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TRAJECTORY OF A MOVING CURVEBALL IN VISCID FLOW

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Abstract. A dynamical system to determine the trajectory of a moving curveball is proposed. It is the combination of the Navier-Stokes equations for the fluid and Newton's 2nd law for the ball. The dynamical system is rewritten in coordinates moving with the ball. Numerical simulations demonstrate how the curved path of the ball is caused by the interaction between the flow and the ball.

1. **Introduction.** The curved flight path of a spinning ball has been noticed for over 300 years dating back to Sir Isaac Newton. Newton [8] (1672) had noted how the flight of a tennis ball was affected by spin and his explanation is: "For, a circular as well as a progressive motion..., its parts on that side, where the motions conspire, must press and beat the contiguous air more violently than on the other, and there excite a reluctancy and reaction of the air proportionably greater.". In 1686, *Philosophiae Naturalis Principia Mathematica* was published. Newton set up the first mathematical system to describe the dynamics of the universe. However, he still could not explain the curved flight path from his three basic laws of motion.

The association of this effect with the name of Magnus was due to Rayleigh. His paper [9] in 1877 is credited as the first "true explanation" of the so-called Magnus effect. Magnus found that a rotating cylinder moved sideway when mounted perpendicular to the flow. Rayleigh gave a simple analysis that the side force was proportional to the free-stream velocity and the spinning speed of the cylinder. This was all before the introduction of the boundary-layer concept by Prandtl in 1904. Since then, the Magnus effect has been attributed to asymmetric boundary-layer separation. On the other hand, Kutta and Joukowski used complex variables to describe the irrotational steady Euler flow, and their theorem states that the drag force is zero and the side force is proportional to the circulation around the object and its velocity.

However, the explanation by asymmetric boundary-layer separation is still not clear enough. Kutta-Joukowski theorem is correct for a very special case only. Without viscosity, the spin of a ball or cylinder will never produce circulation. With viscosity, the flow around an object is not a Euler flow and the drag force is not zero. Furthermore, the flow around a moving ball is not irrotational and steady.

Many people have used computers to compute the motion of the 2D Navier-Stokes flow around a moving 2D ball (cylinder) when the ball's constant velocity and rotational speed are specified to verify the Magnus effect. The dynamical system is:

Dynamical System 1:

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$$
\begin{aligned}
\text{The moving ball} \quad &: \quad B = \{\mathbf{x} : |\mathbf{x} - \mathbf{v}_0 t| \le R\} \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} \quad &= \quad -\nabla\left(\frac{p}{\rho}\right) + \nu \Delta \mathbf{u} \qquad (1) \\
\nabla \cdot \mathbf{u} \quad &= \quad 0 \qquad (2)
\end{aligned}
$$

Boundary conditions:

$$
\begin{aligned}
\mathbf{u}\,|_{\partial B} \quad &= \quad \mathbf{v}_0 + R\omega_0 \mathbf{s} \\
\mathbf{u}\,|_{\infty} \quad &= \quad 0
\end{aligned}
$$

Here $\mathbf{u}$, p, ρ are the velocity, pressure and constant density of the fluid. R, $\mathbf{v}_0$, ω_0 are the radius, constant velocity and rotational speed of the ball. ν is the viscosity constant. $\mathbf{s}$ is tangent of the ball's boundary. This approach provides pressure and stress tensor around the ball, from which we can get both drag and side forces:

$$
\begin{aligned}
\mathbf{f} \quad &= \quad -\int_{\partial B} (p\mathbf{n} - \sigma \cdot \mathbf{n})\, ds \qquad (3) \\
\sigma \quad &\equiv \quad \rho\nu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)
\end{aligned}
$$

Here $\mathbf{n}$ is normal of the ball's boundary. However, this approach is not sufficient for calculating the trajectory of a moving spinning ball. The velocity and rotational speed of a ball will change with time, and we can not obtain the trajectory of a ball from this dynamical system.

