PRICING AND HEDGING

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FINANCIAL MATHEMATICS PROGRAM

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Contents

1 Three Laws of Derivatives

A **derivative** security W is a financial contract which derives its value from an underlying asset U. Consequently, there must exist a mathematical relationship between the derivative and its underlying,

$$W = W(t, U)$$

The course "Pricing and Hedging" will focus on this functional relationship. The **pricing** part of the course will be concerned with deriving the expression and calibrating it to the market, while the **hedging** part will analyze how to dynamically replicate the derivative with a portfolio consisting the underlying and cash.

Derivative pricing theory is governed by a triumvirate of basic principles which are analogous to Newton's Laws of Mechanics:

- **Time Value of Money**
  The forward price of a money market account increases monotonically with time.
- **No-Arbitrage**
  All riskless portfolios must earn the same return as a money market account.
- **Risk Aversion**
  Investors demand a premium above the return of a money market account for incurring risk.

2 Asset Pricing

An object posesses **value** if it is capable of providing utility (possibly negative) to its owner. The set of **assets** $\mathcal{A}$ consists of all objects A that possess a quantifiable value. This implies the existence of a **value function**,

$$V : \mathcal{A} \longrightarrow \mathcal{R}^1,$$

which assigns a value $V(A)$ for each $A \in \mathcal{A}$. The space $\mathcal{A}$ of assets equipped with a value function V is an example of normed vector space $(\mathcal{A}, V)$.
In theory, the value function V must have the property that,

$$V(A_2) > V(A_1),$$

implies that all market participants would be willing to trade asset A_1 for asset A_2. In practice, this only holds for assets for which a **liquid** market exists. Common examples of assets are commodities, real estate, durable goods, contracts and money. We will discuss the latter in great detail in the next chapter.

The **price** of an asset is its value denominated in terms of a **numeraire** asset. The pricing function for the choice of asset A_0 as numeraire is,

$$P(A; A_0) = \frac{V(A)}{V(A_0)}.$$

We will confine our attention to three types of numeraires. Choosing the **local currency** gives the domestic price of an asset in a currency based economy. When a **foreign currency** is used as numeraire we get the price of the asset in the foreign country. Finally, the choice of a **commodity** as numeraire gives us the price of the asset in terms of that commodity in a barter economy.

A **spot pricer** is a function $P^s(t, A; A_0) \in \mathcal{R}^1$ that computes the instantaneous price of an asset A in terms of a numeraire asset A_0,

$$S = P^s(t, A; A_0).$$

When the numeraire asset is blank, always assume that we have chosen the local currency.

If the asset A is liquid the market will determine its clearing price. It not, then a theoretical pricing model is required to compute the hypothetical price at which the a spot trade would clear. We shall see that the guiding principle of such theoretical pricing models is the prevention of arbitrage.

A **forward pricer** is a function $P^f(t, \tau, A; A_0) \in \mathcal{R}^1$ which determines the price at which one would agree to sell asset A for forward delivery at time τ in units of A_0. The case $\tau = t$ simply corresponds to spot pricing so that we must have,

$$P^f(t, t, A; A_0) \equiv P^s(t, A; A_0).$$

The **forward curve** $\{F_\tau\}$ for asset A is define by,

$$F_\tau \equiv P^f(t, \tau, A; A_0) \quad \forall \, \tau \geq t.$$

For assets with forwards which trade on an exchange or over-the-counter (OTC) the forward curve is determined by market forces. Otherwise, the forward pricer must compute the forward relative to the spot using the no-arbitrage principle. We will see in a later chapter that this relationship takes the general form,

$$F_\tau = \text{Spot} + \text{Costs} - \text{Benefits}.$$

3 Time Value of Money

Money is an asset issued by sovereign governments in the form of **currency**. It serves as a numeraire in currency based economies where it is a convenient **medium of exchange**. Money derives its value from the belief that the issuing government will support the current pricing structure by trading in the foreign exchange markets.

The price of money depends on the choice of numeraire. The **nominal** price of money P_{nom} is obtained by choosing the local currency itself as the numeraire. This must always lead to a price of unity because the price of a dollar is one dollar,

$$P_{nom} = 1 \left(\frac{\$}{\$} \right)$$

Alternatively, we can measure the **real** price of money P_{real} in terms of another asset such as a basket of commodities. This choice of numeraire leads to the amount of a commodity that can purchased with a unit of currency. The term real is appropriate here because this price measures the "real" buying power of the currency. As an example, consider the price of a dollar denominated in terms of one barrel of oil,

$$P_{real} = 0.0825 \left(\frac{\text{barrels}}{\$} \right)$$

Finally, we can denominate the local money in terms of a foriegn currency to arrive at the **exchange rate** x between the two currencies. For example, the price of a local dollar in terms of British pounds gives us the number of pounds we need to exchange for one dollar,

$$x = 0.64 \left(\frac{\$}{\$} \right)$$

In addition to allowing immediate consumption via its property as a medium of exchange, money can act a **store of value** permitting possible future consumption. This value storage is accomplished via an abstract construct known as a **money market account** which can be represented physically as a currency deposit made at $t = \tau_0$ that can be withdrawn at any time $t \geq \tau_0$ upon demand.

We will restrict our attention to the **unit** money market account which satisfies the initial condition,

$$M(\tau_0) = 1.$$

We can now price this unit account forward from $t = \tau_0$ to each time τ to obtain the **forward curve** for the currency,

$$\{F_\tau\} \equiv P^f(\tau_0, \tau, \tilde{M};) \qquad \tau \geq \tau_0.$$

The **time value of money** postulates that F_τ is a monotonically increasing function of τ,

$$F_{\tau_2} > F_{\tau_1} \qquad \tau_2 > \tau_1$$

Standard Curve

A standard forward curve F_τ is provided on the next page. We will assume this forward curve for all examples and exercises in the course.

A **term deposit** is a spot money market account which is sold forward at time τ for the amount F_τ. The time period $\tau_0 < t < \tau$ is referred to as the **tenor** of the of term deposit.

Define a **discount bond** D_τ which pays \$1 at maturity τ. If we invest an amount D_τ in a term deposit, we must also withdraw \$1 at τ to prevent arbitrage,

$$D_\tau \times F_\tau = 1$$

Solving for the discount bond price,

$$D_\tau = \frac{1}{F_\tau}$$

Exercise 3.1:

Compute the discount bond prices $\{D_{\tau_i}\}$ for $0 \leq i \leq 20$.

The name discount is derived from the from the fact that $F_\tau \geq 1$ so that

$$D_\tau \leq 1.$$

According to this result a dollar today is worth more than a dollar in the future.

We would now like to compute the stochastic value $\tilde{M}(t)$ of a money market account. The fact that a withdrawal can be made at any time from the account allows us to model it as a term deposit over the infinitesimal tenor period $(\tau_0, \tau_0 + \Delta\tau)$. Hence its value at the end of the tenor period must approximately equal the forward price,

$$M(\tau_0 + \Delta\tau) \simeq F_{\tau_0+\Delta\tau}.$$

Define the **over-night rate** r_0 by,

$$r_0 \equiv \frac{1}{F_{\tau_0}} \frac{dF_\tau}{d\tau}(\tau_0).$$

The instantaneous rate of return of the money market account is,

$$\frac{1}{M(\tau_0)} \lim_{\Delta\tau\to 0} \frac{M(\tau_0 + \Delta\tau) - M(\tau_0)}{\Delta\tau} = \frac{1}{F_{\tau_0}} \lim_{\Delta\tau\to 0} \frac{F_{\tau_0+\Delta\tau} - F_{\tau_0}}{\Delta\tau}.$$

Recalling the definition of a derivative,

$$\frac{1}{M_0} \frac{dM}{dt}(\tau_0) = r_0.$$

Integrating the above equation from τ_0 to time t,

$$\int_{\tau_0}^t \frac{dM}{M(s)} = \int_{\tau_0}^t r_0(s)ds,$$

$$\log M(t) - \log M(\tau_0) = \int_{\tau_0}^t r_0(s)ds,$$

Finally, exponentiating both sides,

$$M(t) = M(\tau_0) \times \exp\left(\int_{\tau_0}^t r_0(s)ds\right).$$

We interpret this result to mean that the deposit is rolled continuously at the "overnight" rate. The predictability of the process comes from the fact at time t knowing the deposit rate $r_0(t)$ is enough to predict with certainty the price of the account at $t + \Delta t$,

$$M(t + \Delta t) = M(t) \times e^{r_0(t)\Delta t}.$$

This predictability makes the money market account a riskless investment with respect to changes in interest rates.

The forward curve F_τ or the discount bond prices D_τ provide a coodinate free description of the time value of money. In order to compute the time rate of change of the value of money over a finite tenor period we must introduce the concept of interest rates which will require choosing a coordinate system.

4 Spot Interest Rates

A **spot rate** measures the local rate of return of a term deposit with a specified tenor. While the forward price F_τ of the deposit is coordinate free and depends only on the tenor period (τ_0, τ) the spot interest rate will depend on both the daycount convention and compounding rule we choose.

The **daycount convention** provides us with a metric for measuring the length of a tenor period,

$$\Delta \mathrm{t} = \frac{\text{Tenor Days}}{\text{Year Days}}.$$

In the numerator we need a rule for counting the number of days in the tenor period which begins on the date $d_1 = Y_1 M_1 D_1$ and ends on $d_2 = Y_2 M_2 D_2$. Now define a function which computes the actual number of days between any two dates d_1 and d_2,

$$\text{Actual Days} = ActDays(d_1, d_2).$$

and a function which calculates the number of days in a year Y,

$$\begin{aligned} YearDays(Y) &= 365 \quad \text{non leap year} \\ &= 366 \quad \text{leap year.} \end{aligned}$$

There are two choices for the tenor daycount,

$$\begin{aligned} \textbf{Act}: \quad \text{Tenor Days} &= ActDays(d_1, d_2) \\ \textbf{30}: \quad \text{Tenor Days} &= (Y_2 - Y_1) \times 360 + (M_2 - M_1) \times 30 + (D_2 - D_1) \times 1. \end{aligned}$$

In the denominator we need a convention for the number of days in a year so that we can determine the length of the tenor period expressed as a fraction of a year. There are three possible rules,

$$\begin{aligned} \textbf{360}: \quad \text{Year Days} &= 360, \\ \textbf{365}: \quad \text{Year Days} &= 365, \\ \textbf{Act}: \quad \text{Year Days} &= YearDays(Y) \qquad \text{if } Y_2 = Y_1, \\ &= \frac{ActDays(Y_1 0101, Y_2 1231)}{1 + Y_2 - Y_1} \quad \text{if } Y_2 > Y_1. \end{aligned}$$

There are three standard daycount conventions:

- **Act/Act** - Treasury,
- **Act/360** - Money Market,
- **30/360** - Corporate Bonds and Swaps.

Exercise 4.1:
Compute the length of each of the tenor periods $\{\Delta\tau_i = \tau_{i+1} - \tau_i\}$ for all three daycount conventions.

The **compounding rule** tells us how often the interest earned by the deposit can be hypothetically reinvested at the spot rate over the tenor period. For example, **simple** compounding allows no reinvestment, **discrete** compunding calls for periodic reinvestment and **continuous** compounding permits instantaneous reinvestment.

The **simple** spot rate r^s_τ is defined as a finite difference,

$$r^s_\tau \equiv \frac{1}{F_{\tau_0}} \frac{F_\tau - F_{\tau_0}}{\tau - \tau_0}.$$

Exercise 4.2:
Compute the simple spot rate r^s_τ for each of the three daycount conventions where $\tau = 19990215$.

Solving for the forward price shows that,

$$F_\tau = 1 + r^s_\tau(\tau - \tau_0)$$

where we have invoked the initial condition $F_0 = 1$.

The 1-period discount bond then becomes,

$$D_\tau \equiv F_\tau^{-1} = \frac{1}{1 + r^s_\tau(\tau - \tau_0)}.$$

To define a **discrete** spot rate we must choose an n-period partition $\mathcal{P}^n_\tau$ for the tenor period τ,

$$\tau_0 < \tau_1 < \tau_2 < \cdots \tau_n = \tau.$$

The intervals are then defined by,

$$\Delta\tau_i \equiv \tau_{i+1} - \tau_i \quad 0 \leq i < n..$$

Associated with the partition $\mathcal{P}_\tau^n$ is a discrete spot rate r_τ^n which when rolled over each of the intervals gives the correct forward price,

$$F_\tau \equiv \prod_{i=0}^{n} \left(1 + r_\tau^n \Delta\tau_i\right).$$

To solve this nonlinear implicit equation for the discrete rate r_τ^n we must employ either the Newton-Raphson or bisection method.

Exercise 4.3:
Compute the discrete spot rate r_τ^n for all three daycounts assuming a monthly partition ($n = 5$) where $\tau = 19990215$.

For the special case of a uniform n-period partition $\hat{\mathcal{P}}_\tau^n$ each interval has length,

$$\Delta\tau^n \equiv \Delta\tau_i = \frac{\tau - \tau_0}{n} \quad \forall i,$$

and we can write,

$$F_\tau = \left(1 + \hat{r}_\tau^n \Delta\tau^n\right)^n,$$

where $\hat{r}_\tau^n$ is the discrete rate associated with the uniform partition $\hat{\mathcal{P}}_\tau^n$.

Solving for the discrete spot rate,

$$\hat{r}_\tau^n = \frac{F_\tau^{1/n} - 1}{\Delta\tau^n}.$$

The **continuous** spot rate is achieved by taking the limit as $n \to 0$,

$$F_\tau = \lim_{n\to\infty} \left(1 + r_\tau^n \frac{\tau - \tau_0}{n}\right)^n = e^{r_\tau^c(\tau-\tau_0)},$$

where the continuous spot rate r_τ^c is defined by,

$$r_\tau^c \equiv \lim_{n\to\infty} r_\tau^n.$$

Expressing the 1-period discount bond in terms of the continuous rate,

$$B_\tau = F_\tau^{-1} = e^{-r_\tau^c(\tau-\tau_0)}.$$

Setting equal the forward prices given in term of the spot and continuous rates gives us a relation between the two rates,

$$e^{r_\tau^c(\tau-\tau_0)} = 1 + r_\tau^s(\tau - \tau_0).$$

Solving for r_τ^c gives,

$$r_\tau^c = \frac{\log\left[1 + r_\tau^s(\tau-\tau_0)\right]}{\tau - \tau_0}.$$

Exercise 4.4:
Compute the continuous spot rate r_τ^c for all three daycounts where again $\tau = 19990215$.

5 Forward Interest Rates

Define the **forward rate** f_n as a simple deposit rate for forward tenor period,

$$\tau_n < \tau < \tau_{n+1}.$$

The **forward rate curve** is the collection of all forward rates,

$$\vec{f} = \{f_i\}.$$

An n-period **forward rate agreement** (FRA) $\mathcal{F}_n$ is an cash-settled OTC contract which allows the long to deposit \$1 at the rate K for the forward period (τ_n, τ_{n+1}),

$$\mathcal{F}_n(t) = \mathcal{F}_n(t, f_n; K).$$

The definition of forward rates requires that,

$$\mathcal{F}_n = \mathcal{F}_n(t, f_n; K = f_n) = 0.$$

Assume that we go long an FRA struck at K and short and FRA struck at f_n. Our net position is,

$$\mathcal{F}_n(t, f_n; K) - \mathcal{F}_n(t, f_n; f_n) = D_{\tau_{n+1}} \times (K - f_n)\Delta\tau_n.$$

Recalling that $\mathcal{F}_n(t, f_n; f_n) = 0$ yields the following FRA pricing equation,

$$\mathcal{F}_n(t, f_n; K) = D_{\tau_{n+1}} \times (K - f_n)\Delta\tau_n$$

We can model a discount bond D_T as a strip of FRA's with notionals designed to match the forward prices of the associated money market account with initial investment $N_0 = B_T$.

Assume the forward interval is partitioned as follows,

$$t = \tau_0 < \tau_1 < \tau_2 \cdots < \tau_n = T.$$

Next, we need to go long N_i contracts $\mathcal{F}_i$ where the notionals are given by,

$$N_i = N_0 \times \prod_{j=0}^{i-1}(1 + f_i\Delta\tau_i).$$

If we now invest an amount $A_0 = D_T$ at the spot rate we have the following amount \$1 at time τ_1,

$$A_1 = (1 + f_0 \Delta\tau_0) \times A_0 = N_1.$$

The contract $\mathcal{F}_1$ enables us to effectively invest the amount A_1 at the rate f_1 for the interval (τ_1, τ_2) so that at time τ_2 we have the amount,

$$A_2 = (1 + f_1 \Delta\tau_1) \times A_1 \equiv N_2.$$

Proceeding inductively until time $\tau_n = T$ when we wind up with the certain amount,

$$A_n = D_T \times \prod_{i=0}^{n-1} (1 + f_i \Delta\tau_i).$$

Since the alternate strategy of paying B_T for the discount bond maturing at time T results in a certain payoff of \$1 at time T, we must have $A_n = 1$ to prevent arbitrage,

$$A_n = D_T \times \prod_{i=0}^{n-1} (1 + f_i \Delta\tau_i) = 1.$$

Solving for the price of the discount bond,

$$D_T = \prod_{i=0}^{n-1} (1 + f_i \Delta\tau_i)^{-1}.$$

Exercise 5.1:
Derive the forward price $D^f_{\tau_i, T}$ of discount bond D_T,

$$D^f_{\tau_i, T} \equiv P^f(0, \tau_i, D_T;)$$

The philosophy behind estimating the forward curve $\vec{f}$ is to imply the discount bond prices D_{τ_i} from the market prices of bonds and then to compute the forward rates according to,

$$f_i = \left(\frac{D_{\tau_i}}{D_{\tau_{i+1}}} - 1 \right) / \Delta\tau_i \quad \forall\ i.$$

Exercise 5.2:
Compute the forward rate curve $\vec{f} = \{f_i\}$ for all three daycount conventions.

