Atmospheric Dynamics

Lecture 10: Tropical Dynamics

1 Introduction

The equatorial region differs fundamentally from the mid-latitudes in that the Coriolis parameter vanishes on the equator. This implies that quasi-geostrophic theory is inappropriate and the geostrophic flow transitions to a direct pressure gradient flow on the equator. Additionally many phenomena in this region are able to be understood (to first order at least) by the linearized equations. Since the mean state is often not large our results from Lecture 6 are directly relevant. The vertical analysis performed there still holds however we need to reconsider the shallow water equation solutions.

2 The equatorial β-plane

In the tropics it is a reasonable approximation to set the Coriolis parameter $f = 2\Omega \sin\varphi$ to the approximate expression

$$f = \beta y$$

where $\beta = 2.3 \times 10^{-11} m^{-1} s^{-1}$ is exact at the equator and approximately constant for most of the tropics. y is the distance from the equator. The shallow water equations now read

$$\begin{aligned} U_t - \beta y V &= -h_x \\ V_t + \beta y U &= -h_y \\ h_t + c^2(U_x + V_y) &= 0 \end{aligned} \tag{1}$$

The usual vorticity equation can be obtained by subtracting the y derivative of the first equation from the x derivative of the second. When the third equation is used this becomes

$$\left(U_y - V_x + \frac{f}{c^2} h\right)_t - \beta V = 0$$

These four equations may be reduced to a single equation in V by applying $\frac{-f}{c^2}\partial_t$ to the first; $\frac{1}{c^2}\partial_{tt}$ to the second; $-\frac{1}{c^2}\partial_{yt}$ to the third and ∂_x to the fourth (!). This is

$$(V_{xx} + V_{yy})_t + \beta V_x - c^{-2} V_{ttt} - \frac{\beta^2 y^2}{c^2} V_t = 0 \tag{2}$$

This is a linear pde with coefficients only dependent on y due to the beta effect i.e. the variation of the Coriolis parameter with latitude. General solutions to these equations can be found as usual by separating variables (again) and can be written as sums of terms of the form

$$V = V(y)\exp(ikx - i\omega t) \tag{3}$$

where we need to derive a dispersion relation for k and ω. Substitution of (3) into (2) leads to the equation

$$V_{yy} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)V = 0 \tag{4}$$

Now the operator $-\partial_{yy} + \frac{\beta^2}{c^2}y^2$ is easily shown to be Hermitian and positive and we again have a Sturm Liouville problem this time on an infinite domain. The solutions to this eigenproblem are well known (those of you who have done quantum mechanics will recognize this equation as the Schroedinger wave equation for the harmonic oscillator which has been much studied). The eigenvalues are discrete and (obviously) bounded below and we have

$$\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = \frac{\beta}{c}(2n+1) \quad for \quad n = 0, 1, 2, 3...... \tag{5}$$

In addition there is a solution which has $V = 0$. Substitution of this in (1) gives the dispersion relation $\omega^2 = c^2k^2$ and the negative root here gives unbounded solutions for U and h as $y \to \infty$ so we are left with the additional relation

$$\omega = kc \tag{6}$$

The solutions to both dispersion relations are plotted in Figure 1.

The upper (i.e. high frequency) curves are gravity waves (qf Lecture 6); the curves on the lower left hand side are *Rossby waves* (qf Lecture 7) while the curve corresponding to (6) is for the so-called *Kelvin wave*[1]. We shall consider all these solutions in more detail below.

The eigensolutions of (4) (as well as the additional $V = 0$) are the well known *parabolic cylinder functions*

$$D_n(\xi) \equiv D_n(\sqrt{\beta/c}y) = \exp\left(-\xi^2/2\right) H_n(\xi) \tag{7}$$

where $H_n(\xi)$ are the *Hermite polynomials* for which we have the following

$$\begin{aligned} H_0(\xi) &= 1 \\ H_1(\xi) &= 2\xi \\ H_2(\xi) &= 4\xi^2 - 2 \\ \frac{\partial H_n}{\partial y} &= 2nH_{n-1} \\ \xi H_n &= nH_{n-1} + 0.5H_{n+1} \end{aligned} \tag{8}$$

[1]The other solution which has $n = 0$ is called a *mixed Rossby-gravity wave* since it resembles the former for $k \to -\infty$ and the latter for $k \to \infty$.