The curved flight path is caused by the complex interaction between the ball and the fluid. Another classic problem about interaction between an object and a fluid is the oscillation of a falling paper. As yet, there is no explanation for this phenomenon. However, some phenomenological models have been proposed. Tanabe & Kaneko [10] (1994) assumed that the drag force is proportional to the velocity (which is correct for Stokes flows, Reynolds number $\ll 1$) and used Kutta-Joukowski theorem (Reynolds number $= \infty$) to compute the motion of a falling paper. Thus the velocity and rotational speed of a falling paper can determine the drag and side forces and it becomes a closed dynamical system "without" the fluid – the fluid just produces the drag and side forces to the falling paper. But it is suspect to combine the results from two different kinds of flows (see Mahadevan, Aref & Jones [6]).

The general problem of interaction between a solid body and an *inviscid* flow has been studied for over a century following the work by Kelvin and Kirchhoff around 1870 and the resulting equations are presented in many places (see Lamb [5]). However, the trajectory will not be curved unless the spin of a ball can change the behavior of the flow around it, which requires *viscosity.*

Motivated by the curved path of a moving ball, this paper considers the motion of a ball and an viscid fluid around it *simultaneously.* To reduce the complexity of this dynamical system, the flow is assumed to be incompressible, and we restrict our attention to *two-dimensions.* So, the "ball" in this paper is a disk in 2D or a *cylinder* floating in a fluid in 3D. Navier-Stokes equations are used for the fluid and Newton's 2nd law is used for the "ball". The moving coordinate methods are introduced to facilitate numerical simulations. A curveball is achieved and so is the complicated behavior of the fluid around the ball.

2. **The Interacting Dynamical System.**

The dynamical system (PDEs for the fluid and ODEs for the ball) we study here is:

Dynamical System 2:

$$\begin{aligned}
\text{The moving ball} &: \quad B = \{\mathbf{x} : |\mathbf{x} - \mathbf{q}| \le R\} \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} &= -\nabla\left(\frac{p}{\rho}\right) + \nu \Delta \mathbf{u} && (4) \\
\nabla \cdot \mathbf{u} &= 0 \\
\mathbf{q}_t &= \mathbf{v} \\
M\mathbf{v}_t &= -\int_{\partial B} (p\mathbf{n} - \sigma \cdot \mathbf{n})\, ds + M\mathbf{g} + \mathbf{F} && (5) \\
I\omega_t &= R\int_{\partial B} \mathbf{s} \cdot (\sigma \cdot \mathbf{n})\, ds + L && (6) \\
\sigma &= \rho\nu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)
\end{aligned}$$

Boundary conditions:

$$\begin{aligned}
\mathbf{u}\,|_{\partial B} &= \mathbf{v} + R\omega\mathbf{s} \\
\mathbf{u}\,|_{\infty} &= 0
\end{aligned}$$

Here $\mathbf{q}$, $\mathbf{v}$, ω, M, I are the position, velocity, rotational speed, mass and inertia tensor of the ball. $\mathbf{F}$ and L are the force and torque from other sources (for example, from a pitcher's hand).

Note that no new assumption is introduced in the dynamical system. It is simply the combination of a system of PDEs for the fluid and a system of ODEs for Newtonian dynamics. The boundary condition $\mathbf{u}\,|_{\partial B} = \mathbf{v} + R\omega\mathbf{s}$ serves as the connection between them. Without gravity, when M and $I \to \infty$, $\mathbf{v}_t$ and $\omega_t \to 0$ and the system reduces to Dynamical System 1 when $\mathbf{v}$ and ω are constants.

For convenience, we also introduce the vorticity $\xi = \nabla \times \mathbf{u}$, which satisfies

$$\xi_t + (\mathbf{u} \cdot \nabla)\,\xi = \nu \Delta \xi \tag{7}$$

This allows us to avoid evaluating pressure in the computation. We can recover $\mathbf{u}$ from solving a Poisson's equation for the stream function.

The trajectory $\mathbf{q}(t)$ of the ball is determined by the complex system. It is difficult to determine this in the original formulations because the boundary between the ball and fluid is moving and the motion is *unknown*. In Kalthoff *et.al.* [4], the interaction between a falling cylinder and the flow around it is studied. They integrated the stress tensor around the cylinder to get the force. However, because the mesh they used for the fluid is fixed, which conflicts with the free boundary of the falling cylinder, an analytical expansion for the fluid fields based on *low* Reynolds number flow is introduced.