6 Coupon Bonds

A **coupon bond** B makes a series of coupon payments on the dates,

$$\tau_0 \leq t < \tau_1 < \cdots < \tau_N = T.$$

and a principal payment at maturity T. Hence, we can think of a coupon bond as a portfolio of discount bonds representing the individual coupons and principal.

The tenor period is expressed as a fraction of a year such as 1/4 for **quarterly** bonds, 1/2 for **semi-annual** bonds and 1 for **annual** bonds. The coupons are paid on a fixed day of the month (e.g. the 15^{th} for treasury bonds). This means that the actual number of days in the tenor period will vary from coupon to coupon. However, the cashflows CF are independent of the actual number of days in the period and is given by,

$$CF = C \times tenor.$$

The **invoice price** $IP(t; C, T)$ of the bond is amount one would be charged in the market to purchase the bond. Because the holder of the bond is entitled to the principal and all future coupon payments the invoice price it is given by the sum of the present values of these cashflows,

$$IP(t; C, T) = \sum_{i=1}^{N} D_{\tau_i} \times CF + D_{\tau_N} \times 1.$$

The invoice price tends to increase linearly until a coupon is paid at which time it drops discontinuously by the amount of the coupon payment.

Exercise 6.1:
Compute the invoice price $\hat{IP}$ of a semi-annual bond $\hat{B}$ with coupon rate $C = 0.0625$ and maturity date $T = 20080815$.

The **accrued interest** earned since last coupon at time τ_0 must be paid by the purchaser in order to compensate the seller for the loss of coupon interest.

$$AI = \gamma \times CF,$$

where γ is the **accrual fraction** defined by,

$$\gamma \equiv \frac{\text{Daycount in period } (\tau_0, t)}{\text{Daycount in period } (\tau_0, \tau_1)}.$$

To smooth out the "sawtooth" behavior exhibited by the invoice price we subtract out the accrued interest to create the **quoted price**,

$$QP \equiv IP - AI.$$

Exercise 6.2:
Compute the quoted price $\hat{QP}$ of bond $\hat{B}$ for the Act/Act daycount basis.

For notational simplicity, unless otherwise specified, we will always compute the quoted price QP when referring to the price B of a coupon bond,

$$B \equiv QP(t; C, T).$$

7 Bond Yield

The **yield** of a bond B is designed to measure its internal rate of return. If we define N to be the number of coupons, the bond yield y is then defined implicitly by the **price-yield formula**,

$$B(t,y;C,T) = \sum_{i=1}^{N} \frac{C \times tenor}{(1 + y \times tenor)^{i-\gamma}} + \frac{1}{(1 + y \times tenor)^{N-\gamma}} - AI.$$

where γ is the accrual period defined in the previous chapter.

We often wish to invert the above non-linear formula to find the yield as a function of the bond price,

$$y = y(t, B; C, T).$$

The price-yield formula can be inverted using a non-linear equation solver such as the Newton-Raphson or bisection methods.

Exercise 7.1:
Compute the yield y of the bond $\hat{B}$ for the Act/Act daycount basis.

A **par bond** is a coupon bond B which is equal to par,

$$B(t, y; C_{par}, T) = 1,$$

where C_{par} is called the **par coupon**.

Exercise 7.2:
Show that $C_{par} = y$ on cashflow dates.

The **par curve** is a graph of the par coupon as a function of maturity,

$$C_{par} = C_{par}(T).$$

Bonds must asymptote to par as they approach maturity. This **pull-to-par** property can holds for all yields as expressed by,

$$\lim_{t \to T} B(t; C, T) = 1 \quad \forall\ y,$$

which makes the use of y as the stochastic underlying variable particularly convenient.

Define the **duration** of a coupon bond B by,

$$Dur(B) \equiv -\frac{1 + y \times tenor}{B}\frac{\partial B}{\partial y}.$$

Let D_T be a discount bond with maturity T and yield y so that the price-yield formula with $C = 0$ becomes,

$$D_T = B(t, y; 0, T) = (1 + y \times tenor)^{-(N-\gamma)}.$$

Computing its duration,

$$\begin{aligned} Dur(D_T) &= -\frac{1 + y \times tenor}{D_T}\frac{\partial D_T}{\partial y} \\ &= -\frac{1 + y \times tenor}{D_T}\left[-\frac{(N-\gamma) \times tenor}{1 + y \times tenor} \times D_T\right] \\ &= (N-\gamma) \times tenor = T - t. \end{aligned}$$

This result shows that the duration of a discount bond is equal to its time to maturity and explains the origen of the term.

Writing a coupon bond B as a portfolio of discount bonds D_i having yield y,

$$B = \sum_{i=1}^{N} D_i(y)$$

It is important to realize that these discount bonds are not equal to the present value of the cashflows but that their sum is equal to the price of the bond. Computing the duration of B,

$$\begin{aligned} Dur(B) &= -\frac{1 + y \times tenor}{B}\sum_{i=1}^{N}\frac{\partial D_i}{\partial y} \\ &= \sum_{i=1}^{N}\frac{D_i}{B}Dur(D_i) = \sum_{i=1}^{N}\frac{D_i}{B}(T_i - t) \end{aligned}$$

When the yield curve is constant the discount bonds are equal the discounted cashflows. In this case we can interpret the duration to be the value weighted time to maturity of its cashflows.

Expanding the bond in a Taylor series about the current yield y to compute its change in price if the yield changes by a small amount Δy over a short time Δt,

$$\Delta B = \frac{\partial B}{\partial t}\Delta t + \frac{\partial B}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 B}{\partial y^2}(\Delta y)^2 + hot.$$

Now define the **modified duration** by,

$$MDur(B) \equiv \frac{Dur(B)}{1 + y \times tenor} = -\frac{1}{B}\frac{\partial B}{\partial y},$$

and the **convexity** according to,

$$Con(B) \equiv \frac{\partial^2 B}{\partial y^2}.$$

The change in bond price can now be expressed as,

$$\Delta B = \frac{\partial B}{\partial t}\Delta t - B \times MDur(B)\Delta y + \frac{1}{2}Con(B)(\Delta y)^2 + hot.$$

Exercise 7.3:
Compute the modified duration and convexity for the bond $\hat{B}$.

8 The Money Market

The **money market** refers to the collection of all high grade debt instruments with one year or less to maturity. The money market therefore defines the forward curve $\mathcal{F}_\tau$ of the money market account. We will discuss six components of the money market. All insturments described use the Act/360 daycount basis.

Treasury bills are discount bonds issued by the US Treasury with 3 month, 6 month and 52 week maturities. They are quoted in terms of a discount rate R. If τ is the time to maturity in the Act/360 daycount basis the bill price becomes,

$$P = 1 - R \times \tau$$

<u>**Exercise 8.1**</u>:
Calculate the discount rate R of a T-bill maturing on $T = 19990215$.

A **repurchase agreement** is a transaction involving the sale of Treasury securities by a dealer together his forward purchase of the security. It amounts to borrowing money using the security as collateral. Repos are typically done overnight, but term repos do exist. The interest rate implied by the repurchase is called the **repo rate**. The mirror image transaction is called **reverse repo** or **matched sales**.

Banks in the Federal Reserve system are required to maintain a certain level of reserves. Banks having excess reserves can lend them to banks needing reserves. This is typically done over-night at the **Fed Funds** rate. The Federal Reserve controls this rate by buying and selling Treasury bonds through its **open market operations**. If it wants to lower rates it increases the money supply by buying bonds outright or doing matched sales. Conversely, it raises rates by selling bonds or repo.

Certificates of Deposit (CD) are negotiable certificates issued by commercial banks that promise to pay then notional at maturity. They typically are in units of $100,000. Similarly, **commercial paper** (CP) is issued by investment grade corporations with a maturity not to exceed 9 months. CP is generally backed by the company's unused bank credit lines.

Bankers acceptances (BA) are designed to facilitate trade. They are discount drafts written on the banks of importers that are initially given to the exporters to pay for the goods prior to shipment. They are fully negotiable in the secondary market. The issuing bank holds claim on the imported goods. Finally, a **Eurocurrency deposit** is a bank deposit denominated in a foreign currency. For example, a dollar deposit in a London bank would constitute a **Eurodollar** deposit. Rates are quoted as an index called the **London Interbank Offer Rate** (LIBOR) which is the average rate offered by five London banks. The LIBOR rate serves as the index for the floating side of an interest rate swap.

9 Discrete Time Stochastic Calculus

A **state space** $\boldsymbol{\Omega}$ is a set whose points $\omega \in \boldsymbol{\Omega}$ represent states of the world. We wish to interpret certain subsets $E \subset \boldsymbol{\Omega}$ of the state space as probabalistic **events**. To maintain this interpretation the collection of all events $\mathcal{F}$ must have the following properties,

- $\boldsymbol{\Omega} \in \mathcal{F}$
- $E \in \mathcal{F} \Longleftrightarrow E^c \in \mathcal{F}$
- $E_i \in \mathcal{F} \quad 1 \leq i \leq N \Longrightarrow \bigcup_{i=1}^{N} E_i \in \mathcal{F}$

If $\mathcal{F}$ satisfies these criterion it is called a **σ-algebra** and the pair $(\boldsymbol{\Omega}, \mathcal{F})$ becomes a **measurable space**.

We now define a **probability measure** Q,

$$Q : \mathcal{F} \rightarrow \mathcal{R}^1$$

which assigns a probability of occurence to each event $E \in \mathcal{F}$ and satisfies,

- $P(\emptyset) = 0$
- $P(\boldsymbol{\Omega}) = 1$
- $E_i \bigcap E_j = \emptyset \Longrightarrow Q(E_i \bigcup E_j) = Q(E_i) + Q(E_j)$

A probability measure Q' is **equivalent** to Q if they agree on the sets of measure zero,

$$Q(E) = 0 \Longleftrightarrow Q'(E) = 0$$

The triple $(\boldsymbol{\Omega}, \mathcal{F}, Q)$ is called a **probability space**.

A **random variable** f is real-valued function,

$$f : \boldsymbol{\Omega} \rightarrow \mathcal{R}^1$$

which is measureable wrt the σ-algebra $\mathcal{F}$,

$$f^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}$$

where $\mathcal{B}$ is the collection of all Borel sets in $\mathcal{R}^1$.

Define a **discrete stochastic process** as a collection of random variables indexed by time,

$$\mathbf{X} = \{X_n; 0 \leq n < \infty\}$$

Each $\omega \in \Omega$ gives a **realization** $\mathbf{X}(\omega)$ of the stochastic process X.

A **filtration** $(\mathcal{F}_i)$ is an ascending sequence $\{\mathcal{F}_i\}$ of σ-subalgebras of $\mathcal{F}$ that are indexed by time,

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}$$

The filtration is said to be **adapted** to the stochastic process $\mathbf{X}$ if each X_n is a $\mathcal{F}_n$-measurable process. This means that by time n we know with certainty whether an event $E \in \mathcal{F}_n$ has occurred.

The **probability distribution** of a stochastic process $\mathbf{X}$ is defined by,

$$P(t_n, x; X_0, X_1, \cdots, X_m)\Delta x = Q\left[\tilde{X}_n \in (x, x + \Delta x)\right]$$

A stochastic process $\mathbf{X}$ is called a **martingale** if the future expectation equal to its current value,

$$\mathbf{E}\left[\tilde{X}_n | \mathcal{F}_m\right] = X_m$$

Furthermore, a process $\mathbf{X}$ is **predictable** if $\tilde{X}_n$ is $\mathcal{F}_{n-1}$-measurable. Therefore, a process is predictable if at time t_{n-1} we know the value of $\tilde{X}_n$ with certainty. Recall this was the case for a money market account which grows at the spot rate.

Finally, a process is **Markovian** if its future behavior depends only on current state. This property can be expressed via the probability distribution as follows,

$$P(t_n, X; X_0, X_1, \cdots, X_m) = P(t_n, X; X_m)$$

The **quadratic variation** QV of a discrete sequence,

$$\mathbf{x} = \{x_i\} \quad 0 \le i < \infty$$

is defined at time t_n by,

$$QV(t_n, \mathbf{x}) \equiv \sum_{i=1}^{n} (x_i - x_{i-i})^2$$

A **random walk** is a stochastic process,

$$\mathbf{w} = \{w_i\} \quad 0 \le i < \infty$$

with the following properties,

- Initial Condition: $w_0 = 0$
- Increments: $w_i = w_{i-1} \pm 1$
- Probabilities: $p_\pm = P(t_i, w_{i-1} \pm 1; w_{i-1}) = \frac{1}{2}$

Exercise 9.1:
Construct a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ for a 2-period random walk and compute the probability $Q(E)$ for each event $E \in \mathcal{F}_2$.

The quadratic variation of each realization $\mathbf{w}(\omega)$ of a random walk at time t_n is,

$$QV[t_n, \mathbf{w}(\omega)] = \sum_{i=1}^{n} (x_i - x_{i-i})^2 = \sum_{i=1}^{n} (\pm 1)^2 = n \quad \forall\, \omega \in \Omega$$

Now let i be the number of up moves of the random walk from t_0 to t_n. The number of down moves is then $n - i$ and the value of the process is,

$$w_n = w_0 + i - (n - i) = 2 \times i - n \equiv x_i$$

The probability of w_n having the value x_i is then given by the binomial probability,

$$P(t_n, x_i; w_0) = \frac{n!}{i! \times (n-i)!}\; p_+^i\; p_-^{n-i}$$

The expectation of any function $f(\tilde{w})$ is given by,

$$\mathbf{E}\left[f(\tilde{w}|w_0\right] = \sum_{i=0}^{n} f(x_i) \times P(0, w_n = x_i; w_0)$$

The **mean** m_n of the random walk corresponds to the choice $f(\tilde{w}) = \tilde{w}$,

$$m_n \equiv \mathbf{E}\left[\tilde{w}_n|w_0\right] = \sum_{i=0}^{n} x_i \times P(0, w_n = x_i; w_0)$$

The symmetry of the random walk about zero yields the result,

$$m_n \equiv \mathbf{E}\left[\tilde{w}_n|w_0\right] = 0 = w_0,$$

which demonstrates that the random walk a martingale.

The higher moments $\gamma \geq 2$ corresponds to the choice,

$$f(\tilde{w}) = (\tilde{w} - m_n)^{\gamma}$$

The 2^n moment $\gamma = 2$ is called the **variance**,

$$v_n \equiv \mathbf{E}\left[(\tilde{w}_n - m_n)^2|w_0\right]$$

Exercise 9.2:
Show that $v_n = n$ using an induction proof.

The standard deviation sd_n is defined as the square-root of the variance,

$$(sd)_n \equiv \sqrt{v_n} = \sqrt{n}$$

This means that the distance travelled by the random walk, in an expected value sense, grows slower than linearly in time. Intuitively, this is due to the fact that the shocks to the random walk can be both positive and negative which is in contrast to a constant drift term which is unidirectional. The fact that for long times the drift term always dominates the stochastic one is illustrated by the following analogy with a children's fable.

Example: "The Tortoise and the Hare"
Add a small drift $\mu \ll 1$ to the random walk,

$$\tilde{w}' = \underbrace{n \times \mu}_{\text{Tortoise}} + \underbrace{\tilde{w}}_{\text{Hare}}$$

$$sd_n = \sqrt{(n)} \Longrightarrow \mathbf{E}\left[|\tilde{w}(n)|\right] = O(\sqrt{n})$$

$$\text{Breakeven :} \qquad n \times \mu = \sqrt{n} \Longrightarrow n = \frac{1}{\mu^2}$$

$$\begin{aligned} n < \frac{1}{\mu^2} \quad &\Longrightarrow \quad n \times \mu < \mathbf{E}\left[|\tilde{w}|\right] \qquad \text{Hare Wins} \\ n > \frac{1}{\mu^2} \quad &\Longrightarrow \quad n \times \mu > \mathbf{E}\left[|\tilde{w}|\right] \qquad \text{Tortoise Wins} \end{aligned}$$

10 No-Arbitrage Principle

Arbitrage is the process of profiting from the trading of assets with varying degrees of risk. There are three basic types of arbitrage. **Pure arbitrage** is the simultaneous purchase and sale of an asset in different markets at different prices. As the name implies, pure abitrage results in a certain profit. By contrast, **risk arbitrage** results in a certain profit only upon the occurence of an independent event. The classic example of risk-arbitrage is the expected takeover of company A by company B. Here the arbitrageur will buy the stock of A and sell the stock of B and earn a certain profit contingent upon the acquistion taking place.

