Continuous Time Finance Final Exam

Spring 2004 – Professor Kohn

May 5, 2004

This was administered as a closed-book exam, however students were permitted to bring two sheets of notes (both sides, any font). Problems 1-10 require relatively short answers and are worth 10 points each. Problems 11 and 12 require longer answers and are worth 25 points each. Thus the total possible score is 150.

PART I: SHORT ANSWER QUESTIONS (10 points each).

(1) Is the following statement true or false? *"If the interest rate r is constant then the market price of risk must also be constant."*

(2) Suppose S is the price of a risky tradeable. Can S^2 also be the price of a tradeable?

(3) In the Cox-Ingersoll-Ross short-rate model, the SDE for the short rate under the risk-neutral probability is $dr = (\theta - ar)\,dt + \sigma\sqrt{r}dw$, where σ and a are constant and $\theta(t)$ is a deterministic function of time. The prices of zero-coupon bonds under this model satisfy $P(t,T) = V(r(t),t)$ where $V(x,t)$ solves the PDE

$$V_t + \_\_\_\_V_x + \_\_\_\_V_{xx} + \_\_\_\_V = 0, \quad \text{with} \quad V(x,t) = \_\_ \text{ at } t = \_\_.$$

Fill in the blanks.

(4) Consider options on an underlying whose actual volatility is a function of time alone, i.e. whose subjective process is

$$dS = \mu(S,t)S\,dt + \sigma(t)S\,dw$$

where $\sigma(t)$ is a deterministic function of time. Assume the interest rate r is constant. What is the Black-Scholes implied volatility $\sigma_I(K,T)$?

(5) Consider the one-factor HJM model under which infinitesimal forward rates $f(t,T)$ solve

$$d_t f(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dw$$

under the risk-neutral measure. Recall that the SDE satisfied by $P(t,T)$ is then $d_t P(t,T) = rP(t,T)\,dt + \Sigma(t,T)P(t,T)\,dw$ where $\Sigma(t,T) = -\int_t^T \sigma(t,u)\,du$. Fill in the blank: if

$$d\overline{w} = dw + \_\_\_\_\,dt$$

then $\overline{w}$ is a Brownian motion under the forward measure associated with maturity T (i.e. the measure with respect to which the price of any tradeable divided by $P(t,T)$ is a martingale).

(6) Consider a one-factor HJM model as in Problem 5. What is the SDE for the associated short rate? Explain the statement: "HJM models usually lead to non-Markovian short rates."

(7) In discussing Libor Market Model we considered a discrete set of maturity dates $T_1, T_2, \ldots$. Therefore it was natural to consider as numeraire the "rolling-forward CD", i.e. an account which is reinvested at each maturity date T_k, at the term rate for investing from T_k to T_{k+1}. What is the relation between the volatility of the rolling-forward CD and the volatilities of zero-coupon bonds?

(8) Let's consider constructing a trinomial tree for the process $dx = -ax^3\,dt + \sigma\,dw$, with time increment Δt and spatial increment Δx. Suppose that from the node at time $t_j = j\Delta t$ and spatial value $x_k = k\Delta x$ the process goes up to $x_{k+1} = (k+1)\Delta x$ with probability p_u, goes down to $x_{k-1} = (k-1)\Delta x$ with probability p_d, and stays the same (i.e. remains at $x_k = k\Delta x$) with probability p_m. What linear system should be solved to find p_u, p_m, and p_d? (You need not actually solve it. Do not assume any special relation between Δx and Δt.) Do you think it will be necessary to truncate this tree?

(9) Consider the jump-diffusion process

$$dy = -ay\,dt + \sigma dw + J\,dN,$$

with $y(0) = 0$. Assume a and σ are constant; the occurrence of a jump is a Poisson event with rate λ, and the jumps are drawn independently from a specified distribution with mean m. What ODE should you solve to find the mean value $\overline{y} = E[y]$? (You need not actually solve it.)

(10) Consider a local vol model, in which the risk-neutral dynamics of the underlying is $dS = rSdt + \sigma(S,t)Sdw$ for some deterministic function $\sigma(S,t)$. Assume the interest rate r is constant, and the present time is $t = 0$. Let $P(K,T)$ be the value of a put option with strike K and maturity T, and let $\rho(\xi, T)$ be the probability (under the risk-neutral measure) that the underlying has value ξ at time T. Give a formula for P_{KK} in terms of ρ.

PART II: LONGER-ANSWER QUESTIONS (25 points each).

(11) This problem concerns the pricing and hedging of a quanto option. The underlying is a Euro tradeable, whose price (in Euros) satisfies

$$dS = \mu_S S\,dt + \sigma_S S\,dw_S$$

under the subjective measure. The exchange rate C in dollars per Euro satisfies

$$dC = \mu_C C\,dt + \sigma_C C\,dw_C.$$

The dollar risk-free rate is r and the Euro risk-free rate is u. Assume all parameters $(\mu_S, \mu_C, \sigma_S, \sigma_C, r, u)$ are constant, and $dw_S\,dw_C = \rho\,dt$. Notice that the natural dollar tradeables are the stock valued in dollars (value SC), the Euro money market account valued in dollars (value CD where $D = e^{ut}$), and the dollar money market account (value $B = e^{rt}$).

Let's price and hedge the option whose payoff at time T is $f(S_T)$ dollars.

(a) What is the SDE for S under the dollar investor's risk-neutral measure?

(b) What PDE should the dollar investor solve to price the option?

(c) How can a dollar investor replicate this option, by trading only in the stock (valued in dollars), the Euro money market account (valued in dollars), and the dollar money market account?

(12) This problem concerns the pricing and hedging of certain interest-based exchange options using the Hull-White model. So we assume short rate satisfies $dr = (\theta(t) - ar)\,dt + \sigma\,dw$ where a and σ are constant and θ is a deterministic function of t. Consider the option whose payoff is $[P(T, T_2) - KP(T, T_1)]_+$ at time $t = T$, where $T < T_1 < T_2$ and K is a fixed constant.

(a) Explain the sense in which this is an "exchange option."

(b) What is the SDE for $P(t, T_2)/P(t, T_1)$ under the forward measure associated with maturity T_1?

(c) Value the option using the result of (b) and Black's formula.