3. **The Moving Coordinates Methods.** In this paper, we address this problem by transferring the system to coordinates moving with the ball. Even though $\mathbf{q}(t)$ is unknown, we can introduce this coordinate transformation formally [1] :

$$\left\{\begin{array}{l} \mathbf{x}' = \mathbf{x} - \mathbf{q}(t) \\ t' = t \end{array}\right.$$

$$\partial_t = \partial_{t'} - \mathbf{v} \cdot \nabla', \; \nabla = \nabla'$$

Let $\mathbf{u}' = \mathbf{u} - \mathbf{v}$ be the fluid velocity in the moving coordinates. Then we rewrite the PDEs as

[1] Note that this is not a Galileo transformation.

$$\begin{aligned}
(\mathbf{u}' + \mathbf{v})_{t'} + (\mathbf{u}' \cdot \nabla)\,\mathbf{u}' &= -\nabla\left(\frac{p}{\rho}\right) + \nu\Delta\mathbf{u}' \qquad (8)\\
\nabla \cdot \mathbf{u}' &= 0\\
\mathbf{u}'\,|_{\partial B} &= R\omega\mathbf{s}\\
\mathbf{u}'\,|_{\infty} &= -\mathbf{v}
\end{aligned}$$

Because $\xi = \nabla \times \mathbf{u} = \nabla' \times \mathbf{u}'$, Eq.(7) is not changed:

$$\xi_{t'} + (\mathbf{u}' \cdot \nabla)\,\xi = \nu\Delta\xi \qquad (9)$$

Now the boundary is fixed and it's straightforward to use complex variables and polar coordinates:

$$Re^{s+i\theta} = x_1' + ix_2'$$

Re^s is used instead of r as it is a more convenient coordinate for the dynamics. In particular, if the emphasis is on the motion of the ball more than the fluid, the behavior of the fluid near the ball is more important and it requires finer mesh spacing.

Let ψ' be the stream function in the moving coordinates. Eq.(9) becomes

$$\begin{aligned}
\xi_{t'} + \frac{\psi_s'\xi_\theta - \psi_\theta'\xi_s}{R^2e^{2s}} &= \nu\Delta\xi \qquad (10)\\
\Delta\psi' &= \xi \qquad (11)
\end{aligned}$$

While we want to use Eq.(10) to compute the flow around the ball, pressure on the boundary is necessary (see Eq.(5)) to compute the motion of the ball. However, it is unnecessary to solve a Poisson's equation to evaluate pressure everywhere.

The only term including p we need is

$$\int_{\partial B} p\mathbf{n}ds = R\int_0^{2\pi} p\,|_{\partial B}\, e^{i\theta}d\theta = iR\int_0^{2\pi} p_\theta\,|_{\partial B}\, e^{i\theta}d\theta.$$

Because $p_\theta\,|_{\partial B} = R\nabla p\,|_{\partial B} \cdot \mathbf{s}$, by Eq.(8),

$$-R^2\omega_{t'} + iR \cdot \mathbf{Re}\left(e^{-i\theta}\mathbf{v}_{t'}\right) = \left(\frac{p_\theta}{\rho} - \nu\xi_s\right)|_{\partial B}\,. \qquad (12)$$

On the other hand,

$$\sigma \cdot \mathbf{n}\,|_{\partial B} = i\rho\nu e^{i\theta}\left(\xi\,|_{\partial B} - 2\omega\right) \qquad (13)$$

in polar coordinates. Using this, we rewrite Eq.(5) and (6) as

$$\begin{aligned}
\left(M - \pi R^2\rho\right)\mathbf{v}_{t'} &= iR\rho\nu\int e^{i\theta}\left(\xi - \xi_s\right)|_{\partial B}\, d\theta + \mathbf{F} \qquad (14)\\
I\omega_{t'} &= R^2\rho\nu\int \xi\,|_{\partial B}\, d\theta - 4\pi R^2\rho\nu\omega + L \qquad (15)
\end{aligned}$$

Note that the terms $-\pi R^2\rho\mathbf{v}_{t'}$ and $-\pi R^2\rho\mathbf{g}$ in Eq.(14) are from the integral of pressure around the ball.

Integrating Eq.(12) over θ, we get

$$2\pi R^2\omega_{t'} = \nu \int \xi_s \mid_{\partial B} d\theta \tag{16}$$

While it appears peculiar that $\omega_{t'}$ is determined by two different equations when $\mathbf{v}_{t'}$ is determined by one only, the reason will become clear later.