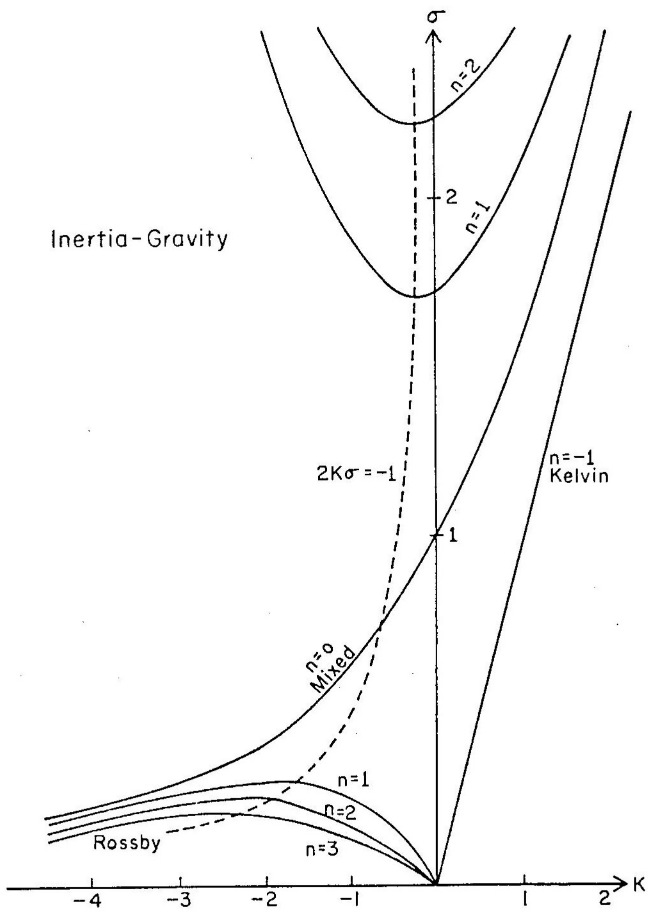

Figure 1: Equatorial beta plane dispersion relation curves. The unit of frequency is $\sqrt{\beta c}$ while that of the wavenumber k is $\sqrt{\beta/c}$.

The eigensolutions satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} D_n(\xi) D_m(\xi) d\xi = 2^n n! \pi^{1/2} \delta_{mn}$$

Note that we have introduced a natural scaling distance $a = \sqrt{c/\beta}$ which is known as the equatorial radius of deformation.

The first few parabolic cylinder functions are displayed in Figure 2.

Note that they are symmetric (about the equator) for n even and antisymmetric for n odd. Solutions for V are proportional to these functions while the solutions for U and h can be obtained by substituting for the V solution back in (1) and using (3), (8) and (7):

$$\begin{aligned} V &= C\exp\left(ikx - i\omega t\right) D_n(\xi) \\ U &= C\exp\left(ikx - i\omega t\right)\left[\frac{n^{1/2} D_{n-1}(\xi)}{\omega + ck} + \frac{(n+1)^{1/2} D_{n+1}(\xi)}{\omega - ck}\right] \\ h &= C\exp\left(ikx - i\omega t\right)\left[\frac{n^{1/2} D_{n-1}(\xi)}{\omega + ck} - \frac{(n+1)^{1/2} D_{n+1}(\xi)}{\omega - ck}\right] \end{aligned} \tag{9}$$

Notice that when U and h are symmetric then V is antisymmetric and conversely. The Kelvin wave solution has $V = 0$ and U and h proportional to $D_0(\xi)$ (see below for further detail). We now consider the solutions in detail.