The third example is **statistical arbitrage** which is designed to result in a profit on average if repeated many times. An example of statistical arbitrage is the purchase of an option with an implied volatility below the forecasted volatility. The hedging profits will exceed the option premium only if the realized underlying volatility is greater than the implied volatility.

A **riskless portfolio** $\tilde{\mathcal{P}}(t_0)$ has the same value at time $t_1 = t_0 + \Delta t$ in all possible states of the world. The **no-arbitrage principle** states that a riskless portfolio must appreciate at the same rate as a money market account,

$$\mathcal{P}_{t_1} = (1 + r^s_{t_1} \Delta t) \times \mathcal{P}_{t_0}$$

Therefore, the value of a riskless portfolio at t_{n+1} is known t_n and $\tilde{\mathcal{P}}_i$ is a predictable stochastic process. A portfolio $\hat{\mathcal{P}}$ is **self-financing** if it has zero value, i.e. $\hat{\mathcal{P}} = 0$. For the case of self-financing portfolios the no-arbitrage principle reduces to,

$$\Delta \hat{\mathcal{P}} = 0$$

The intuition behind this condition is that a riskless portfolio requiring no initial investment and having a certain non-zero value at the next time step would produce an arbitrage opportunity.

We shall now state an equivalent version of the no-arbitrage principle in terms of martingales.

Theorem:
Consider an economy consisting of the following $N+1$ stochastic securities,

$$\tilde{U}_0, \tilde{U}_1, \tilde{U}_2, \cdots, \tilde{U}_i, \cdots \tilde{U}_N.$$

Changing the numeraire to $\tilde{U}_0$,

$$\tilde{U}_i^* \equiv \frac{\tilde{U}_i}{\tilde{U}_0} \quad 0 \leq i \leq N.$$

The no-arbitrage requirement is equivalent to the existence of an equivalent measure Q' such that these prices are martingales,

$$\mathbf{E}'\left[\tilde{U}_i^*(n)|\mathcal{F}_m\right] = U_i^*(m) \quad n \geq m.$$

Physically this theorem states that for any choice of numeraire there must exist a risk preference such that all assets have the same expected return as the numeraire asset.

Proof:
To prove this theorem we consider a simple world consisting of one time step and two states. Now construct a self-financing portfolio at t_0 composed of one unit of asset N and α units of asset A,

$$\mathcal{P}(t_0) = N(t_0) + \alpha A(t_0) = 0.$$

Solving for the α which makes $\mathcal{P}(t_0)$ self-financing,

$$\alpha = -\frac{N(t_0)}{A(t_0)}.$$

We choose N to be the numeraire,

$$A^*(t) \equiv \frac{A(t)}{N(t)}.$$

The renormalized portfolio becomes,

$$\mathcal{P}^*(t_0) \equiv \frac{P(t_0)}{N(t_0)} = 1 + \alpha A^*(t_0).$$

At time t_1 the following two states are possible,

$$\begin{aligned}\mathcal{P}^*_-(t_1) &= 1 + \alpha A^*_-(t_1) \\ \mathcal{P}^*_+(t_1) &= 1 + \alpha A^*_+(t_1).\end{aligned}$$

We must now introduce the concept of an **Arrow-Debreu** security which is one in one-state and zero in all others. Assume that A is an Arrow-Debreu securty with value only in the plus-state,

$$\begin{aligned}A_-(t_1) &= 0 \\ A_+(t_1) &= 1.\end{aligned}$$

The states of the portfolio become,

$$\begin{aligned}\mathcal{P}^*_-(t_1) &= 1 \\ \mathcal{P}^*_+(t_1) &= 1 + \alpha A^*_+(t_1).\end{aligned}$$

Since the portfolio is positive in the minus-state, it must be negative in the plus-state in order to prevent arbitrage,

$$\mathcal{P}^*_+(t_1) = 1 + \alpha A^*_+(t_1) < 0.$$

Solving for $A^*_+(1)$,

$$A^*_+(t_1) > -\frac{1}{\alpha} = \frac{A(t_0)}{N(t_0)} \equiv A^*(t_0).$$

This proves the existence of a constant λ such that,

$$A^*(t_0) = \lambda A^*_+(t_1) \quad 0 < \lambda < 1.$$

Repeating the argument for the complimentary Arrow-Debreu security $\bar{A}$ which is one in the minus-state leads to the analogous result,

$$\bar{A}^*(t_0) = \bar{\lambda} \bar{A}^*_-(t_1) \quad 0 < \bar{\lambda} < 1.$$

It is easy to show that the Arrow-Debreu securities form a linear basis that spans the space of all securties. Therefore, we can write any asset U as a linear combination of the two Arrow-Debreu securites,

$$U = \beta A + \bar{\beta} \bar{A}.$$

Renormalize U using the numeraire N,

$$U^* = \beta A^* + \bar{\beta} \bar{A}^*.$$

Valuing $U^*(t_0)$ at time t_0 using the above expressions for $A^*(t_0)$ and $\bar{A}^*(t_0)$,

$$\begin{aligned} U^*(t_0) &= \beta A^*(t_0) + \bar{\beta} \bar{A}^*(t_0) \\ &= \beta \left(\lambda A^*_+(t_1) \right) + \bar{\beta} \left(\bar{\lambda} A^*_-(t_1) \right) \\ &= \lambda \left(\beta A^*_+(t_1) \right) + \bar{\lambda} \left(\bar{\beta} A^*_-(t_1) \right) \\ &= \lambda U^*_+(t_1) + \bar{\lambda} U^*_-(t_1). \end{aligned}$$

Interpreting λ and $\bar{\lambda}$ as probabilities proves the existence of a measure Q' such that U^* is a martingale,

$$U^*(t_0) = \mathbf{E}' \left[\tilde{U}^*(t_1) \right]. \quad Q.E.D.$$

(The fact that in general the probabilities don't add up to one can be corrected by an appropriate dilation of the numeraire.)

11 Risk-Neutral Pricing

In this section we will derive the one-step pricing condition which says that the value of a derivative at time t is the discounted risk-neutral expectation of its prices at time $t + \Delta t$. We will see that the risk-neutral measure is the equivalent martingale measure corresponding to the choice of the money market account as our numeraire.

Consider a underlying asset $\tilde{U}$ which has the following binomial distribution,

$$P\left[\tilde{U}_{t+\Delta t} = U_{\pm}; U_t\right] = p_{\pm}.$$

We are interested invaluing an arbitrary derivative W on U,

$$W = W(t, U).$$

Create a riskless portfolio $\mathcal{P}$ consisting of the derivative W and α units of the underlying U,

$$\mathcal{P}_t = W_t + \alpha U_t.$$

To be riskless portfolio $\mathcal{P}$ must have the have the same value in both states of the world at time $t + \Delta t$,

$$\mathcal{P}_- = W_- + \alpha U_- = W_+ + \alpha U_+ = \mathcal{P}_+.$$

Solving for the hedge ratio α,

$$\alpha = -\frac{W_+ - W_-}{U_+ - U_-}.$$

Since the portfolio $\mathcal{P}$ is riskless, the no-arbitrage condition requires that it appreciate at the risk-free rate,

$$\mathcal{P}_{t+\Delta t} = (1 + f_0 \Delta t) \times \mathcal{P}_t.$$

In the plus state this equation becomes,

$$W_+ + \alpha U_+ = (1 + f_0 \Delta t)(W_t + \alpha U_t)$$

This allows us to write the W_t as a function of the up and down states at time $t + \Delta t$,

$$W_t = (1 + f_0\Delta t)^{-1}\left[W_+ + \alpha\left(U_+ - (1 + f_0\Delta t)U_t\right)\right].$$

Substituting for α and rearranging terms,

$$W_t = (1 + f_0\Delta t)^{-1}\left[\hat{p}_- W_- + \hat{p}_+ W_+\right],$$

where $\hat{p}_-$ and $\hat{p}_-$ are defined by,

$$\begin{aligned}\hat{p}_- &\equiv \frac{U_+ - (1 + f_0\Delta t)U_t}{U_+ - U_-}\\ \hat{p}_+ &\equiv 1 - \frac{U_+ - (1 + f_0\Delta t)U_t}{U_+ - U_-}.\end{aligned}$$

Since $\hat{p}_- + \hat{p}_+ = 1$, we can interpret $\hat{p}_\pm$ as binomial probabilities.

We interpret this equation as the pricing of a derivative in a world which is indifferent between a stochastic asset and it expectation. We call this the risk-neutral world for U because it demands no risk-premium for the uncertainty in the stochastic underlying.

Hence, we refer to $\hat{p}_\pm$ as the risk-neutral probabilities and define the corresponding risk-neutral expectation operator,

$$\hat{\mathbf{E}}\left[X\right] \equiv \hat{p}_- X_- + \hat{p}_+ X_+.$$

We can now write the derivative W_t as the discounted risk-neutral expectation at time $t + \Delta t$ in the following one-step pricing equation,

$$W_t = (1 + f_0\Delta t)^{-1}\hat{\mathbf{E}}\left[\tilde{W}_{t+\Delta t}\right].$$

Exercise 13.1:
Show that under risk-neutral measure assets priced relative to the money market account $\tilde{M}(t)$ are martingales. Hence, risk-neutral measure is the equivalent martingale measure Q' for the choice of the money market account M as numeraire.

To employ this risk-neutral pricing equation we need to transform to the risk-neutral world by changing the probability measure. According to Girsonov's Theorem this can be accomplished by adjusting the drift the the underlying U.

As an example, let us assume that the underlying U obeys a lognormal diffusion equation with drift,

$$\frac{\Delta U}{U} = \mu \Delta t + \sigma \Delta \tilde{w}.$$

To preserve the volatility σ the binomial states must be,

$$U_{\pm} = U_0 e^{\pm \sigma \sqrt{\Delta t}}.$$

The risk-neutral drift $\hat{\mu}$ of U is defined by,

$$\begin{aligned} \hat{\mu} &= \hat{\mathbf{E}}\left[\frac{\Delta U}{\Delta t}\right] = \frac{\hat{p}_- U_- + \hat{p}_+ U_+ - U_t}{\Delta t} \\ &= \frac{U_+ - U_t - \hat{p}_-(U_+ - U_-)}{\Delta t}. \end{aligned}$$

Substituting for $\hat{p}_-$,

$$\hat{\mu} = \frac{U_+ - U_t - [U_+ - (1 + f_0 \Delta t) U_t]}{\Delta t} = f_0.$$

Hence, the change in drift is given by,

$$\Delta \mu = \hat{\mu} - \mu = f_0 - \mu,$$

and risk-neutral process for U is,

$$\frac{\Delta U}{U} = f_0 \Delta t + \sigma \Delta \tilde{w}.$$

An important feature of the risk-neutral probabilities is that they are independent of the specific nature of the derivative W. This is evident from the form of the above expressions for the risk-neutral probabilities. However, one can also reason intuitively that this must be the case. The argument goes as follows. If in the risk-neutral we earn no risk premium for holding the underlying, then we can not earn risk premium for any derivative of the underlying because in that case we could instantaneously hedge out the underlying risk and create a riskless portfolio earning something other than the risk-free rate.

12 Forward Contracts

A **forward contract**,

$$\mathcal{F} = \mathcal{F}(t, U; K, T)$$

is a derivative that obligates the short to deliver an underlying asset U to the long on the maturity date T and obligates the long to pay the strike price K to the short upon delivery. The strike which makes the forward contract worth zero is called the forward price,

$$\mathcal{F}(t, U; F, T) = 0$$

Some forward contracts also have an imbedded **quality option** where $\mathcal{U}$ is the set of deliverable assets and the short can deliver any $U \in \mathcal{U}$.

We will determine the relationship between the spot S and forward prices F by employing the no-arbitrage condition. Assume that the short in the future contract purchases the U at the spot price S and sells it at T for F as prescribed by the contract. While holding the asset U the short incurs costs such as financing the position and enjoys benefits such as coupon payments or dividends. To prevent arbitrage the forward price must satisfy,

$$\mathrm{F} = \mathrm{S} + \mathrm{Costs} - \mathrm{Benefits}$$

The **basis** is defined as the difference between the spot and forward prices,

$$\begin{aligned} \mathrm{Basis} &\equiv \mathrm{F} - \mathrm{S} \\ &= \mathrm{Costs} - \mathrm{Benefits} \end{aligned}$$

The negative of the basis is sometimes referred to as the **carry**,

$$\mathrm{Carry} \equiv -\mathrm{Basis} = \mathrm{Benefits} - \mathrm{Costs},$$

so that a position with positive carry pays its holder. We shall see that owners of American options exercise them early in order to enjoy the positive carry of the underlying.

Exercise 12.1:

Compute the forward price of XYX stock where $S = 50$, $T = 19990815$, assuming quarterly dividends of \$1 paid on the 15^{th} of the month for a (Feb,May,Aug,Nov) cycle.

Exercise 12.2:
Compute the forward invoice price of the coupon bond $\hat{B}$ for the maturity date $T = 19990815$.

If we go long a contract with strike K and short a contract at the current forward price F the value of the portfolio is the discounted value of the certain payoff $F - K$ at maturity T,

$$\mathcal{F}(t, U; K, T) - \mathcal{F}(t, U; F, T) = B_T \times (F - K)$$

Since $\mathcal{F}(t, U; F, T) = 0$ by construction, the value of the contract with an arbitrary strike K is given by,

$$\mathcal{F}(t, U; K, T) = B_T \times (F - K)$$

13 Options on Point Underlyings

A **point underlying** U_t is one whose state can be characterized at each time by a single number real-number,

$$U_t \in \mathcal{R}^1 \quad \forall\, t.$$

This is in contrast to an underlying such as the forward rate curve which is described by a n-tuple $\vec{f} = \{f_i(t)\}$ of values at each time t such that,

$$\vec{f} = \{f_i(t)\} \in \mathcal{R}^n \quad \forall\, t.$$

Examples of point underlyings are equities, foreign exchange rates, commodities and bond yields. We usually assume that U

An **option contract** is a derivative which provides the long the right, but not obligation, to buy or sell the underlying U at the **strike** price K. Contrast this with the definition of a forward contract which obligates both parties to exchange the asset at maturity. This additional optionality makes the long option position more valuable than the corresponding forward contract. We refer to this difference as the **volatility** or **insurance value** of the option.

Options typically come in two flavors. **Call** options enable the long to buy U at the strike from the short while **put** options allow the long to sell U at the strike to the short. Options also come in a variety of exercise styles. In the case of **European** options the option can only be exercised at expiration. However, in the case of **American** options they can be exercised at any time up to expiration. Finally, **Bermudan** options allow exercise on discrete dates.

14 Put-Call Parity

The put-call parity formula establishes a relation between puts and call with the same strike and maturity by constructing a position equivalent to a forward position in the underlying. This relationship is independant of volatility and hence depends only on the level of interest rates. In the early days of option markets violations of put-call parity enabled traders to earn arbitrage profits by doing conversions and reversals which are described below.

Being simultaneously long a call and short a put of the same strike K and maturity T means that I will purchase the underlying for K at T in all states of the world. In other words, this is a **synthetic forward** position,

$$F_{\text{syn}}(t, U; T, K) = C(t, U; T, K) - P(t, U; T, K)$$

The forward price F of an underlying U paying a series of dividends $(Div)_i$ was shown earlier to be,

$$F = \frac{U}{D_T} - \sum_i (Div)_i^f$$

where $(Div)_i^f$ are the forward values of the dividends,

$$(Div)_i^f = \frac{(Div)_i}{D_{t_i,T}^f}$$

Since a forward contract struck at F is worth zero, the value of the synthetic forward must equal the discounted difference between F and the strike K to arrive at the **put-call parity** formula,

$$C - P = D_T \left(F - K \right)$$

If a call C is dear relative to the corresponding put,

$$C - P > D_T \left(F - K \right)$$

an arbitrageur will put on a **reversal** or short synthetic forward postion $P - C$ by going long the put P and short the call C. This position will enable the trader to borrow or lend at a favorable interest rate. Conversely, if the call is cheap relative to the put the arbitrageur will put on a **conversion** or long synthetic forward position $C - P$.

15 Distribution Probes

- **Reversal** - Mean

$$\text{Reversal} = (P - C)_{\Delta=50}$$

- **Straddle** - Variance

$$\text{Straddle} = (P + C)_{\Delta<50}$$

- **Risk Reversal** - Skew

$$\text{RiskReversal} = (P - C)_{\Delta<50}$$

- **Strangle** - Kurtosis

$$\text{Strangle} = (P + C)_{\Delta<50}$$

16 Continuous Time Stochastic Calculus

A **continuous** stochastic process is a collection of random variables X_t indexed by $t \in \mathcal{R}^1$ together with a probability measure Q and adapted filtration $\mathcal{F}_t$,

$$\mathbf{X} = (X_t, \mathcal{F}_t, Q) \qquad 0 \leq t < \infty.$$

The definitions of probability distribution, martingale, predictable, and Markov extend naturally to the continuous domain.

