal and rotate the entire space around it (recall that space-time is rotationally symmetric in the (r, ϕ) plane, which is the same as the (x_1, x_2) plane, so we can rotate). Then the point of intersection of V and the line is fixed, but since V is not orthogonal to this line, it is not sent into itself. This shows that U is not sent into itself (rotation does not affect the x_3 coordinate so it is sufficient to show V is not taken into itself), but the positive direction of time is taken into itself (since rotation preserves the direction of the world lines of matter). Finally, it is insightful to look at the geodesics of the solution. The description of the geodesics of M was carried out by Chandrasekhar and Wright in 1961[11]. First note that the manifold (M, g) can be written as the direct sum of the manifolds $M_1 = \mathbb{R}^3$ with metric $ds_1^2 = a^2(dx_0^2 - dx_1^2 + \frac{e^{2x_1}}{2} dx_2^2 + 2e^{x_1} dx_0 dx_2)$, and $M_2 = \mathbb{R}$ with metric $ds_2^2 = a^2 dx_3^3$. It is enough to discuss the geometry on M_1 alone, since the M_2 component of M is flat. We can therefore visualize the Gödel universe in three space by suppressing the x_3 or y variable. The following figure from Hawking and Ellis demonstrates the behavior of time-like geodesics emanating from a point p on the $r = 0$ axis.

[11]cf. Raychaudhuri, pg. 92

The time-like geodesics from p all initially diverge, but reach a maximum radius of divergence and then converge once again to a point p' also on the $r = 0$ axis. All of these curves lie within the circle of radius R = log(1 + $\sqrt{2}$). If we look at the light cone structure of M_1, we see that at r = 0 it points directly upwards. As r increases, the light cone tips over, until at r = R it is tangent to the plane $t =$ constant. For larger values of r, this plane intersects the interior of the light cone, which is how we constructed the closed time-like curves. This suggests (perhaps) the relationship between rotation and the peculiar time properties of the solution. Through all the peculiar properties we have seen, it is maybe a little comforting to hear (though we won't prove this) that no closed time-like geodesics occur in this solution.

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E-mail address: al.momin@utoronto.ca