3 Rossby waves

The low frequency solutions of the dispersion relation (5) are the Rossby waves which we met in Lecture 7 and depend for their existence on the fact that the Coriolis parameter varies with latitude (i.e. that $\beta \neq 0$). Their approximate dispersion relation can be obtained by neglecting the term involving ω^2 in (5) and solving for ω :

$$\omega = \frac{-\beta k}{k^2 + \frac{\beta(2n+1)}{c}} \tag{10}$$

Note the similarity to dispersion equation derived in Lecture 7 for the quasi-geostrophic equation (set $l^2 + \frac{f_0^2}{c^2} = \frac{\beta(2n+1)}{c}$). The limits of $|k|$ small (long waves) and $|k|$ large (short waves) are of significant interest physically. In the former case we can neglect k^2 from the denominator in (10) and obtain

$$\omega = \frac{-ck}{2n+1} \tag{11}$$

which is nondispersive and indicates that these waves propagate towards the west. Note that as n increases this speed slows. The first (or gravest) symmetric Rossby wave has $n = 1$ and the long wave horizontal structure is plotted in Figure 3. Note the cyclonic couplet structure. It travels at a speed which is one third of the shallow water velocity.

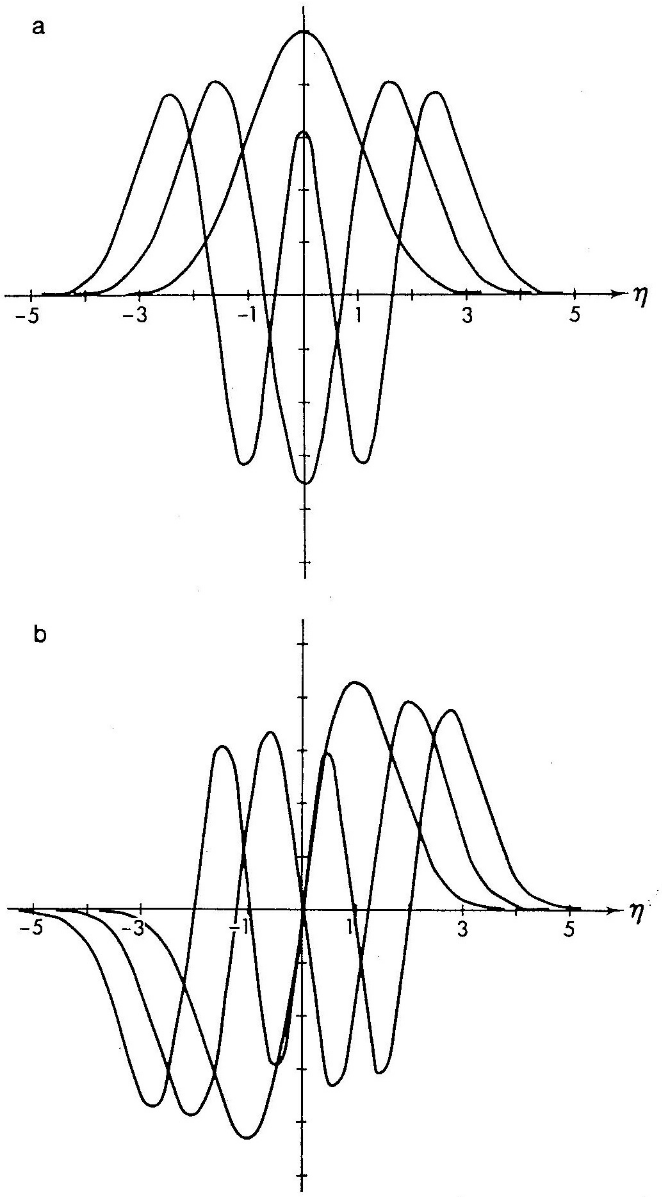

Figure 2: Parabolic cylinder functions. Top panel shows the symmetric functions and the bottom the anti-symmetric functions.