A **Brownian motion** is a stochastic process,

$$\mathbf{w} = (w_t, \mathcal{F}_t, Q)\,,$$

with the properties listed below,

- Initial Condition: $w_0 = 0$
- Independent Increments: $P(\tilde{w}_t - \tilde{w}_s | \mathcal{F}_s) = P(\tilde{w}_t - \tilde{w}_s)$
- Normal Increments: $\quad P(\tilde{w}_t - \tilde{w}_s) = \mathcal{N}(0, t-s).$

Consider the following uniform partition,

$$0 \leq t_1 \leq t_2 \cdots t_n = t \qquad \text{where,}\ \ \Delta t_i \equiv t_{i+1} - t_i = \frac{t}{n}.$$

Now define the **quadratic variation** by,

$$QV(t,\omega) = \lim_{n\to\infty} \sum_{i=0}^{n-1} (w_{i+1}(\omega) - w_i(\omega))^2$$

The expectation of $\tilde{QV}(t)$ is given by,

$$\begin{aligned} \mathbf{E}\left[\tilde{QV}(t)\right] &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{E}\left[(w_{i+1} - w_i)^2\right] \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i) = t. \end{aligned}$$

Computing the variance of $QV(t, \omega)$,

$$\begin{aligned}
\mathbf{Var}\left[\tilde{QV}(t)\right] &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{Var}\left[(w_{i+1} - w_i)^2\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{Kur}\left[w_{i+1} - w_i\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} 3(t_{i+1} - t_i)^2 = 0
\end{aligned}$$

Since its variance is zero, the quadratic variation is equal to its mean $\forall\, \omega$,

$$QV(t, \omega) = QV(t) = t.$$

From this result it follows that the infinitesimal increments are deterministic to first order,

$$(\Delta w)^2 = \Delta t + O\left(\Delta t^2\right).$$

This is the continuous analogy to the fixed shock size of the random walk.

Exercise 16.1:
Compute the quadratic variation $QV(t)$ of a deterministic $f \in \mathcal{C}^1$.

Exercise 16.2:
Show that a Brownian motion is a Martingale,

$$\mathbf{E}\left[\tilde{w}_t | \mathcal{F}_s\right] = w_s.$$

Let $\tilde{X}_t$ be an $\mathcal{F}_t$-measurable stochastic process (i.e. **non-anticipating**) and consider a stochastic integral which we represent symbolically as follows,

$$I(t) = \int_0^t X(s)\, d\tilde{w}.$$

We need to determine how to interpret an integral of this form.

We begin by attempting to treat it a **Riemann integral**. Consider the following partition $\mathcal{P}$,

$$0 < t_0 < t_1 \cdots < t_n = t.$$

Now define the sum,

$$S(t, \mathcal{P}) = \sum_{i=0}^{n} X(\tau_i)[\tilde{w}(t_{i+1}) - \tilde{w}(t_i)].$$

For each i let $\bar{\tau}_i$ be the time which maximizes the function X on the interval (t_i, t_{i+1}) and define the Riemann upper sum by,

$$\bar{S}(t, \mathcal{P}) = \sum_{i=0}^{n} X(\bar{\tau}_i)[\tilde{w}(t_{i+1}) - \tilde{w}(t_i)].$$

The Riemann upper integral $\bar{\mathcal{R}}(t)$ is computed by minimizing the upper sum over all partitions,

$$\bar{\mathcal{R}}(t) = \min_{\mathcal{P}} \bar{S}(t, \mathcal{P})$$

The Riemann lower integral $\underline{\mathcal{R}}(t)$ is defined analogously. If $\bar{\mathcal{R}}(t) = \underline{\mathcal{R}}(t)$ we say the Riemann integral $\mathcal{R}(t)$ exists and its value is given by,

$$\bar{\mathcal{R}}(t) \equiv \mathcal{R}(t) \equiv \underline{\mathcal{R}}(t)$$

If it exists the Riemann integral can be computed numerically according to,

$$\mathcal{R}(t) \equiv \int_0^t X(s)ds = \lim_{n\to\infty} \sum_{i=0}^{n} X(\tau_i)[\tilde{w}(t_{i+1}) - \tilde{w}(t_i)].$$

where we can choose any $\tau_i \in (t_i, t_{i+1})$.

In the stochastic case the Riemann integral will diverge because the increments are $O(\sqrt{\Delta t})$ instead of $O(\Delta t)$ as in the case of a deterministic integral.

Instead, we define the Ito n-sum,

$$S_n = \sum_{i=0}^{n} X(t_i)[\tilde{w}(t_{i+1}) - \tilde{w}(t_i)].$$

The **Ito integral** is then defined as,

$$I(t, \omega) = (ms) \lim_{n\to\infty} S_n.$$

where we take the **mean-square** limit which is defined by,

$$I(\omega) = (ms) \lim_{n\to\infty} S_n \Longleftrightarrow \lim_{n\to\infty} \int q(\omega)[I(t,\omega) - S_n(\omega)]^2 d\omega = 0.$$

where $q(\omega)$ is the probability density associated with the measure Q.

The stochastic Ito integral $I(t)$ is a martingale,

$$\mathbf{E}\left[I(t)|\mathcal{F}_s\right] = I(s) \quad t > s.$$

A **stochastic differential equation** (SDE) is an equation of the form,

$$X_t = \underbrace{\int_0^t \mu(t, X_t)dt}_{\text{Riemann}} + \underbrace{\int_0^t \sigma(t, X_t)d\tilde{w}_t}_{\text{Ito}}.$$

This integral equation is generally written in the following differential form,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)d\tilde{w}_t.$$

Now consider a function a real-valued function,

$$F : \mathcal{R}^2 \to \mathcal{R}$$

According to **Ito's lemma** F satisfies the following SDE,

$$dF(t, X_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X_t}dX_t + \frac{1}{2}\sigma(t, X_t)^2 \frac{\partial^2 F}{\partial X_t^2}dt.$$

Ito's lemma is often referred to as the chain rule for stochastic calculus.

Exercise 16.3:
Use Ito's Lemma to compute the following Ito integral,

$$I(t) = \int_0^t \tilde{w}_s d\tilde{w}_s$$

Let Q be a martingale measure for the numeraire asset U and define the equivalent martingale measure Q' for the numeraire asset U' according to the prescription,

$$Q'(t, E) = \mathbf{E}\left[1_E \times \rho(t, \lambda)|\mathcal{F}_0\right]; \quad E \in \mathcal{F}_t$$

where, $1_E : \Omega \rightarrow \mathcal{R}^1$ is the indicator function,

$$\begin{aligned} 1_E(\omega) &= 1; \quad \omega \in E \\ &= 0; \quad \omega \notin E, \end{aligned}$$

and $\rho(t, \lambda_t)$ is the martingale,

$$\rho(t, \lambda) \equiv \exp\left[\int_0^t \lambda(s) d\tilde{w}'_s - \frac{1}{2}\int_0^t |\lambda(s)|^2 ds\right].$$

Notice that $\rho(0) = 1$ so that the measures Q' and Q agree on all events $E \in \mathcal{F}_0$. Furthermore, we require $\rho(t, \lambda_t)$ to be a martingale to satisfy the consistency condition,

$$Q'(T, E) = Q'(t, E), \quad E \in \mathcal{F}_t, \quad 0 \leq t \leq T.$$

We interpret the function $\lambda(t)$ as the relative volatility of the numeraire assets giving rise to the equivalent measures.

We call $\rho(t)$ the **Radon-Nikodyn derivative** of measure Q' with respect to the equivalent measure Q and denote it by,

$$\rho(t) \equiv \frac{dQ'}{dQ}(t).$$

Let $\tilde{w}_t$ be a Brownian motion under the measure Q. Then **Girsonov's Theorem** states that the process defined by,

$$\tilde{w}'_t \equiv \tilde{w}_t - \int_0^t \lambda(s) ds.$$

is a Brownian motion under the measure Q'.

Assume that under measure Q an asset $\tilde{X}_t$ obeys the following SDE,

$$dX_t = \mu(t)dt + \sigma(t)d\tilde{w}_t.$$

Exercise 16.4:
Show that if we transform the measure to Q' the SDE now becomes,

$$dX_t = \mu'(t)dt + \sigma(t)d\tilde{w}'_t,$$

where the new drift is given by,

$$\mu'(t) \equiv \mu(t) + \lambda(t)\sigma(t).$$

This demonstrates that the above change in measure is equivalent to adjusting the drift. We interpret $\lambda(t)$ as the **market price of risk**,

$$\lambda(t) = \frac{\mu'(t) - \mu(t)}{\sigma(t)}.$$

The continuous time analog of the one-step pricing equation is the Feynman-Kac formula. To prove their result we choose the numeraire asset to be the money market account $M(t)$,

$$W^*(t, U_t^*) \equiv \frac{W(t, U_t)}{M(t)} \quad \text{where,} \quad U_t^* \equiv \frac{U_t}{M(t)}.$$

Assume the following processes for U^* and W^*,

$$\begin{aligned} dU_t^* &= \mu_{U^*} dt + \sigma_{U^*} d\tilde{w}_t \\ dW_t^* &= \mu_{W^*} dt + \sigma_{W^*} d\tilde{w}_t \end{aligned}$$

Applying Ito's lemma to the derivative price,

$$\begin{aligned} dW_t^* &= \frac{\partial W^*}{\partial t} dt + \frac{\partial W^*}{\partial U^*} dU^* + \frac{1}{2}\sigma_{U^*}^2 \frac{\partial^2 W^*}{\partial U^{*2}} dt \\ &= \left(\frac{\partial W^*}{\partial t} + \mu_{U^*} \frac{\partial W^*}{\partial U^*} + \frac{1}{2}\sigma_{U^*}^2 \frac{\partial^2 W^*}{\partial U^{*2}} \right) dt + \sigma_{U^*} \frac{\partial W^*}{\partial U^*} d\tilde{w}_t \end{aligned}$$

Hence, the drift μ_{W^*} of the derivative is,

$$\mu_{W^*} = \frac{\partial W^*}{\partial t} \mu_{U^*} \frac{\partial W^*}{\partial U^*} + \frac{1}{2}\sigma_{U^*}^2 \frac{\partial^2 W^*}{\partial U^{*2}}.$$

According to the no-arbitrage principle, the prices of both the underlying U^* and derivative W^* are martingales under the risk-neutral measure.

Exercise 16.5:
Assume asset $\tilde{A}$ satisfies the following SDE,

$$dA = \mu_A dt + \sigma_A d\tilde{w}_t,$$

and prove that $\mu_A = 0$ iff $\tilde{A}$ is a martingale.

This means that $\mu_{U^*} = \mu_{W^*} = 0$ so that W^* satisfies the following PDE,

$$\frac{\partial W^*}{\partial t} + \frac{1}{2}\sigma_{U^*}^2 \frac{\partial^2 W^*}{\partial U^{*2}} = 0.$$

We recognize this parabolic PDE as the the heat equation that governs temperature distribution $T(t, x)$ in one dimensional solid with conductivity κ,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

Since W^* is a martingale, its value today is equal to expectation at expiration,

$$W^*(t_0) = \hat{\mathbf{E}}\left[\tilde{W}^*(T)\right].$$

This is the celebrated Feynman-Kac formula which gives the solution to a PDE as an expecation of the solution at some initial time thus demonstrating the deep connection between parabolic PDE's and probability theory.

Recalling the expression for the money market account $M(t)$,

$$W^*(t) \equiv \frac{W(t)}{M(t)} = \exp\left(-\int_{t_0}^{t} \tilde{r}_0(s)ds\right) \times W(t),$$

allows us to write the solution in the form,

$$W(t) = \hat{\mathbf{E}}\left[\exp\left(-\int_{t_0}^{T} \tilde{r}_0(s)ds\right) \tilde{W}_T\right].$$

which is the continuous time analog of our one-step risk-neutral pricing equation.

17 Point Underlying Dynamics

Assume tha a point underlying U follows a generalized Brownian motion process with drift,

$$\frac{\Delta U}{U^\alpha} = \mu(t, U)\Delta t + \sigma(t)\Delta\tilde{z}$$

The three most common choices of α are $\alpha = 0$ corresponding to normal dynamics, $\alpha = \frac{1}{2}$ which results in square-root dynamics, and the $\alpha = 1$ case which yields lognormal dynamics.

The drift $\mu(t, U)$ corresponds to the **real world** measure and is the drift we would observe if we attempted to estimate based on historical data. When we actually value derivatives on the underlying U we will need the **risk-neutral** drift that is appropriate to a world requiring no risk premium.

The **volatility term structure** describes how volatility is expected to change deterministically through time,

$$\vec{\sigma} = \{\sigma_i\} \quad \text{where,} \;\; \sigma_i = \sigma(t_i)$$

We shall see later what happens when the volatility term structure becomes stochastic and propagates in time.

We assume that assets such as equities, commodities, and exchange rates follow the lognormal process because it gives scale invariant returns. Although bond yields are not required to be scale invariant we often assume they are lognormal also in order to insure that they remain positive. Consequently, in this course we focus primarily on the lognormal case.

Set $\alpha = 1$ to obtain lognormal dynamics,

$$\frac{\Delta U}{U} = \mu(t)\Delta t + \sigma\Delta\tilde{w}.$$

Exercise 17.1:
Show that in the risk-neutral world $\mu(t) = r_0(t)$

Make the change of variable,

$$u_t \equiv \log(U_t),$$

and apply Ito's Lemma,

$$\Delta u = \frac{\partial u}{\partial t}\Delta t + \frac{\partial u}{\partial U}\Delta U + \frac{1}{2}\frac{\partial^2 u}{\partial U^2}(\Delta U)^2.$$

Computing the derivatives,

$$\frac{\partial u}{\partial t} = 0, \quad \frac{\partial u}{\partial U} = \frac{1}{U}, \quad \frac{\partial^2 u}{\partial U^2} = -\frac{1}{U^2}.$$

and substituting into Ito gives,

$$\Delta u_t \;\; = \;\; \frac{\Delta U}{U} - \frac{1}{2}\left(\frac{\Delta U}{U}\right)^2 = \mu^{'}\Delta t + \sigma\Delta\tilde{w}.$$

This demonstrates that u_t follows a normal Brownian motion process with modified the modified drift,

$$\mu^{'} = \mu - \frac{1}{2}\sigma^2.$$

Computing the mean and variance of x at time t,

$$\begin{aligned} m_t &= u_0 + \int_0^t \mu^{'}(s)ds \\ v_t &= \int_0^t \sigma^2(s)ds. \end{aligned}$$

Since $\tilde{u}$ is normally distributed with mean m_t and variance v_t its transition probability is given by,

$$P(0, u_0; t, u) = \frac{1}{\sqrt{2\pi v_t}} \exp\left(\frac{-(u - m_t)^2}{2v_t}\right).$$

Computing the mean of U_t,

$$\begin{aligned} M_t \equiv \mathbf{E}\left[U_t\right] &= \int_{-\infty}^{+\infty} e^{u'} P(0, u_0; t, u')dx' \\ &= \frac{1}{\sqrt{2\pi v_t}} \int_{-\infty}^{+\infty} \exp\left(u' - \frac{(u' - m_t)^2}{2v_t}\right)du'. \end{aligned}$$

Completing the square of the exponent,

$$
\begin{aligned}
u' - \frac{(u'-m_t)^2}{2v_t} &= -\frac{(u'-(m_t+v_t))^2 + 2m_t v_t + v_t^2}{2v_t} \\
&= -\frac{(u'-(m_t+v_t))^2}{2v_t} + m_t + \frac{1}{2}v_t.
\end{aligned}
$$

Substituting above,

$$
M_t = \frac{\exp\left(m_t + \frac{1}{2}v_t\right)}{\sqrt{2\pi v_t}} \int_{-\infty}^{+\infty} \exp - \left(\frac{(u'-(m_t+v_t))^2}{2v_t}\right) du' = \exp\left(m_t + \frac{1}{2}v_t\right).
$$

Recall that m_t is given by,

$$
\begin{aligned}
m_t &= u_0 + \int_0^t \mu'(s)ds \\
&= u_0 + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s)ds = u_0 - \frac{1}{2}v_t + \int_0^t \mu(s)ds
\end{aligned}
$$

Substituting above,

$$
M_t = U_0 \times \exp\left(\int_0^t \mu(s)ds\right)
$$

Exercise 17.2:
Show that the variance V_t is given by,

$$
\begin{aligned}
V_t &= \exp(2m_t + v_t) \times [\exp(v_t) - 1] \\
&= U_0^2 \exp\left(\int_0^t 2\mu(s)ds\right) \times [\exp(v_t) - 1]
\end{aligned}
$$