Eq.(10), (11), (14), (15) and (16) represent the dynamics in the moving coordinates and the pressure p and velocity $\mathbf{u}$ do not appear in these equations. Furthermore, the integrals in Eq. (14), (15) and (16) are Fourier transformation of ξ and ξ_s over θ. Rewriting all of the equations in the transform space gives us (the symbol $'$ is ignored)

Dynamical System 2':

$$\begin{aligned}
R^2e^{2s}\hat{\xi}^n_t &= l\left(\hat{\psi}^l\hat{\xi}^{n-l}_s - \hat{\psi}^{n-l}_s\hat{\xi}^l\right) + \nu\left(\hat{\xi}^n_{ss} - n^2\hat{\xi}^n\right) && (17)\\
R^2e^{2s}\hat{\xi}^n &= \hat{\psi}^n_{ss} - n^2\hat{\psi}^n && (18)\\
\left(M - \pi R^2\rho\right)\mathbf{v}_t &= 2\pi i R\rho\nu\left(\hat{\xi}^{-1} - \hat{\xi}^{-1}_s\right)\mid_0 + \mathbf{F} && (19)\\
I\omega_t &= 2\pi R^2\rho\nu\left(\hat{\xi}^0\mid_0 - 2\omega\right) + L && (20)\\
R^2\omega_t &= \nu\hat{\xi}^0_s\mid_0 && (21)
\end{aligned}$$

From these equations we can see that the accelerations of the ball's velocity and rotational speed are determined by the Fourier modes -1 and 0 of vorticity around the ball separately.

The boundary conditions for $\hat{\psi}^n$ are listed in the table:

n	$\hat{\psi}^n\mid_0$	$\hat{\psi}^n_s\mid_0$	$e^{-s}\hat{\psi}^n\mid_{s\to\infty}$	$e^{-s}\hat{\psi}^n_s\mid_{s\to\infty}$
0	0	$R^2\omega$	$Unknown$	0
-1	0	0	$iR\mathbf{v}/2$	$iR\mathbf{v}/2$
1	0	0	$-iR\bar{\mathbf{v}}/2$	$-iR\bar{\mathbf{v}}/2$
Others	0	0	0	0

We can see that $e^{-s}\hat{\psi}^0\mid_{s\to\infty}$ is unknown when $e^{-s}\hat{\psi}^n\mid_{s\to\infty}$ are known for all other n. This explains why we need two equations for ω_t to close the system.

Solving the linear equation (18) for $\hat{\psi}^n$, we get:

$$\begin{aligned}
\hat{\psi}^n(s) &= \frac{1}{2n}\left(e^{ns}J_n(s) - e^{-ns}\bar{J}_{-n}(s)\right) \text{ when } n \neq 0\\
\hat{\psi}^0(s) &= s\left(J_0(s) + R^2\omega\right) - K(s)\\
J_n(s) &\equiv R^2\int_0^s e^{(2-n)\tau}\hat{\xi}^n(\tau)\,d\tau\\
K(s) &\equiv R^2\int_0^s \tau e^{2\tau}\hat{\xi}^0(\tau)\,d\tau
\end{aligned}$$

From the boundary conditions $e^{-s}\hat{\psi}^n\mid_{s\to\infty}$, $n > 0$,

$$\begin{aligned}
J_n(\infty) &= 0 \text{ when } n \geq 2\\
J_1(\infty) &= -iR\bar{\mathbf{v}}
\end{aligned}$$

So,

$$\mathbf{v} = -\frac{i\bar{J}_1(\infty)}{R} = -iR\int_0^\infty e^\tau\hat{\xi}^{-1}(\tau)\,d\tau \tag{22}$$

By Eq.(21) and (17), we can verify that

$$R^2 \left(\omega + \int_0^\infty e^{2\tau} \hat{\xi}^0 (\tau) \, d\tau \right)_t = 0$$

We get

$$\omega = - \int_0^\infty e^{2\tau} \hat{\xi}^0 (\tau) \, d\tau + Constant \tag{23}$$

4. **Numerical Simulations.** Now we begin the experiment by pushing and spinning the ball impulsively: when $t < 0$, everything is at rest; when $t = 0^+$, $\mathbf{v}(0^+) = \mathbf{v}_0$ and $\omega(0^+) = \omega_0$ and we want to see what will happen when $t > 0$. More formally, we begin the experiment by

$$\begin{aligned} \mathbf{F}(t) &= \left(M + \pi R^2 \rho \right) \mathbf{v}_0 \delta_t \\ L(t) &= I \omega_0 \delta_t \end{aligned}$$

where δ_t is Delta function. Even though it is impossible in the "real" world, we can imagine that the ball is given a constant force $\left(M + \pi R^2 \rho \right) \mathbf{v}_0 / \varepsilon$ and torque $I\omega_0/\varepsilon$ in the time interval $[0, \varepsilon]$ and we are looking for the limit when $\varepsilon \to 0$.