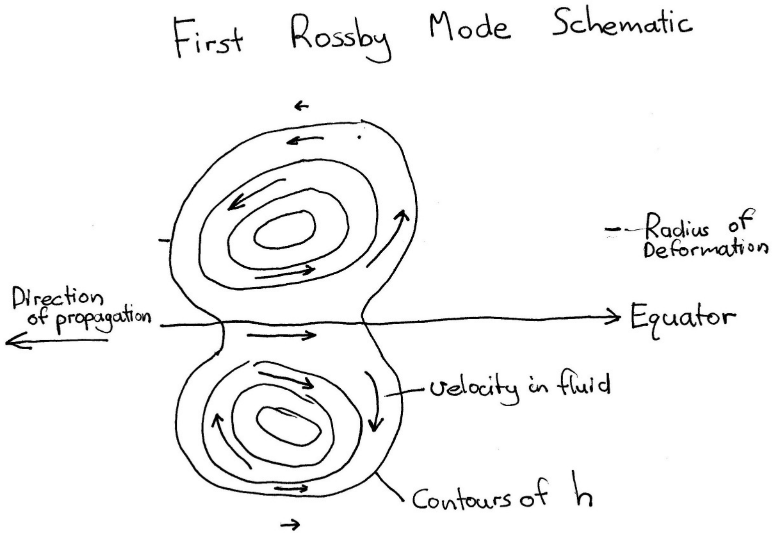

Figure 3: Gravest equatorial Rossby wave schematic.

Another very important property of Rossby waves is that they have a maximum frequency. This may be obtained by finding the turning point of ω in relation (10):

$$\omega_{\max}^2 = \frac{\beta c}{4(2n+1)}$$

Note how this value increases with shallow water speed and decreases as n increases i.e. as the waves have more off-equatorial projection.

4 Kelvin waves

These waves have no meridional velocity and satisfy the nondispersive relation

$$\omega = kc$$

which means they propagate towards the east with the shallow water speed. Their zonal velocity and h structure is given by simply substituting $V = 0$ into (1) and using (3):

$$\begin{aligned} U &= U_0 \exp\left(ikx - i\omega t\right) D_0(\xi) \\ h &= cU_0 \exp\left(ikx - i\omega t\right) D_0(\xi) \end{aligned}$$

Notice that these waves are trapped in the region of the equator (scaled by the deformation radius). This was the reason physically for their absence from the quasi-geostrophic equations.

5 Gravity waves

An approximate dispersion relation for these waves can be obtained by assuming ω large in the dispersion relation (5). If this is done then the third term on the left hand side may be neglected and so we obtain

$$\omega^2 = c^2 \left(k^2 + \frac{\beta}{c}(2n+1) \right) \tag{12}$$

This relation should be compared to that which was derived in Lecture 7. The comparison is formally the same as for Rossby waves.

6 Forced equations

Diabatic heating of the atmosphere can be taken into account in the linearized primitive equations by incorporating a forcing term into the temperature equation (see Lecture 5 equation 4). When the separation of variables occurs as in Lecture 7 then each vertical mode acquires a forcing term obtained by projecting the heating onto the vertical mode eigenfunctions for temperature (vertical velocity eigenfunction multiplied by $S(p)$). This projection coefficient then appears in the shallow water equations which describe the horizontal flow. Thus we have

$$\begin{aligned} U_t - \beta y V &= -h_x \\ V_t + \beta y U &= -h_y \\ h_t + c^2(U_x + V_y) &= -sQ \end{aligned} \tag{13}$$

where Q is the diabatic heating. The coefficient s is determined by the projection onto the vertical modes of the heating.

The forced shallow water equations can be solved now by projecting the forcing onto the parabolic cylinder functions. For example we can write

$$Q = \sum_{n=1}^{\infty} Q_n(x,t) D_n(y)$$

In order to simplify the presentation we now introduce non-dimensionalization of our equations. Here we assume that the natural velocity scale is c and the natural length scale is the equatorial radius of deformation $a = \sqrt{c/\beta}$. Scaling our variables by these we obtain the simplified equations

$$\begin{aligned} U_t - yV &= -h_x \\ V_t + yU &= -h_y \\ h_t + (U_x + V_y) &= -sQ \end{aligned} \tag{14}$$

A further change of variables is convenient. We introduce the ancillary variables

$$\begin{aligned} q &= h + U \\ r &= h - U \end{aligned}$$

Adding and subtracting the first and third equations of (14) while retaining the second leads then easily to the equations

$$
\begin{aligned}
q_t + q_x + V_y - yV &= -sQ \\
r_t - r_x + V_y + yV &= -sQ \\
q_y + yq + r_y - yr + 2V_t &= 0
\end{aligned}
\tag{15}
$$

We now decompose all variables into their parabolic cylinder function components which we label with an integer superscript. Now a particularly useful set of relations for the parabolic cylinder functions are those of the raising and lowering operators:

$$
\begin{aligned}
(\partial_y + y)\, D_n(y) &= 2nD_{n-1}(y) \\
(\partial_y - y)\, D_n(y) &= -D_{n+1}(y)
\end{aligned}
$$