18 Black-Scholes Equation

The derivation of the Black-Scholes equation assumes that we have a point underlying, lognormal dynamics, deterministic spot rate and volatility, zero transaction costs, and continuous hedging. To derive the Black-Scholes equation we start by constructing a riskless portfolio,

$$\mathcal{P} = W(t, U) + \alpha U$$

The portfolio $\mathcal{P}$ must grow at risk-free rate,

$$\Delta \mathcal{P} = \Delta W + \alpha \Delta \tilde{U} = r_0(t) \mathcal{P} \Delta t$$

Applying Ito's Lemma,

$$\frac{\partial W}{\partial t} \Delta t + \left(\frac{\partial W}{\partial U} + \alpha \right) \Delta \tilde{U} + \frac{1}{2} \sigma(t)^2 U^2 \frac{\partial^2 W}{\partial U^2} \Delta t = r_0(t) \left(W + \alpha U \right) \Delta t$$

In order to satisfy this equation in all states of the world we must set the coefficent of the stochastic term $\Delta \tilde{U}$ equal to zero,

$$\left(\frac{\partial W}{\partial U} + \alpha \right) \Delta \tilde{U} = 0$$

Solving for the hedge ratio α,

$$\alpha = - \frac{\partial W}{\partial U}$$

Substituting for α above leads to the Black-Scholes partial differential equation for the option $W(t, U)$,

$$\frac{\partial W}{\partial t} + r_0(t) U \frac{\partial W}{\partial U} + \frac{1}{2} \sigma(t)^2 U^2 \frac{\partial^2 W}{\partial U^2} - r_0(t) W(t, U) = 0$$

subject to the initial conditions,

$$\begin{aligned} W(T, U) &= \max \left(U - K, 0 \right) \quad \text{Call} \\ &= \max \left(K - U, 0 \right) \quad \text{Put} \end{aligned}$$

We will now show that the second derivative term in the BS equation respresents the rate at which profits are earned delta hedging the option. Assume that we live in a 2-step binomial world where at t_0 our riskless portfolio $\mathcal{P}_0$ is given by,

$$\mathcal{P}_0 = W_0 + \alpha_0 \times U_0$$

If U goes up at t_0 we have,

$$U_0 \Longrightarrow U_1 = U_0 \times e^{\sigma\sqrt{\Delta t}}$$

and we must rebalance the hedge ratio by the amount,

$$\Delta\alpha = \frac{\partial \alpha}{\partial U}\Delta U = -\frac{\partial^2 W}{\partial U^2} U_0 \times \left(e^{\sigma\sqrt{\Delta t}} - 1\right)$$

This results in cashflow at time t_1 of,

$$C_1 = -\Delta\alpha \times U_1$$

Now assume U goes back down at t_1,

$$U_1 \Longrightarrow U_2 = U_1 \times e^{-\sigma\sqrt{\Delta t}} = U_0$$

which results in the opposite rebalancing,

$$\Delta\alpha_1 = -\Delta\alpha$$

providing the following cashflow,

$$C_2 = \Delta\alpha \times U_0$$

The hedging profit ΔP_h is simply the sum of the cashflows,

$$\begin{aligned}\Delta P_h &= C_1 + C_2 = -\Delta\alpha \times (U_1 - U_0) \\ &= U_0^2 \frac{\partial^2 W}{\partial U^2}\left(e^{\sigma\sqrt{\Delta t}} - 1\right)^2\end{aligned}$$

Taylor expanding the exponential,

$$e^{\sigma\sqrt{\Delta t}} = 1 + \sigma\sqrt{\Delta t} + O\left(\Delta t^2\right)$$

enables us to write,

$$\Delta P_h = U_0^2 \frac{\partial^2 W}{\partial U^2} \sigma^2 \Delta t$$

Computing the hedging profit rate,

$$\frac{\partial P_h}{\partial t} \equiv \frac{\Delta P_h}{2 \times \Delta} = \frac{1}{2} U_0^2 \sigma^2 \frac{\partial^2 W}{\partial U^2},$$

which we recognize as the 2^{nd} order term in the Black-Scholes equation.

Define the following sensitivities,

$$\Delta \equiv \frac{\partial W}{\partial U}, \quad \Gamma \equiv \frac{\partial^2 W}{\partial U^2}, \quad \Theta \equiv \frac{\partial W}{\partial t}, \quad \rho \equiv \frac{\partial W}{\partial r}, \quad \kappa \equiv \frac{\partial W}{\partial \sigma}.$$

and consider the following physical interpretation of each term in the Black-Scholes equation,

$$\underbrace{\Theta}_{\text{Time Decay}} + \underbrace{r_0(t) U \Delta}_{\text{Underlying Carry}} + \underbrace{\frac{1}{2} U^2 \sigma^2 \Gamma}_{\text{Hedging Profits}} - \underbrace{r_0(t) W}_{\text{Option Carry}} = 0$$

<u>**Exercise 18.1**</u>:
Draw a physical analogy between the BS equation and the **heat equation**,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

19 Black-Scholes Solution

We will apply the Feynman-Kac formula to solve the Black-Scholes equation. This involves taking the risk-neutral expectation of the terminal payoff. Recall that the risk-neutral process for U is,

$$\frac{\Delta U}{U} = r_0(t)\Delta t + \sigma(t)\Delta\tilde{z}.$$

Define the normal variable $u \equiv \log(U)$ and apply Ito's lemma,

$$\begin{aligned} du &= \frac{\partial u}{\partial U}\Delta U + \frac{1}{2}\sigma(t)^2\frac{\partial^2 u}{\partial U^2}\Delta t, \\ &= \left(r_0(t) - \frac{1}{2}\sigma(t)^2\right)\Delta t + \sigma(t)\Delta\tilde{z}. \end{aligned}$$

The transition probability for the process u is given by,

$$P(t,u;T,u') = \frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(u'-\bar{u})^2}{2v}\right).$$

where $\bar{u}$ is the mean,

$$\bar{u} = u + \int_t^T \left[r_0(s) - \frac{1}{2}\sigma^2(s)\right] ds,$$

and v is the total variance,

$$v = \int_t^T \sigma^2(s) ds.$$

According to the Feynman-Kac formula we can write the solution in the form,

$$W(t,U;T,K) = \hat{\mathbf{E}}\left[exp\left(-\int_t^T r_0(s)ds\right)W_T\left(\tilde{U}\right)\right].$$

Since the spot rate $r_0(t)$ is deterministic under the Black-Scholes assumptions we can write,

$$exp\left(-\int_t^T r_0(s)ds\right) = D_T(t_0),$$

and the solution becomes,

$$
\begin{aligned}
W(t,U;T,K) &= D_T(t) \times \hat{\mathbf{E}}\left[W_T\left(\tilde{U}\right)\right], \\
&= D_T(t) \times \int_{-\infty}^{\infty} P(t,u;T,u')W_T(u')du',
\end{aligned}
$$

where the initial condition is,

$$
\begin{aligned}
W(T,u';T,K) &= \max\left(e^{u'} - K, 0\right) \quad \text{Call}, \\
&= \max\left(K - e^{u'}, 0\right) \quad \text{Put}.
\end{aligned}
$$

The expression for the call becomes,

$$
C(t,U;T,K) = B_T \frac{1}{\sqrt{2\pi v}} \int_{\log K}^{\infty} \exp\left(-\frac{(u'-\bar{u})^2}{2v}\right)(\exp(u') - K)du'.
$$

Define the two integrals,

$$
\begin{aligned}
I_1 &\equiv \frac{1}{\sqrt{2\pi v}} \int_{\log K}^{\infty} \exp\left(-\frac{(u'-\bar{u})^2}{2v}\right)du', \\
I_2 &\equiv \frac{1}{\sqrt{2\pi v}} \int_{\log K}^{\infty} \exp\left(-\frac{(u'-\bar{u})^2}{2v}\right)\exp(u')du',
\end{aligned}
$$

so that the solution can be written as,

$$
C(t,U;T,K) = D_T\left(I_2 - K \times I_1\right).
$$

Make the following change of variable in I_1,

$$
\begin{aligned}
\omega &\equiv \frac{u'-\bar{u}}{\sqrt{v}}, \\
u' &= \sqrt{v}\omega + \bar{u} \Longrightarrow du' = \sqrt{v}d\omega.
\end{aligned}
$$

The first integral becomes,

$$
I_1 = \frac{1}{\sqrt{2\pi}} \int_{\frac{\log K - \bar{u}}{\sqrt{v}}}^{\infty} \exp\left(-\frac{\omega^2}{2}\right)d\omega = N\left(\frac{\bar{u} - \log K}{\sqrt{v}}\right).
$$

The second integral has the form,

$$I_2 = \frac{1}{\sqrt{2\pi v}} \int_{\log K}^{\infty} e^{p(u')} du',$$

where the polynomial $p(u')$ is defined by,

$$p(u') \equiv -\frac{(u' - \bar{u})^2}{2v} + u'.$$

Complete the square in $p(u')$,

$$\begin{aligned}
p(u') &= -\frac{(u' - \bar{u})^2 - 2vu'}{2v} = -\frac{u'^2 - 2(\bar{u} + v)u' + \bar{u}^2}{2v}, \\
&= -\frac{(u' - (\bar{u} + v))^2}{2v} + \frac{(\bar{u} + v)^2 - \bar{u}^2}{2v}, \\
&= -\frac{(u' - (\bar{u} + v))^2}{2v} + \left(\bar{u} + \frac{1}{2}v\right).
\end{aligned}$$

Now make the change of variable,

$$\begin{aligned}
\omega &\equiv \frac{u' - (\bar{u} + v)}{\sqrt{v}}, \\
u' &= \sqrt{v}\omega + (\bar{u} + v) \Longrightarrow du' = \sqrt{v} d\omega.
\end{aligned}$$

Substitute into the second integral,

$$\begin{aligned}
I_2 &= \exp\left(\bar{u} + \frac{1}{2}v\right) \frac{1}{\sqrt{2\pi v}} \int_{\frac{\log K - (\bar{u} + v)}{\sqrt{v}}}^{\infty} \exp\left(-\frac{\omega^2}{2}\right) d\omega, \\
&= F \times N\left(\frac{(\bar{u} + v) - \log K}{\sqrt{v}}\right).
\end{aligned}$$

Substituting into our expression for a BS call gives,

$$C(t, U; T, K) = D_T \times [F N(d_1) - K N(d_2)],$$

where the cum norm arguments are defined by,

$$\begin{aligned}
d_1 &\equiv \frac{\bar{u} + v - \log K}{\sqrt{v}}, \\
d_2 &\equiv \frac{\bar{u} - \log K}{\sqrt{v}}.
\end{aligned}$$

If we use the fact that,

$$F = \exp\left(\bar{u} + \frac{v}{2}\right),$$

we can write the arguments in the familiar form,

$$\begin{aligned} d_1 &= \frac{\log \frac{F}{K} + \frac{v}{2}}{\sqrt{v}}, \\ d_2 &= d_1 - \sqrt{v}. \end{aligned}$$

Exercise 19.1:
Show the BS put price is given by,

$$P(t, U; T, K) = D_T \times \left[K N(-d_2) - F N(-d_1)\right].$$

Exercise 19.2:
Show that $N(d_1)$ is equal to the hedge ratio α and that $N(d_2)$ is the probability of exercise.

Since we have,

$$F = \frac{U}{D_T} \Longrightarrow D_T \times F = U$$

we can write the call solution in the form,

$$C(t, U; K, T) = \underbrace{N(d_1) \times U}_{\text{Hedge Ratio} \times \text{Underlying}} - \underbrace{N(d_2) \times B_T K}_{\text{Exer Prob} \times \text{Strike PV}}.$$

We interpret the solution as a replicating portfolio composed of cash and the underlying which must be dynamically rebalanced.

20 Future Contracts

A **future contract** $\hat{\mathcal{F}}$ is an exchange traded derivative whose price $\hat{F}(t)$ is **marked-to-market** at the end of each trading day and marked to the spot value of its underlying U at maturity T,

$$\hat{F}(T) = U_T.$$

This ensures that the contract has the same economic consequences in the case of **physical** settlement which requires the short to deliver the underlying at the closing future price $\hat{F}_T$.

The exchange marks-to-market by withdrawing the change in the future price ΔF from the margin account of the **short** and depositing it in the account of the **long**. The forward price adjusts itself so that the **equity** value of the contract is always zero. Contrast this with the forward contract which initially has zero equity but acquires value as the market changes. This means that there is no cost of carrying a future contract.

Assume the underlying U satisfies the following lognormal SDE,

$$\frac{\Delta U}{U} = \mu(t, U)\Delta t + \sigma(t)\Delta\tilde{w}.$$

To value the future price we construct the riskless portfolio,

$$\mathcal{P} = \hat{\mathcal{F}} + \alpha U \quad \text{where} \quad \hat{\mathcal{F}} = 0.$$

According to the no-arbitrage principle,

$$\Delta\mathcal{P} = \Delta\hat{\mathcal{F}} + \alpha\Delta U = r_0(t)\mathcal{P}\Delta t = r_0(t)\alpha U_t\Delta t.$$

Since the account is marked-to-market we have

$$\Delta\hat{\mathcal{F}} = \Delta\hat{F},$$

which upon substitution above gives,

$$\Delta\hat{F} + \alpha\Delta U = r_0(t)\alpha U_t\Delta t.$$

Applying Ito's Lemma,

$$\frac{\partial \hat{F}}{\partial t}\Delta t + \frac{\partial F}{\partial U}\Delta U + \frac{1}{2}\sigma^2(t)U^2\frac{\partial^2 \hat{F}}{\partial U^2}\Delta t + \alpha\Delta U = r_0(t)\alpha U_t\Delta t.$$

The portfolio is riskless when,

$$\left(\frac{\partial \hat{F}}{\partial U} + \alpha\right)\Delta U = 0 \Longrightarrow \alpha = -\frac{\partial \hat{F}}{\partial U}.$$

Substituting above for α gives,

$$\frac{\partial \hat{F}}{\partial t} + r_0(t)U\frac{\partial \hat{F}}{\partial U} + \frac{1}{2}\sigma^2(t)U^2\frac{\partial^2 \hat{F}}{\partial U^2} = 0.$$

To eliminate the first derivative term make the change of variable,

$$U' \equiv \exp\left(-\int_{t_0}^{t} r_0(s)ds\right) \times U.$$

where we have assumed that the spot interest rate is a deterministic function of time.

Computing the derivatives,

$$\begin{aligned}
\frac{\partial \hat{F}}{\partial t} &= \frac{\partial \hat{F}}{\partial t} + \frac{\partial \hat{F}}{\partial U'}\frac{\partial U'}{\partial t} = \frac{\partial \hat{F}}{\partial t} - r_0(t)\frac{\partial \hat{F}}{\partial U'}, \\
\frac{\partial \hat{F}}{\partial U} &= \frac{\partial \hat{F}}{\partial U'}\frac{\partial U'}{\partial U} = \exp\left(-\int_{t_0}^{t} r_0(s)ds\right) \times \frac{\partial \hat{F}}{\partial U'}, \\
\frac{\partial^2 \hat{F}}{\partial U^2} &= \frac{\partial}{\partial U'}\left[\exp\left(-\int_{t_0}^{t} r_0(s)ds\right) \times \frac{\partial \hat{F}}{\partial U'}\right]\frac{\partial U'}{\partial U} = \exp\left(-2\int_{t_0}^{t} r_0(s)ds\right) \times \frac{\partial^2 \hat{F}}{\partial U'^2}.
\end{aligned}$$

Substituting into the equation for $\hat{F}$ above,

$$\frac{\partial \hat{F}}{\partial t} + \frac{1}{2}\sigma^2(t)U'^2\frac{\partial^2 \hat{F}}{\partial U'^2} = 0,$$

which we recognize as the heat equation. According the the Feynman-Kac formula, this equation has the solution,

$$\hat{F}(t_0) = \mathbf{E}\left[\hat{F}_T(U')\right],$$

where U' is a martingale under measure Q,

$$\frac{\Delta U'}{U'} = \sigma(t)\Delta\tilde{w}.$$

We now want to determine the process under measure Q for our underlying U. Writing U in terms of U',

$$U = \exp\left(\int_{t_0}^{t} r_0(s)ds\right) \times U'$$

Apply Ito's Lemma to determine the SDE satisfied by U,

$$\begin{aligned}\Delta U &= \frac{\partial U}{\partial t}\Delta t + \frac{\partial U}{\partial U'}\Delta U' + \frac{1}{2}\sigma^2(t)U'^2\frac{\partial^2 U}{\partial U'^2}\Delta t \\ &= \exp\left(\int_{t_0}^{t} r_0(s)ds\right) \times \left(r_0(t)U'\Delta t + \sigma(t)U'\Delta\tilde{w}\right).\end{aligned}$$

Upon simplification we see that,

$$\frac{\Delta U}{U} = r_0(t)\Delta t + \sigma(t)\Delta\tilde{w}.$$

Since U drifts at the spot rate $r_0(t)$, we recognize that that Q is the risk neutral measure. Alternatively, we could have noticed that U' is simply the underlying U denomimated in terms of the money market account so that it being a martingale means that Q is the risk-neutral measure.