After the system is nondimensionalized, R, ν, $\rho = 1$ and only two parameters are left: M and I. If the ball has uniform density, M and I are determined by one parameter − density d of the ball only. By Newton's 2nd law, if we want to increase the changes of direction, we need to decrease d. So, in order to see Magnus effect clearly, we chose $d = 10$, $\mathbf{v}_0 = 100$ and $\omega_0 = 50$. We achieve a curveball:

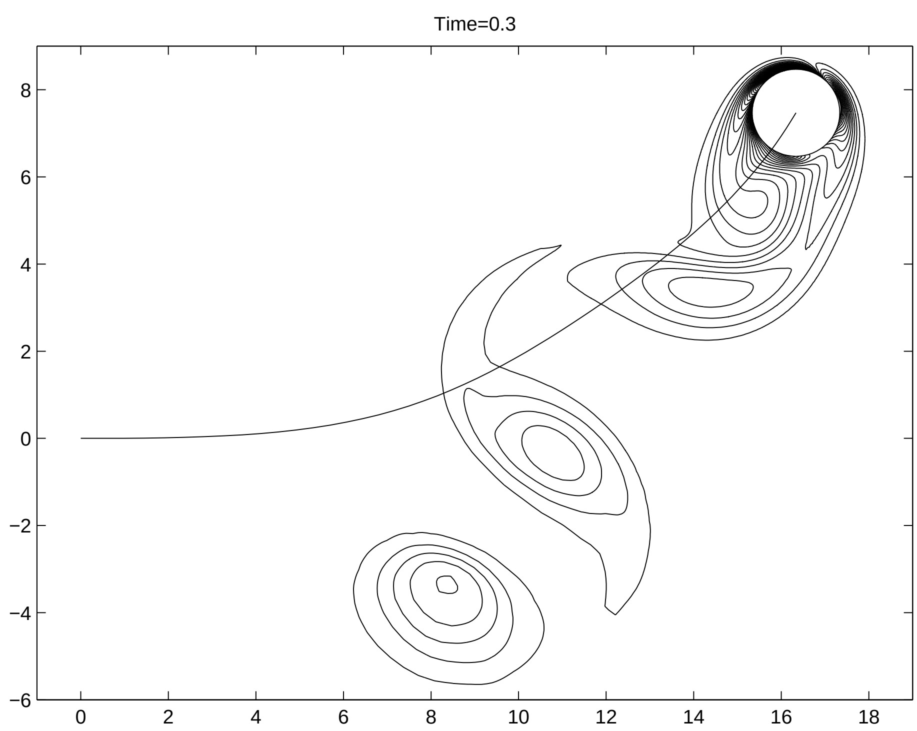

FIGURE 1. The trajectory of the ball and the contour plot of vorticity

5. Conclusion. The simulations show the possibility of computing the interaction between a two-dimensional viscous flow and a rigid body without employing a *simplified* model. We also believe that it is possible to compute the motion of a falling paper by skills similar to what have been developed in this work.

This research was inspired by baseball, but the simulations are still far away from a real moving baseball. The dynamical system we study here is in two-dimensional space. In order to extend the system to three-dimensional space, the vorticity will become a vector and the singularity in spherical polar coordinates will need to be handled carefully. Additionally, a baseball is not a perfect sphere – there are strings around it. There is still no explanation why pitchers can pitch knuckleball, slider, split-finger fastball, forkball, sinker...... Some physicists have tried to understand the trajectory of a baseball by experiment – for example, Briggs [3] for a curveball; Watts & Sawyer [11] for a knuckleball. We are still a long way from being able to accurately simulate a moving baseball and explain how the erratic changes of direction happen.

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