It now follows easily from (15) that we have an infinite set of constant coefficient linear equations for the variables q^n, r^n and V^n :

$$
\begin{aligned}
q^n_t + q^n_x - V^{n-1} &= -sQ^n \\
r^n_t - r^n_x + 2(n+1)V^{n+1} &= -sQ^n \\
2(n+1)q^{n+1} - r^{n-1} + 2V^n_t &= 0
\end{aligned}
\tag{16}
$$

where $n = 0, 1, 2, 3, \ldots$ and if superscripts are negative then the term vanishes. These linear equations can be solved in a straightforward albeit tedious fashion once the Q^n are computed by beginning with the $n = 0$ versions and proceeding iteratively. In practice a further approximation to do with zonal scale is made (see below) before solutions are computed.

In terms of the solutions derived above, the modes q^0 correspond with the Kelvin waves while for $n > 1$ the triple (q^n, r^{n-2}, V^{n-1}) correspond with Rossby and gravity/Poincare waves seen in Figure 1 (the labelling in the Figure is from the V^j in the triple since the dispersion relation is derived for this variable). The pair (q^1, V^0) is the mixed Rossby gravity wave.

7 Linear response to diabatic heating

As we saw in Lecture 2, the main driving force of the tropical circulation is provided by diabatic heating caused primarily by latent heat release which is a consequence of the large amount of precipitation that occurs in this region. We consider now the dynamical consequences of this heat engine.

In the lower layers of the tropical atmosphere adjustment time scales are generally very rapid compared to those in the ocean and are typically of the order of days to weeks. Since climatic phenomenon such as El Nino and the Monsoon occur on much longer time scales (which are really oceanic time scales) it is useful to consider the equilibrium or steady state response of the tropical atmosphere. A useful first model of this was proposed by Gill in 1980 [1]. Here he considered a linearization about a state of rest (as we have done in previous lectures) and equilibriated the response by means of the linear dissipation terms. In general the diabatic heating that occurs in moist convection has a very deep structure associated with it and peaks in the mid-troposphere. For this reason we can consider as a reasonable approximation that the heating projects predominantly onto the first baroclinic mode of the atmosphere which has a shallow water speed c of around $40-60ms^{-1}$. Such a mode has a vertical velocity structure peaking in mid-troposphere hence the reason for the approximate heating projection onto this mode. Horizontal velocity and pressure/geopotential perturbations are roughly the vertical derivative of this and have opposite signs in the lower and upper troposphere. The equations governing these perturbations on an equatorial $\beta-$plane can thus be written as

$$\begin{aligned} \epsilon U - \beta y V &= -h_x \\ \epsilon V + \beta y U &= -h_y \\ \epsilon h + c^2(U_x + V_y) &= -sQ \end{aligned} \tag{17}$$

where we are equilibriating the equations with equal Newtonian cooling and Rayleigh friction. Gill assumed that ϵ was of order $(3-4days)^{-1}$ and contended that this dissipation was caused by vertical mixing processes induced by convective systems. In actual fact the Newtonian cooling term can only be ascribed to radiative cooling which has a significantly longer time scale (order two weeks). Solutions are somewhat distorted with a more realistic choice but qualitatively are similar to those we derive below. In order to easily obtain solutions Gill also assumed that the zonal scale is large which enabled the omission of the first term in the second equation of (17). Such an approximation can be shown to be quite accurate for forcing with a large horizontal zonal scale. With this so called long wave approximation we can derive very similar equations to those in (16)[2]

$$\begin{aligned} (2n-1)\epsilon q^n - q_x^n &= -s\left(Q^{n-2}/2 + (n-1)Q^n\right) \\ r^{n-1} &= 2(n+1)q^{n+1} \\ V^{n-1} &= \epsilon q^n + q_x^n + sQ^n \end{aligned} \tag{18}$$

The Kelvin mode equation is obtained by setting $n = 0$ in the first equation:

$$\epsilon q^0 + q_x^0 = -sQ^0 \tag{19}$$

while the first (symmetric) Rossby mode equation is obtained setting $n = 2$:

$$\begin{aligned} 3\epsilon q^2 - q_x^2 &= -s\left(Q^0/2 + Q^2\right) \\ r^0 &= 4q^2 \\ V^1 &= \epsilon q^2 + q_x^2 + sQ^2 \end{aligned} \tag{20}$$