Therefore, the future price is given by,

$$\hat{F}(t_0) = \hat{\mathbf{E}}\left[\hat{F}_T(U)\right].$$

This equation says that the future price is a martingale under risk neutral measure. This result is follows intuitively from the fact that in a risk neutral world a future contract must be a **fair game**.

Since $\hat{F}_T(U) = U$ the future is given by,

$$\hat{F}(t_0) = \hat{\mathbf{E}}\left[\tilde{U}\right] = U(t_0) \times \exp\left(\int_{t_0}^{T} r_0(s)ds\right) = \frac{U(t_0)}{D_T(t_0)}.$$

This demonstrates that the future is equal to the forward price when interest rates are non-stochastic,

$$\hat{F}(t_0) = F(t_0).$$

Some physically settled future contracts have an imbedded **quality option** which enables the short to choose from a set $\mathcal{U}$ of eligible underlying. Upon delivering $U_i \in \mathcal{U}$ the short invoices the long the amount,

$$(IP)_i = f_i \times \hat{F}_T,$$

where f_i is the **factor** for the i^{th} underlying. Therefore, the cost of delivering U_i is given by,

$$Cost\,(U_i) = U_i - (IP)_i = U_i - f_i \times \hat{F}_T.$$

The short will always choose to deliver the underlying which minimizes the cost of delivery,

$$Cost\,(U_{CTD}) = \min_{i \in I} Cost(U_i),$$

where U_{CTD} is called the **cheapest to deliver**. In addition, to prevent arbitrage the cost of delivering the cheapest must be zero,

$$Cost\,(U_{CTD}) = 0 \Longrightarrow \hat{F}_T = \frac{U_{CTD}}{f_{CTD}}.$$

Since delivering any other underlying would require a higher future price to keep the short from losing money we can conclude that the initial condition for the future price is,

$$\hat{F}_T = \min_{i \in I} \left(\frac{U_i}{f_i} \right).$$

Exercise 20.1:
Typically equity prices are negatively correlated to interest rates. Discuss the effect of this on the S&P equity index future.

21 American Options

An **American** option can be exercised at any time t prior to its expiration at T. This is in contrast to a European option which can only be exercised at expiration. The option to exercise an American option early means it is always more valuable than its European counterpart and the difference in value is called **early exercise value** EEV,

$$EEV \equiv W_a - W_e \Longrightarrow W_a = W_e + EEV$$

The **parity value** is the profit upon immediate exercise,

$$\begin{aligned} Par(t, U; K) &= \max(U - K, 0) \quad \text{Call} \\ &= \max(K - U, 0) \quad \text{Put} \end{aligned}$$

Since an American option can be exercised at anytime for parity, it must satisfy the **parity condition**,

$$W_a(t, U; T, K) \geq Par(t, U; K).$$

This leads to the **first form** of the exercise criterion,

$$W_a(t, U; T, K) \leq Par(t, U; K) \overset{\text{exer}}{\Longrightarrow} W_a(t, U; T, K) = Par(t, U; K).$$

In order to apply this criterion we must know the value W_a of the American option in each state. This will require us to construct a lattice, apply the initial condition at expiration, and work backwards imposing the exercise criterion at each node. Since we act optimally at each epoch this approach is tantamount to the **dynamic programming** method. American options cannot be valued using a Monte-Carlo simulation because that technique doesn't have the ability to look forward and value the option.

The **second form** of the exercise criterion provides more intuition into the exercise decision. However, before we can state it we must define the following three terms.

First, the **underlying carry** is the present value of the benefit of owning the underlying until time t,

$$C_U(t) = \underbrace{\text{Benefits}}_{\text{Dividends}} - \underbrace{\text{Costs}}_{\text{Strike Interest}} = \sum_{t_i \leq t} D_{t_i}(t_0) \times (Div)_i - [1 - D_t(t_0)] \times (\pm K).$$

Second, the **option carry** is minus the present value of the interest paid until time t on the current parity value,

$$C_W(t) = \underbrace{\text{Benefits}}_{\text{None}} - \underbrace{\text{Costs}}_{\text{Parity Interest}} = -\left[1 - D_t(t_0)\right] \times Par(t_0, U; K).$$

Finally, the **volatility value** is the discounted expected profit from hedging the option until t,

$$VV(t) = \hat{\mathbf{E}}\left[\int_{t_0}^{t} e^{-\tilde{r}_0(s)(s-t_0)} \times \frac{1}{2}\sigma^2 U^2 \frac{\partial^2 W}{\partial U^2}(s)ds\right].$$

Deep in-the-money American options resemble forward contracts because they will almost certainly be exercised. The two differences are that the maturity date is flexible and that in the case of a large move in the underlying they might not be exercised. The value of the protection against a large move is given by the volatility value which is sometimes also called the **insurance value**. For deep in-the-money options the value of this insurance is typically small and equal to zero in the limit. American options are exercised early to capture an advantageous carry at the expense of giving up this small insurance value. The decision to early exercise requires the satisfaction of both a global and local condition.

The **global** condition requires that the underlying carry be greater than the volatility value plus option carry until expiration,

$$C_U(T) > VV(T) + C_W(T)$$

while the **local** condition requires that this carry also exceed the volatility value and option carry over-night,

$$C_U(t_0 + \Delta t) > VV(t_0 + \Delta t) + C_W(t_0 + \Delta t).$$

We can model a deep ITM American call option as the optimal forward contract $\mathcal{F}^{\text{opt}}$ plus an OTM option to switch to another contract,

$$W_a = \mathcal{F}^{\text{opt}} + \text{Switching Option}$$

where the optimal contract maximizes forward contract equity,

$$\mathcal{F}^{\text{opt}} = \max_{t_0 \le t \le T} \mathcal{F}_t(t_0) = \max_{t_0 \le t \le T} B_t(t_0) \times [F_t(t_0) - K]$$

Exercise 21.1:
What is the impact of the negative correlation between stock prices and interest rates on the early exercise value of an American equity put option?

22 The Trinomial Lattice

As mentioned above, American options must be valued using a dynamic programming technique in a lattice. We will impose the initial condition at expiration and then work backwards using the one-step pricing equation. The parity condition is then applied at each node.

This solution method is called an **explicit** scheme because we can express the value of the option at time t_k in terms of known option values at t_{k+1}. This is in contrast to an **implicit** scheme which requires the solution of a tridiagonal system of linear equations at each epoch.

We choose a set of lattice **epoch** dates from t_0 until expiration T,

$$t_0 < t_1 < t_2 \cdots t_k \cdots t_{N-1} < t_N = T,$$

where the time between epochs is given by,

$$\Delta t_k = t_{k+1} - t_k \qquad 0 \leq k < N.$$

Assume that the underlying U is lognormal and obeys the following SDE under risk-neutral measure,

$$\frac{\Delta U}{U} = \hat{\mu}(t)\Delta t + \sigma(t)\Delta\tilde{w}.$$

Now define normal variable,

$$u \equiv \log(U),$$

and apply Ito's lemma to show that it satisfies,

$$\Delta u = \hat{\mu}'(t)\Delta t + \sigma(t)\Delta\tilde{w},$$

where the modified risk-neutral drift $\hat{\mu}'(t)$ is defined by,

$$\hat{\mu}'(t) \equiv \hat{\mu}(t) - \frac{1}{2}\sigma^2(t).$$

Define the **spine** of the lattice by,

$$u_k^k = u_0 + \sum_{j=0}^{k-1} \hat{\mu}'(t_j)\Delta t_j,$$

and at each epoch t_k construct a set of $2 \times k+1$ **nodes** u_i^k which are centered on the spine,

$$u_0^k < u_1^k \cdots u_{k-1}^k < \underbrace{u_k^k}_{\text{spine}} < u_{k+1}^k \cdots u_{2k-1}^k < u_{2k}^k.$$

For any node u_i^k at epoch t_k its mean at epoch t_{k+1} is,

$$\bar{x} \equiv \mathbf{E}\left[\tilde{u}^{k+1} | u_i^k\right] = u_i^k + \hat{\mu}'(t_k)\Delta t_k,$$

while its incremental variance over the epoch period Δt_k is,

$$v_k = \sigma^2(t_k)\Delta t_k.$$

Now define the **center** of its distribution as the node $x_0 \equiv u_\iota^{k+1}$ at epoch t_{k+1} closest to the mean $\bar{x}$,

$$|x_0 - \bar{x}| = \min_{0 \le i \le 2k} |u_i^{k+1} - \bar{x}|.$$

The **minus** and **plus** nodes are then defined by,

$$x_- = u_{\iota-1}^{k+1} \quad \text{and,} \quad x_+ = u_{\iota+1}^{k+1}.$$

In a trinomial lattice the node u_i^k epoch t_k can take on the three values x_-, x_0 and x_+ at epoch t_{k+1} with the respective probabilities p_-, p_0, and p_+. We call this a **trinomial distribution**. We must conserve probability as well as preserve the mean and the variance of the corresponding normal distribution. Therefore, the three probabilities must satisfy the following linear system of equations,

$$\begin{aligned}
\sum p_i &= p_- + p_0 + p_+ = 1, \\
\mathbf{E}[\tilde{x}] &= p_- x_- + p_0 x_0 + p_+ x_+ = \bar{x}, \\
\mathbf{E}\left[(\tilde{x} - \bar{x})^2\right] &= p_- x_-^2 + p_0 x_0^2 + p_+ x_+^2 - \bar{x}^2 = v_k.
\end{aligned}$$

Solving the conservation of probability equation for p_0,

$$p_0 = 1 - p_- - p_+,$$

and substituting into the mean and variance equations,

$$\begin{aligned} (x_- - x_0)p_- + (x_+ - x_0)p_+ &= \bar{x} - x_0 \\ (x_-^2 - x_0^2)p_- + (x_+^2 - x_0^2)p_+ &= v_k + \bar{x}^2 - x_0^2. \end{aligned}$$

Use Cramer's rule to solve for p_- and p_+,

$$\begin{aligned} p_- &= \frac{(\bar{x} - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(v_k + \bar{x}^2 - x_0^2)}{(x_- - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(x_-^2 - x_0^2)}, \\ p_+ &= \frac{(x_- - x_0)(v_k + \bar{x}^2 - x_0^2) - (\bar{x} - x_0)(x_-^2 - x_0^2)}{(x_- - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(x_-^2 - x_0^2)}, \end{aligned}$$

In the case where $x_0 = \bar{x}$ and the nodes are equally spaced at t_{k+1},

$$\Delta x_{k+1} = x_0 - x_- = x_+ - x_0,$$

the probabilities simplify to,

$$\begin{aligned} p_- = p_+ &= \frac{1}{2} \frac{v_k}{\Delta x_{k+1}^2}, \\ p_0 = 1 - 2 \times p_- &= 1 - \frac{v_k}{\Delta x_{k+1}^2}. \end{aligned}$$

The following choice of lattice spacing,

$$\Delta x_{k+1} = \sigma(t_k)\sqrt{\Delta t_k},$$

leads to the probabilities,

$$p_- = p_+ = \frac{1}{2} \quad \text{and,} \;\; p_0 = 0.$$

This limiting case is called the **binomial distribution** because only two states are reached with positive probability. Any smaller choice of node spacing will lead to a negative middle probability p_0 so that we have the constraint,

$$\Delta x_{k+1} \geq \sigma(t_k)\sqrt{\Delta t_k}.$$

If the volatility $\sigma(t)$ is a function of time, the binomial shocks $\pm\Delta x$ change from epoch to epoch and the binomial lattice does not recombine. Therefore, a binomial lattice requires constant volatility,

$$\sigma(t) = \sigma_0 \qquad t_0 \le t \le T.$$

As the node spacing is increased the probability on the wings goes to zero and the center probability approaches 1 in order to maintain the finite incremental variance,

$$\lim_{\Delta x \to \infty} p_- = \lim_{\Delta x \to \infty} p_+ = 0 \qquad \text{and,} \quad \lim_{\Delta x \to \infty} p_0 = 1.$$

A convenient choice of spacing is,

$$\Delta x = \sigma(t)\sqrt{3\Delta t},$$

which leads to the probabilities,

$$p_- = p_+ = \frac{1}{6} \qquad \text{and,} \quad p_0 = \frac{2}{3}.$$

This spacing leads to the highest order of accuracy because in addition to matching the mean and variance, it matches the kurtosis of the normal distribution,

$$\mathbf{E}\left[(\tilde{x} - \bar{x})^4\right] = p_- \Delta x^4 + p_+ \Delta x^4 = 3\Delta t^2.$$

Unfortunately, when volatility is time dependent these probabilities cannot be maintained throughout the lattice. Instead, an attempt is made to achieve these probabilities "on average".

To value a derivative at our node u_i^k we apply the one-step pricing condition,

$$W(t_k, u_i^k) = (1 + f_0 \Delta t_k)^{-1} \left(p_- W_- + p_0 W_0 + p_- W_+\right),$$

where we've defined,

$$W_\pm \equiv W(t_{k+1}, x_\pm) \qquad \text{and,} \quad W_0 \equiv W(t_{k+1}, x_0).$$

23 Static Term Structures

A **static term structure** $h(t)$ is a time dependent model parameter,

$$h = h(t) \quad t_0 \leq t \leq T.$$

We will generally specify the term structure at discrete times,

$$t_0 < t_1 < \cdots < t_i < \cdots < t_{N-1} < t_N,$$

and hence represent it as a vector,

$$\vec{h} = h(t_i) \quad 0 \leq i \leq N.$$

We have already encountered two examples of static term structures. The first was the forward rate curve $\vec{f}$ which specifies the spot deposit rate as function of time. The second is the volatility term structure $\vec{\sigma}$ which gives us the volatility of the underlying through time.

The term structures are determined through a process referred to as **calibration** which is equivalent to solving an **inverse problem**. This involves choosing sucessive values of the term structure to match liquid derivatives whose market prices are known. For example, if we are interested in valuing an American option we would compute its volatility term structure by calibrating the Black-Scholes solution to its underlying European options. Similarly, we calculate the forward rate curve we calibrate to Eurodollar futures and interest rate swaps.

Example: American Equity Put
Calibrate American option valued on a lattice by matching variances **implied** by the market prices P_n^{mkt} of the underlying European puts.

$$\sum_{i=0}^{n-1} \sigma_i^2 \Delta t_i = v_n \quad 1 \leq n \leq N,$$

where the implied variance v_n is given implicitly by the nonlinear equation,

$$P^{\text{BS}}(t, U; v_n, t_n, K) = P_n^{\text{mkt}}.$$

Solving for σ_{n-1},

$$\sigma_{n-1} = \sqrt{\left(v_n - \sum_{i=0}^{n-2} \sigma_i^2 \Delta t_i\right) / \Delta t_{n-1}}.$$

In the second part of the course we will consider **dynamic** term structures which evolve stochastically through time according to a family of stochastic differential equations which in general has the form,

$$\Delta h_i = \mu_i(t, \vec{h})\Delta t + \kappa_i(t, \vec{h})\Delta \tilde{w}_i \quad 0 \le i \le N.$$

Even in the static case we will be interested in possible changes in the term structures. We first shock the term structure in a manner consistent with the above dynamics,

$$\vec{h} = \{h_i\} \Longrightarrow \vec{h}^+ = \{h_i + \kappa_i \times \delta\} \quad 0 \le i \le N.$$

and then compute the sensitivity $\Delta_{\vec{\kappa}}$ of the derivative with respect to the parameter δ,

$$\Delta_{\vec{\kappa}} = \lim_{\delta \to 0} \frac{W(t, U; \vec{h}^+, T) - W(t, U; \vec{h}, T)}{\delta}$$

In the case where the term structure volatility is constant,

$$\kappa_i = \kappa_0 \quad 0 \le i \le N.$$

we refer to the shock as a **parallel shift**.

If we are interested in hedging our derivative against changes in shape of the term structure or in knowing the time distribution of the sensitivity we compute a **bucket hedge**. This requires partitioning the term structure into M buckets,

$$t_0 =< t_1^b < t_2^b \cdots < t_m^b < \cdots < t_M^b.$$

To compute the m^{th} bucket delta Δ_m^b we parallel shift the term structure over the interval (t_m^b, t_{m+1}^b),

$$\begin{aligned} h_i^+ &= h_i + \delta \quad & t_m^b \le t_i < t_{m+1}^b, \\ &= h_i & t_i < t_m^b, \;\; t_i \ge t_{m+1}^b, \end{aligned}$$

and compute the derivative sensitivity,

$$\Delta_m^b = \lim_{\delta \to 0} \frac{W(t, U; \vec{h}^+, T) - W(t, U; \vec{h}, T)}{\delta}.$$

We usually choose the bucket boundaries to coincide with the expirations of liquid derivatives. For example, if we were trying to hedge a 10yr American option we would choose the boundaries to coincide with 6mos, 1yr, 2yr, 3yr, 5yr, 7yr, and 10yr ATM European options. Assume the n^{th} liquid derivative has the following bucket deltas,

$$\left(\Delta_m^b\right)_n = \hat{\Delta}_{n,m} \quad 1 \leq m \leq M.$$

To eliminate the risk in the M^{th} bucket we must purchase α_M of the liquid derivatives expiring at t_M^b,

$$\alpha_M = -\frac{\Delta_M^b}{\hat{\Delta}_{M,M}}.$$

The deltas in the remaining buckets become,

$$\Delta_m^b \Longrightarrow \Delta_m^b + \alpha_M \times \hat{\Delta}_{M,m} \quad 1 \leq m \leq M - 1.$$

We then continue by purchasing the right amount of liquid derivatives expiring at the edge of the last risky bucket until the risk in all buckets is zero.

