Continuous Time Finance Notes, Spring 2004 – Section 5, *Feb. 25, 2004*
Notes by Robert V. Kohn, Courant Institute of Mathematical Sciences. For use in connection with the NYU course Continuous Time Finance.

This section discusses the Hull-White model. The continuous-time analysis is not much more difficult than Vasicek – everything is still quite explicit. The basic paper is Rev. Fin. Stud. 3, no. 4 (1990) 573-592, downloadable via JSTOR; my treatment is much simpler though because I keep the parameters a and σ constant rather than letting them (as well as θ) vary with time.

The real importance of Hull-White is that while it's rich enough to match any forward curve, it's also simple enough to be approximated by a (recombining, trinomial) tree. This topic is covered very clearly in Sections 23.11-23.12 of Hull (5th edition), so I won't cover it separately in these notes.

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The Hull-White model. Also sometimes known as "extended Vasicek," this model assumes that the risk-neutral process for the short rate has the form

$$dr = (\theta(t) - ar)\,dt + \sigma\,dw \tag{1}$$

where a and σ are constant but θ is a function of t. (Actually the 1990 paper by Hull and White also considers taking $a = a(t)$ and $\sigma = \sigma(t)$.) We'll show that

(a) for a given choice of $\theta(t)$, the situation is a lot like Vasicek;

(b) there is a unique choice of θ that matches the term structure observed in the marketplace at $t = 0$.

Solving for $r(t)$. The calculation is entirely parallel to Vasicek: we have

$$d(e^{at}r) = e^{at}\,dr + ae^{at}r\,dt = \theta(t)e^{at}\,dt + e^{at}\sigma\,dw,$$

so

$$e^{at}r(t) = r(0) + \int_0^t \theta(s)e^{as}\,ds + \sigma\int_0^t e^{as}dw(s).$$

which simplifies to

$$r(t) = r(0)e^{-at} + \int_0^t \theta(s)e^{-a(t-s)}\,ds + \sigma\int_0^t e^{-a(t-s)}\,dw(s). \tag{2}$$

That calculation could have started at any time; thus

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(\tau)e^{-a(t-\tau)}\,ds + \sigma\int_s^t e^{-a(t-\tau)}\,dw(\tau).$$

Notice that $r(t)$ is still Gaussian.

Solving for $P(t,T)$. We use the same PDE method that worked for Vasicek. We know that $P(t,T) = V(t, r(t))$ where V solves the PDE

$$V_t + (\theta(t) - ar)V_r + \tfrac{1}{2}\sigma^2 V_{rr} - rV = 0$$

with final-time condition $V(T,r) = 1$ for all r at $t = T$. Let's look for a solution of the form

$$V = A(t,T)e^{-B(t,T)r(t)}. \tag{3}$$

To satisfy the PDE, A and B should satisfy

$$A_t - \theta(t)AB + \tfrac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

with final-time conditions

$$A(T,T) = 1 \quad \text{and} \quad B(T,T) = 0.$$

The equation for B doesn't involve θ, so the solution is the same as for Vasicek:

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{4}$$

The equation for A is different only in the fact that θ is no longer constant; not surprisingly, the θ-dependent part of the solution formula requires doing an integration:

$$A(t,T) = \exp\left[-\int_t^T \theta(s)B(s,T)\,ds - \frac{\sigma^2}{2a^2}(B(t,T) - T + t) - \frac{\sigma^2}{4a}B(t,T)^2\right]. \tag{5}$$

Determining θ from the term structure at time 0. Our goal is to demonstrate that following relation between the infinitesimal forward rate and the function $\theta(t)$:

$$\theta(t) = \frac{\partial f}{\partial T}(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{6}$$

When we get to HJM we'll find a simple proof of this relation. But we can also prove it now, by using the explicit representation of $P(t,T)$ given by (3)-(5). Recall that $f(t,T) = -\partial \log P(t,T)/\partial T$. We have

$$-\log P(0,T) = \int_0^T \theta(s)B(s,T)\,ds + \frac{\sigma^2}{2a^2}(B(0,T) - T) + \frac{\sigma^2}{4a}B(0,T)^2 + B(0,T)r_0.$$

Differentiating and using that $B(T,T) = 0$ and $\partial_T B - 1 = -aB$, we get

$$f(0,T) = \int_0^T \theta(s)\partial_T B(s,T)\,ds - \frac{\sigma^2}{2a}B(0,T) + \frac{\sigma^2}{2a}B(0,T)\partial_T B(0,T) + \partial_T B(0,T)r_0.$$

Differentiating again, we get

$$\begin{aligned}\partial_T f(0,T) \quad &= \quad \theta(T) + \int_0^T \theta(s)\partial_{TT}B(s,T)\,ds - \frac{\sigma^2}{2a}\partial_T B(0,T) \\ &\qquad + \frac{\sigma^2}{2a}[(\partial_T B(0,T))^2 + B(0,T)\partial_{TT}B(0,T)] + \partial_{TT}B(0,T)r_0.\end{aligned}$$

Combining these equations, and using the fact that $a\partial_T B + \partial_{TT} B = 0$, we get

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(aB + \partial_T B) + \frac{\sigma^2}{2a}[aB\partial_T B + (\partial_T B)^2 + B\partial_{TT} B].$$

Substituting the formula for B and simplifying, we finally get

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(1 - e^{-2aT}),$$

which is equivalent to (6).

A convenient representation. Looking at (6), it seems at first that we must use the *differentiated* term structure $\partial_T f(0,T)$ to calibrate the model. That would be unfortunate, because differentiation amplifies the effect of observation-error. Actually, we can make do with f alone. Indeed, let's look for a representation of the form

$$r(t) = \alpha(t) + x(t) \tag{7}$$

where $\alpha(t)$ is deterministic and $x(t)$ solves

$$dx = -ax\, dt + \sigma\, dw \quad \text{with } x(0) = 0.$$

A brief calculation reveals that if

$$\alpha' + a\alpha = \theta \quad \text{and } \alpha(0) = r_0$$

then $\alpha(t)+x(t)$ solves the SDE for $r(t)$, and has the right initial condition, so (by uniqueness) it equals $r(t)$. The ODE for α is easy to solve: we have $(e^{at}\alpha)' = e^{at}\theta$, so

$$\alpha(t) = r_0 e^{-at} + \int_0^t e^{-a(t-s)}\theta(s)\, ds.$$

Substituting (6) on the right, we get

$$\alpha(t) = r_0 e^{-at} + \int_0^t \partial_s[e^{-a(t-s)} f(0,s)] + \frac{\sigma^2}{2a} e^{-a(t-s)}(1 - e^{-2as})\, ds.$$

This simplifies to

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

Thus the decomposition (7) expresses r as the sum of two terms: a deterministic $\alpha(t)$ reflecting the term structure at time 0, and a random process $x(t)$ that's entirely independent of market data.

Validity of Black's formula. The situation is exactly the same as for Vasicek. The SDE for the interest rate under the forward-risk-neutral measure is

$$dr = [\theta(t) - ar - \sigma^2 B(t,T)]\, dt + \sigma\, d\overline{w}$$

where $d\overline{w}$ is a Brownian motion under this measure. This is simply a version of Hull-White with a different choice of θ. So bond prices are lognormal and Black's formula is valid.

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The trinomial tree version of Hull-White. This topic is discussed quite clearly in Hull's book *Options, Futures, and other Derivatives* (5th edition), sections 23.11 and 23.12. Please read it there.