As can be easily shown the forcing induces Kelvin and Rossby modes *of the same sign.* Solutions fall off exponentially away from forcing in opposite zonal directions reflecting the fact that the two (long wave) modes propagate

[2]The term V_t disappears from the third equation and partial derivatives with respect to t are replaced by products by the dissipation parameter ϵ. The long wave approximation also eliminates gravity waves.

in opposite directions from the forcing. An interesting first solution may be obtained by setting

$$Q_s = AD_0(y)\sin\left(\frac{\pi x}{L}\right) \qquad 0 < x < L \tag{21}$$

and zero for other values of x. This forcing represents closely the anomalous latent heating forcing during an El Nino event which occurs around the equator in the international dateline region. Solutions are obtained in a straightforward fashion from equations (19) and (20). The solution for pressure and wind is sketched in Figure 4. Note the symmetric Rossby mode to the west of the forcing and the Kelvin mode to the east.

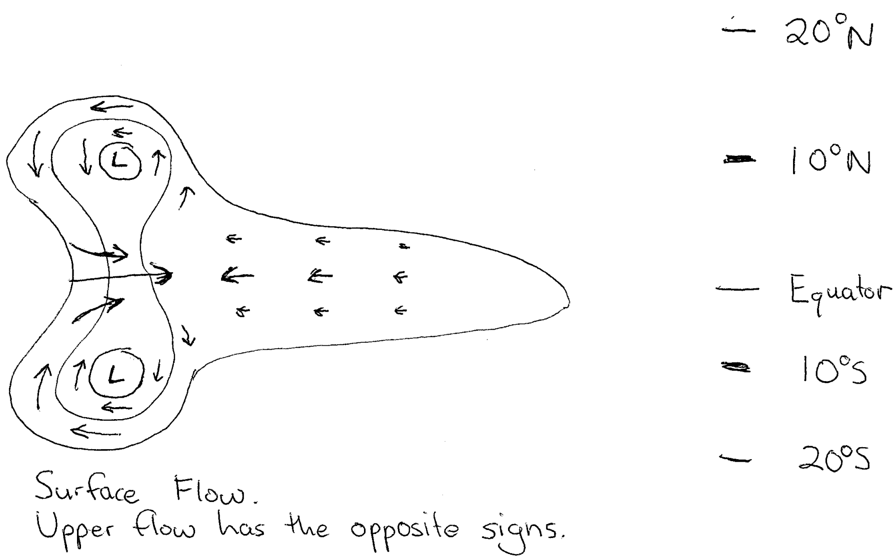

Figure 4: Schematic of Gill solution to equatorial diabatic heating.

The observed circulation anomalies typically seen during an El Nino are displayed in Figure 5.

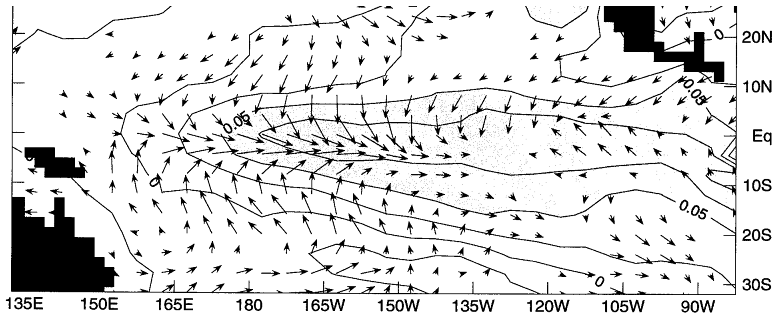

Figure 5: Observed wind anomalies during an El Nino. The anomalous diabatic heating during such events looks like the idealized form chosen for the Gill model.

The agreement is quite good given the simplicity of the model used. Notice the cyclone doublet centered to the west of the forcing and the strong zonal equatorial wind anomalies pointing down the equatorial pressure gradient.

References

[1] A.E. Gill. Some simple solutions for heat-induced tropical circulation. *Quarterly Journal of the Royal Meteorological Society*, 106(449):447–462, 1980.