Project I

Build a trinomial lattice to value an American equity put.

Assume the following parameters,

$$\begin{aligned} t_0 &= 19980915 \\ T &= 20000915 \\ U(t_0) &= = 55 \\ K &= 50 \end{aligned}$$

Value the option for the 3 volatility term structures below,

$$\begin{aligned} \sigma(t) &= 0.20 \\ &= 0.25 - 0.005 \times (t - t_0) \\ &= 0.15 + 0.005 \times (t - t_0) \end{aligned}$$

and provide analysis of these results.

Compute the interest rate and volatility risk in semi-annual buckets.

24 Spot Rate Models

We will consider the class of interest rate models that allow us to write discount bonds as functions of only the spot interest rate,

$$D_T = D(t, r; T).$$

For the following time partition,

$$t = t_0 < t_1 < \cdots t_i \cdots < t_{N-1} < t_N = T.$$

the forward curve becomes,

$$f_i = \left(\frac{D(t, r; t_i)}{D(t, r; t_{i+1})} - 1 \right) / \Delta t_i \quad 0 \leq i < N.$$

We call this a **spot rate** model because the entire forward curve can be written in terms of the spot rate,

$$\vec{f} = \{ f_i(t, r_t) \} \quad 0 \leq i < N.$$

Spot rate models are **Markovian** in the spot rate because at time t the entire state of the world is imbedded in r_t. This is in contrast with the interest rate model we will be focusing on in this course which in general requires specification of the entire forward curve.

Let the spot rate obey an **Ornstein-Uhlenbeck** process,

$$\Delta r = \underbrace{\alpha (\bar{r} - r)}_{\mu_r} \Delta t + \sigma_0 \Delta \tilde{w}_t.$$

The drift term insures that the spot rate will revert to the long term mean $\bar{r}$ at the rate α. To determine the spot rate distribution we make the change of variable,

$$x(t, r) = -(\bar{r} - r_t) e^{\alpha(t - t_0)}.$$

Applying Ito's lemma,

$$\begin{aligned}
\Delta x &= \frac{\partial x}{\partial t} \Delta t + \frac{\partial x}{\partial r} \Delta r + \frac{1}{2} \sigma_0^2 \frac{\partial^2 x}{\partial r^2} \Delta t, \\
&= \left[-\alpha (\bar{r} - r) \Delta t + \alpha (\bar{r} - r) \Delta t + \sigma_0 \Delta \tilde{w}_t \right] e^{\alpha(t - t_0)}, \\
&= \sigma_0 e^{\alpha(t - t_0)} \Delta \tilde{w}_t.
\end{aligned}$$

The variable x is normally distributed with transition probability,

$$P(t_0, x_0; t, x) = \frac{1}{\sqrt{2\pi s(t)}} \exp\left(-\frac{(x - x_0)^2}{2s(t)}\right),$$

where $s(t)$ is the variance of x,

$$\begin{aligned} s(t) &= \sigma_0^2 \int_{t_0}^{t} e^{2\alpha(t'-t_0)} dt', \\ &= \frac{\sigma_0^2}{2\alpha}\left[e^{2\alpha(t-t_0)} - 1\right]. \end{aligned}$$

Substituting for x gives the distribution of r,

$$P(t_0, r_0; t, r) = \frac{1}{\sqrt{2\pi v(t)}} \exp\left[-\frac{\left((\bar{r} - r_0)\, e^{-\alpha(t-t_0)} + \bar{r} - r\right)^2}{2v(t)}\right],$$

where $v(t)$ is defined by,

$$v(t) \equiv e^{-2\alpha(t-t_0)} s(t) = \frac{\sigma_0^2}{2\alpha}\left[1 - e^{-2\alpha(t-t_0)}\right].$$

and can be interpreted as the variance of the spot rate. This demonstrates that the variance of the Ornstein-Uhlenbeck process approaches a finite asymptote,

$$\lim_{t\to\infty} v(t) = \frac{\sigma_0^2}{2\alpha}.$$

The instantaneous forward rates obey the following normal dynamics,

$$\Delta f_\tau = \mu(\tau)\Delta t + \sigma(\tau)\Delta\tilde{w}_t$$

Since the forward rate f_τ today becomes the spot rate at time $t = t_0 + \tau$ they must have the same variance,

$$\int_0^\tau \sigma(\tau')^2 d\tau' = \frac{\sigma_0^2}{2\alpha}\left[1 - e^{-2\alpha\tau}\right].$$

Taking the derivative wrt τ,

$$\frac{\partial}{\partial \tau} \int_0^\tau \sigma^2(\tau')d\tau' = \frac{\partial}{\partial \tau} \frac{\sigma_0^2}{2\alpha} \left[1 - e^{-2\alpha\tau}\right],$$

leads to the result,

$$\Longrightarrow \sigma(\tau) = e^{-\alpha\tau}.$$

This demonstrates that a mean reverting spot rate model is equivalent to a forward rate model with an exponentially decaying volatility term structure.

To derive the PDE satisfied by a discount bond we construct a riskless portfolio consisting of two bonds with different maturities,

$$\mathcal{P} = D(t, r; T_1) + \beta D(t, r; T_2).$$

According to the no arbitrage condition,

$$\Delta\mathcal{P} = \Delta D_1 + \beta \Delta D_2 = r\mathcal{P}\Delta t.$$

Applying Ito's lemma,

$$\begin{aligned} \frac{\partial D_1}{\partial t}\Delta t \quad &+ \quad \frac{\partial D_1}{\partial r}\Delta r + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D_1}{\partial r^2}\Delta t \\ &+ \quad \beta\left(\frac{\partial D_2}{\partial t}\Delta t + \frac{\partial D_2}{\partial r}\Delta r + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D_2}{\partial r^2}\Delta t\right) = r\left(D_1 + \beta D_2\right)\Delta t. \end{aligned}$$

Choosing the hedge ratio β to make the portfolio riskless,

$$\left(\frac{\partial D_1}{\partial r} + \beta \frac{\partial D_2}{\partial r}\right)\Delta r = 0 \Longrightarrow \beta = -\frac{\partial D_1}{\partial r} / \frac{\partial D_2}{\partial r}.$$

Substituting above for β and separating variables,

$$\left(\frac{\partial D_1}{\partial t} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D_1}{\partial r^2} - rD_1\right) / \frac{\partial D_1}{\partial r} = \left(\frac{\partial D_2}{\partial t} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D_2}{\partial r^2} - rD_2\right) / \frac{\partial D_2}{\partial r}.$$

Since the two sides are equal for arbitrary maturities T_1 and T_2, they must both be equal to a separation function which is independent of maturity T,

$$\left(\frac{\partial D}{\partial t} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D}{\partial r^2} - rD(t, r; T)\right) / \frac{\partial D}{\partial r} = \lambda(t, r).$$

The **Vasicek** model chooses the following separation function,

$$\lambda(t, r) = -\left[\alpha\left(\bar{r} - r\right) + \lambda_0 \sigma_0\right],$$

and the discount bond PDE becomes,

$$\frac{\partial D}{\partial t} + \left[\alpha\left(\bar{r} - r\right) + \lambda_0 \sigma_0\right] \frac{\partial D}{\partial r} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D}{\partial r^2} - rD(t, r; T) = 0.$$

The risk-neutral drift of the spot rate r is defined to be the drift $\hat{\mu}_r$ that makes the discount bond a martingale with respect to the money market account,

$$\Delta r = \hat{\mu}_r \Delta t + \sigma_0 \Delta \tilde{w}_r \Longrightarrow \frac{\Delta D}{D} = r(t)\Delta t + \sigma_D \Delta \tilde{w}_r.$$

To determine $\hat{\mu}_r$ we apply Ito's lemma to the discount bond price,

$$\frac{\Delta D}{D} = \frac{1}{D}\left[\frac{\partial D}{\partial t}\Delta t + \frac{\partial D}{\partial r}\left(\hat{\mu}_r \Delta t + \sigma_0 \Delta \tilde{w}\right) + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D}{\partial r^2}\Delta t\right],$$

and set the coefficent of Δt equal to the spot rate,

$$\frac{1}{D}\left(\frac{\partial D}{\partial t} + \hat{\mu}_r \frac{\partial D}{\partial r} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 D}{\partial r^2}\right) = r(t).$$

Solving for the risk-neutral drift $\hat{\mu}_r$,

$$\hat{\mu}_r = \left(r(t)D - \frac{\partial D}{\partial t} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 D}{\partial r^2}\right) / \frac{\partial D}{\partial r},$$

and using the discount bond equation yields the result,

$$\hat{\mu}_r = \alpha\left(\bar{r} - r\right) + \lambda_0 \sigma_0.$$

Now we can interpret λ_0 as the market price of risk,

$$\Longrightarrow \lambda_0 = \frac{\hat{\mu}_r - \mu_r}{\sigma_0}.$$

The need for a market price of risk arises because we are not automatically matching the discount bond prices. It will be determined when we calibrate the model to match the market. In the forward rate model to be presented in the second part of the course we don't need to specify the risk premium because it is already imbedded in the forward rates. In cases where the risk is untraded, we will need to specify the market price of risk ourselves.

This above equation for discount bonds is subject to the following initial condition,

$$D(T, r; T) = 1,$$

and has the solution,

$$D(t, r; T) = A(t, T) e^{-B(t,T) r},$$

where the functions $A(t, T)$ and $B(t, T)$ are given by,

$$\begin{aligned} B(t,T) &= \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right], \\ A(t,T) &= \exp\left[\frac{(B(t,T) - (T-t))\left(\alpha^2 \bar{R} - \sigma_0^2/2\right)}{\alpha^2} - \frac{\sigma_0^2 B^2(t,T)}{4\alpha}\right], \end{aligned}$$

and the risk-adjusted mean $\bar{R}$ is defined by,

$$\bar{R} \equiv \bar{r} + \frac{\lambda_0 \sigma_0}{\alpha}.$$

To write down the PDE satisfied by an arbitrary interest rate derivative W we simply recognize that the derivation of the discount bond equation was independent of the nature of the derivative until it prescribed the initial condition. Hence, all interest rate derivatives W must also satisfy the the discount bond equation,

$$\frac{\partial W}{\partial t} + \left[\alpha\left(\bar{r} - r\right) + \lambda_0 \sigma_0\right] \frac{\partial W}{\partial r} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 W}{\partial r^2} - r W(t, r; T) = 0.$$

For the case of an American bond option we have the initial condition,

$$\begin{aligned} W(T, r; T, K) &= \max\left[B(T, r; T, C) - K, 0\right] \quad \text{Call}, \\ &= \max\left[K - B(T, r; T, C), 0\right] \quad \text{Put}, \end{aligned}$$

and the parity condition,

$$\begin{aligned} W(t, r : T, K) &= \max\left[W, B(t, r; T, C) - K\right] \quad \text{Call}, \\ &= \max\left[W, K - B(t, r; T, C)\right] \quad \text{Put}. \end{aligned}$$

In order to apply the model to value options we need to calibrate it to match discount bonds and European options. One of the drawbacks of Vasicek is that it only provides four parameters that can be used to calibrate to the market. The **extended** Vasicek or **Hull-White** model overcomes this weakness by making both the market price of risk λ_0 and spot volatility σ_0 functions of time.

$$\underbrace{\lambda_0 \Longrightarrow \lambda_0(t)}_{\text{Bonds}} \qquad \underbrace{\sigma_0 \Longrightarrow \sigma_0(t)}_{\text{Options}}$$

25 Interest Rate Risk

In this section we will use the Vasicek model to incorporate the effect of interest rate risk into the pricing of European options. The fact that Vasicek is a spot rate model means we can write the interest rate dependence entirely in terms of the spot rate,

$$W = W(t, r, U).$$

Assume that r and U obey the following dynamics,

$$\begin{aligned} \Delta r &= \mu_r \Delta t + \sigma_0 \Delta \tilde{w}_r, \\ \frac{\Delta U}{U} &= \mu_U \Delta t + \sigma(t) \Delta \tilde{w}_U, \end{aligned}$$

where the correlation between r and U is given by,

$$\Delta \tilde{w}_r \Delta \tilde{w}_U = \rho \Delta t.$$

The forward price of the underlying is given by,

$$F_T \equiv \frac{U}{D_T},$$

and obeys the lognormal dynamics,

$$\frac{\Delta F_T}{F_T} = \mu_F \Delta t + \sigma_F(t) \Delta \tilde{w}_F,$$

where the volatility of the forward price is given by,

$$\sigma_F(t) = \frac{1}{F_T} \sqrt{\left(U \sigma(t) \frac{\partial F_T}{\partial U} \right)^2 + \rho U \sigma(t) \sigma_0 \frac{\partial F_T}{\partial U} \frac{\partial F_T}{\partial r} + \left(\sigma_0 \frac{\partial F_T}{\partial r} \right)^2}.$$

Computing the partial derivatives,

$$\begin{aligned} \frac{\partial F_T}{\partial U} &= \frac{1}{D_T}, \\ \frac{\partial F_T}{\partial r} &= -\frac{U}{D_T^2} \frac{\partial D_T}{\partial r} = \frac{1}{\alpha} \left(1 - e^{-\alpha(T-t)} \right) F_T, \end{aligned}$$

and substituting into the expression for the forward volatility yields,

$$\sigma_F(t) = \sigma(t)\sqrt{1 + \rho\frac{\sigma_0}{\sigma(t)}\frac{1}{\alpha}\left(1 - e^{-B(t,T)r}\right) + \frac{\sigma_0^2}{\sigma^2(t)}\frac{1}{\alpha^2}\left(1 - e^{-B(t,T)r}\right)^2}.$$

We now construct a riskless self-financing portfolio consisting of the option, the underlying and a discount bond maturing at the expiration T of the option,

$$\mathcal{P} = W(t, r, U) + \beta U + \gamma \underbrace{D(t, r; T)}_{\text{financing bond}} = 0.$$

Choose the numeraire to be the financing bond D_T,

$$\mathcal{P}^* = \frac{\mathcal{P}}{D_T} = W^* + \beta F_T + \gamma.$$

The initial condition transforms to,

$$\begin{aligned} W^*(T, U^* = F_T; T, K) &= \max\left(F_T - K, 0\right) \quad \text{Call}, \\ &= \max\left(K - F_T, 0\right) \quad \text{Put}. \end{aligned}$$

Since the portfolio and initial condition only depend on the forward price F_T we can write,

$$W^* = W^*(t, F_T; T, K).$$

According to the no arbitrage condition,

$$\Delta\mathcal{P}^* = \Delta W^* + \beta\Delta F_T = 0.$$

Applying Ito's Lemma,

$$\frac{\partial W^*}{\partial t}\Delta t + \frac{\partial W^*}{\partial F_T}\Delta F_T + \frac{1}{2}F_T^2\sigma_F^2(t)\frac{\partial^2 W^*}{\partial F_T^2}\Delta t + \beta\Delta F_T = 0.$$

To make the portfolio riskless we require,

$$\left(\frac{\partial W^*}{\partial F_T} + \beta\right)\Delta F_T = 0 \Longrightarrow \beta = -\frac{\partial W^*}{\partial F_T}.$$

Substituting above for β leads to the heat equation,

$$\frac{\partial W^*}{\partial t} + \frac{1}{2} F_T^2 \sigma_F^2(t) \frac{\partial^2 W^*}{\partial F_T^2} = 0.$$

According to the Feynman-Kac formula we can write,

$$W^*(t) = \mathbf{E}_T \left[W^* \left(T, F_T \right) \right],$$

where the expectation is taken with respect to the measure Q_T under which the forward price F_T is a martingale,

$$\frac{\Delta F_T}{F_T} = \sigma_F(t) \Delta \tilde{w}_F.$$

Since both W^* and F_T are martingales under Q_T it must be the equivalent martingale measure associated with the numeraire D_T. We call this the **forward measure** associated with time T.

We recognize that the solution for W^* in terms of F_T is analogous to the Black-Scholes solution with the underlying drift set to zero and no option discounting. Therefore, the variance v and the mean m become,

$$\begin{aligned} v &= \int_t^T \sigma_F^2(t') dt', \\ m &= \log\left(F_T\right) - \frac{1}{2} v, \end{aligned}$$

and the call and put solutions are given by,

$$\begin{aligned} C^*(t, F_T; T, K) &= \underbrace{e^{m+v/2}}_{F_T} N(d_1) - K N(d_2), \\ P^*(t, F_T; T, K) &= K N(-d_2) - \underbrace{e^{m+v/2}}_{F_T} N(-d_1), \end{aligned}$$

where we've defined,

$$\begin{aligned} d_1 &\equiv \frac{m + v - \log(K)}{\sqrt{v}} = \frac{\log(F_T/K) + \frac{1}{2} v}{\sqrt{v}}, \\ d_2 &\equiv d_1 - \sqrt{v}. \end{aligned}$$

Recalling that we chose the discount bond D_T as the numeraire allows us to write the solution as,

$$\begin{aligned} C(t, F_T; T, K) &= D_T \times [F_T N(d_1) - K N(d_2)], \\ P(t, F_T; T, K) &= D_T \times [K N(-d_2) - F_T N(-d_1)]. \end{aligned}$$

Comparing this result to the Black-Scholes solution shows that the only effect of the interest rate risk is to modify the volatility from $\sigma(t)$ to $\sigma_F(t)$.

Example: Equity Options

The **dividend discount** model for valuing equity states that the value of a stock is the present value of all its projected future dividends,

$$S = \sum_{i=1}^{\infty} D_i \times (Div)_i .$$

If we assume the dividends are independent of interest rates, stock prices behaves like coupon bonds and are negatively correlated with rates,

$$\rho_{S,r} = \mathbf{E}\left[\left(S - \bar{S}\right)(r - \bar{r})\right] < 0$$

The negative correlation between equity and rates means that an increase in stock price is likely to accompanied by a decrease in the basis, while a decrease in stock price on average leads to a increase in the basis. We see that the negative correlation provides a natural restoring mechanism because the forward price is not as volatile as the spot rate,

$$\sigma_F < \sigma.$$

This reduction of volatility of course leads to lower European option prices. We can also see this by analyzing the costs associated with hedging an option. Consider a European call which we hedge with α shares of stock,

$$\mathcal{P} = C + \alpha S \quad \text{where, } \alpha = -\frac{\partial C}{\partial S} < 0$$

If the stock price rises by ΔS we must adjust our hedge by the amount,

$$\Delta\alpha = \frac{\partial \alpha}{\partial S}\Delta S = -\frac{\partial^2 C}{\partial S^2}\Delta S < 0,$$

which requires us to sell additional shares of stock. However, because of the negative correlation between interest rates and the stock price the proceeds from the short sale will be invested at a lower interest rate. Conversely, if the stock price goes down we be required to buyback shares with money borrowed at a higher interest rate. Thus in both cases the negative correlation provides a "drag" on hedging profits and consequently leads to a lower option price. There is also a small effect due the fact the interest rate hedge also changes with the level of interest rates. We can derive the mathematical expression for the effect of interest rates on the rate of hedging profits by applying Ito's lemma to the option price,

$$\begin{aligned} \Delta W(t,S,r) &= \frac{\partial W}{\partial t}\Delta t + \frac{\partial W}{\partial S}\Delta S + \underbrace{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2}\Delta t}_{\text{Underlying Hedging Profits}} \\ &+ \frac{\partial W}{\partial r}\Delta r + \underbrace{\frac{1}{2}\sigma_0^2 \frac{\partial^2 W}{\partial r^2}\Delta t}_{\text{IR Hedging Profits}} + \underbrace{\rho S \sigma_0^2 \frac{\partial^2 W}{\partial S \partial r}\Delta t}_{\text{Financing Drag}} \end{aligned}$$

26 Term Structure Risk

In this section we discuss how to approximate the effect stochastic term structures using perturbation theory. Consider a derivative W which depends on its underlying U and a term structure $\vec{h}$,

$$W = W(t, \tilde{U}, \tilde{\vec{h}}).$$

Now assume U and $\vec{h}$ obey the following dynamics,

$$\begin{aligned} \frac{\Delta U}{U} &= \mu \Delta t + \sigma(t) \Delta \tilde{w}_U, \\ \frac{\Delta h_i}{h_i} &= \lambda_i \Delta t + \kappa_i(t) \Delta \tilde{w}_h \qquad 0 \le i < N, \end{aligned}$$

where the correlation ρ between U and $\vec{h}$ is given by,

$$\Delta \tilde{w}_U \Delta \tilde{w}_h = \rho \Delta t.$$

According to Ito's lemma,

$$\begin{aligned} \Delta W(t_k) &= \underbrace{\frac{\partial W}{\partial t}(t_k)\Delta t}_{\text{Time Decay}} + \underbrace{\frac{\partial W}{\partial U}(t_k)\Delta U}_{\text{Underlying Risk}} + \underbrace{\frac{1}{2}\sigma^2 U^2 \frac{\partial^2 W}{\partial U^2}(t_k)\Delta t}_{\text{Underlying Hedging Profits}} \\ &+ \underbrace{\frac{\partial W}{\partial h_0}(t_k)\Delta h_0}_{\text{TS Risk}} + \underbrace{\frac{1}{2}\kappa_0^2 h_0^2 \frac{\partial^2 W}{\partial h_0^2}(t_k)\Delta t}_{\text{TS Hedging Profits}} \\ &+ \underbrace{\rho \sigma U \kappa_0 h_0 \frac{\partial^2 W}{\partial U \partial h_0}(t_k)\Delta t.}_{\text{Correlation Hedging Profits}} \end{aligned}$$

Hedging profits (or losses) arise because we are able to rebalance the hedge at a time when the price is favorable (or unfavorable). In the gamma terms the change in the hedge is perfectly correlated with the change in the price of the hedging vehicle. In the cross-partial term, the hedge ratio wrt the underlying also changes when the term structure shifts. This can lead on average to an expected profit or loss if there is a correlation between the underlying and the term structure.

Our goal is to approximate the additional profits earned by hedging the term structure by computing the above terms under the assumption that the term structure is static. Let W_0 be the value of the derivative when $\vec{h}$ is static,

$$W_0 = W_0(t, \tilde{U}, \vec{h}).$$

Assume we have built a lattice to value W_0. Then at each node (t_k, U_i) we must add to the solution the two "cashflows" due to hedging the term structure. In the case where we have an analytic solution for W_0, we can compute an explicit expression for these additional hedging profits and treat them as small "coupons" in the lattice. When we don't have an analytic expression we will have to construct a **lattice sandwich** consisting of the original lattice in the center and **up** and **down** lattices run with the following term structures,

$$\vec{h}^{\pm} = \{h_i \pm \kappa_i \times \delta\} \qquad 0 \leq i < N,$$

where to preserve time homogeneity we require,

$$\kappa_i = \kappa_0 e^{-\alpha \tau_i}.$$

The correction terms become,

$$\Delta W_0(t_k, U_i) = \frac{1}{2}\kappa_0^2 h_0^2 \frac{\partial^2 W_0}{\partial h_0^2}(t_k, U_i)\Delta t + \rho\sigma U \kappa_0 h_0 \frac{\partial^2 W_0}{\partial U \partial h_0}(t_k, U_i)\Delta t.$$

The second derivative in the first term can be computed from the values at the node, the node directly below, and the node directly above. While the cross-partial derivative in the second term also requires the nodes to the right and the left. In order to continue matching the liquid derivatives which we calibrated to we must **renormalize** the initial static term structure.

Exercise 26.1: Future Contract
Recall that for deterministic interest rates the future price equals the forward price,

$$\hat{F}_0 = F = \exp\left(\sum_{i=0}^{N-1} f_i \Delta t\right) U.$$

For the following choices,

$$
\begin{aligned}
T &= 1yr, \\
\Delta t &= \frac{1}{12}yr \Longrightarrow N = 12 \text{ steps}, \\
f_i &= 0.06 \quad 0 \le i < N-1, \\
U &= 50, \\
\sigma &= 0.30, \\
\kappa_0 &= 0.20, \\
\alpha &= 0.05, \\
\rho &= -0.50,
\end{aligned}
$$

compute the dynamic future price $\hat{F}$.

27 Volatility Smile

When we imply volatilities of options expiring at T from market prices we often find that they exhibit a **strike structure** or **smile**,

$$BS(t, U, \sigma, K, T) = W^{\text{mkt}}(K, T) \Longrightarrow \sigma = \sigma(K, T).$$

The existence of a strike dependent volatility demonstrates that the Black-Scholes methodology is inconsistent with the market. This is due to the fact that Black-Scholes assumes that volatility is a deterministic function of time. This assumption breaks down for the following two reasons. First, implied volatility is in general stochastic and correlated with the underlying. Therefore, eliminating vega risk leads to additional hedging profits. Second, because implied volatilities cannot accurately predict realized volatilities we conclude that the current tstate of volatility is unknown. Contrast this with the analogous interest rate instrument, the term deposit, which allows investors to earn known interest rate. This uncertainty requires us to integrate our solution over a distribution of initial volatility curves.

We begin by analyzing the effect of a stochastic implied volatility. The additional hedging profits at t_0 due to changing vega are given by,

$$\begin{aligned} (\Delta W)_{\text{Smile}}(t_k) &= \underbrace{\frac{1}{2}\sigma_0^2\kappa_0^2\frac{\partial^2 W}{\partial \sigma_0}(t_k)}_{\text{Kurtosis}}\,\Delta t \\ &+ \underbrace{\rho_{U,\sigma}(\sigma_0 U)(\kappa_0\sigma_0)\frac{\partial^2 W}{\partial U \partial \sigma_0}(t_k)\,\Delta t}_{\text{Skew}}. \end{aligned}$$

where we have assumed that the underlying and implied volatility obey by the following coupled dynamics,

$$\begin{aligned} \frac{\Delta U}{U} &= \mu \Delta t + \sigma_0 \Delta \tilde{w}_U, \\ \frac{\Delta \sigma_i}{\sigma_i} &= \lambda_i \Delta t + \kappa_i \Delta \tilde{w}_\sigma, \end{aligned}$$

and the correlation between the underlying and the volatility is given by,

$$\Delta \tilde{w}_U \Delta \tilde{w}_\sigma = \rho_{U,\sigma} \Delta t.$$

The vega hedging term is best illustrated by the following example where we hedge an OTM strangle,

$$(W)_{\text{Strangle}} = P(t, U; K_P < F_U, T) + C(t, U; K_C > F_U, T),$$

which probes the kurtosis with a liquid ATM straddle

$$(W)_{\text{Straddle}} = P(t, U; K_0 = F_U, T) + C(t, U; K_0 = F_U, T),$$

which probes the variance. Assume also that we have chosen the strikes of the strangle so that it is delta neutral like the straddle. To eliminate our vega risk we need to sell just enough straddles to cancel the vega of the strangle,

$$\mathcal{P}(t_0) = (W)_{\text{Strangle}} + \alpha\,(W)_{\text{Straddle}}$$

Solving for the initial hedge ratio,

$$\frac{\partial \mathcal{P}}{\partial \sigma_0}(t_0) = 0 \Longrightarrow \alpha = -\frac{\partial\,(W)_{\text{Strangle}}}{\partial \sigma_0}(t_0) / \frac{\partial\,(W)_{\text{Straddle}}}{\partial \sigma_0}(t_0).$$

As volatility changes we will need to readjust our vega hedge. However, according to Black-Scholes an ATM forward option is linear in volatility so that the straddle has a locally constant vega. This is directly analogous to hedging a stock option where the stock is linear in the underlying while the option is nonlinear.

The change in the vega of the strangle is given by,

$$\Delta\text{Vega} = \underbrace{\frac{\partial}{\partial \sigma_0}\frac{\partial\,(W)_{\text{Strangle}}}{\partial \sigma_0}(t_0)}_{\Gamma_\sigma}\,\Delta\sigma_0.$$

Since $\Gamma_\sigma > 0$ an increase in volatility means we will have to sell more ATM straddles. But since the volatility just went up we can sell them at a higher price. Conversely, a decrease in volatility means we can buy back straddles a lower price. This leads to positive hedging profits just as in the case of hedging the option with the stock. This causes OTM to trade with a higher implied volatility than ATM options and gives rise to the so called volatility smile. These vega hedging profits get imbedded in the implied volatility because the BS methodology doesn't naturally incorporate them.

We now turn our attention to the delta hedging term which is best illustrated by hedging an OTM risk reversal,

$$(W)_{\text{Risk Reversal}} = P(t, U; K_P < F_U, T) - C(t, U; K_C > F_U, T),$$

which probes the skew with an ATM reversal,

$$(W)_{\text{Reversal}} = \underbrace{P(t, U; K_0 = F_U, T) - C(t, U; K_0 = F_U, T)}_{\text{Short Synthetic Forward}},$$

which probes the mean of the distribution. Here we choose the strikes of the risk reversal to eliminate the risk as is already the case for the ATM reversal. We sell just enough of the reversals to hedge the delta risk of the risk reversal,

$$\mathcal{P}(t_0) = (W)_{\text{Risk Reversal}} + \alpha\, (W)_{\text{Reversal}}.$$

$$\Longrightarrow \alpha = -\frac{\partial\, (W)_{\text{Risk Reversal}}}{\partial U}(t_0) / \frac{\partial\, (W)_{\text{Reversal}}}{\partial U}(t_0).$$

As volatility changes the delta of our risk reversal also changes according to the following cross-partial derivative,

$$\Delta\text{Delta} = \underbrace{\frac{\partial}{\partial \sigma_0} \frac{\partial\, (W)_{\text{Risk Reversal}}}{\partial U}}_{\Gamma_{U,\sigma}} \Delta\sigma_0.$$

Since $\Gamma_{U,\sigma} < 0$, an increase in volatility requires buying more synthetic forwards. If we assume that $\rho < 0$ then this on average is accompanied by a decrease in stock price the opportunity to buy at a low price. On the other hand, if volatility decreases then we can short expensive forwards in an expectational sense. These enhancements to the regular delta hedging profits make the risk reversal more valuable,

$$(W)_{\text{Risk Reversal}} > 0 \Longrightarrow P(t, U; K_P < F_U, T) > C(t, U; K_C > F_U, T),$$

so that OTM puts are more valuable than OTM calls. The fact that we must pay a premium for downside protection stems from our assumption that $\rho < 0$ which means that large down moves are more likely than large up moves. Or equivalently, "Nothing travels faster than bad news".

Next, we examine the impact of volatility uncertainty on the smile. The implied volatility of an option is simply the risk-adjusted market consensus of what volatility will be over its lifetime. Therefore, there is no guarantee that the realized volatility will equal the implied volatility, or equivalently, that the gamma hedging profits will equal the option premium. The analogy between forward interest rates and forward implied volatilities breaks down here because in the former case we can lock in a known return.

Consider an OTM call option with one day to expiration. If we value at the ATM implied vol we find that it has virtually worthless. However, if we assign a non-zero probability to much higher volatilities we discover that the option now has a finite value. This is equivalent to observing that OTM options are convex functions of volatility so that an increase in volatility raises the option price more than a decrease in volatility drops it. Therefore, if we integrate OTM over the possible volatility states we expect to obtain a higher option price. This result is given by **Jensen's inequality**,

$$\mathbf{E}\left[f\left(\tilde{\omega}\right)\right] > f(\bar{\omega})$$

where, $f(\omega)$ is a convex function.

The amount of volatility uncertainty is a strong function of the time to expiration of the option. The fact that volatility tends to mean revert to long term levels means that volatility is more predictable over the long term than over the short term. Consequently, long term implied vols have more predictive power than short term vols. This leads to a pronounced smile for short dated options that cannot be explained by the stochastic diffusion of volatility alone.

Assume that we can represent all possible initial volatility states by the following one-parameter family,

$$\vec{\sigma}'(\delta) = \vec{\sigma} + \vec{\rho} \times \tilde{\delta}$$

where $\vec{\rho}$ is the shape vector,

$$\vec{\rho} = \{\rho_i\} \quad 0 \leq i < N \quad \text{with normalization } \rho_0 = 1$$

which according to the above discussion will be monotonically decreasing. Now let the parameter $\tilde{\delta}$ be normally distributed with mean zero and variance v,

$$\tilde{\delta} \sim \mathcal{N}(0, v).$$

To compute the option price we follow the Bayesian approach by taking the risk-neutral expectation of the option price over the distribution of initial volatility states,

$$\begin{aligned} W(t, U; \vec{\sigma}, K, T) &= \hat{\mathbf{E}}\left[W_0(\vec{\sigma}')\right] \\ &= \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} \exp\left(-\frac{\delta^2}{2v}\right) W_0\left(t, U; \vec{\sigma}'(\delta), K, T\right) d\delta \end{aligned}$$

which is calibrated to the initial set of liquid options by choosing $\vec{\sigma}$ appropriately. The shape $\vec{\rho}$ and variance v can be estimated by observing the empirical variance of the realized volatilities over various terms.

In practice we approximate the distribution of initial volatility with a discrete binomial that matches the mean and variance. Define the **up** and **down** volatility states,

$$\vec{\sigma}^{\pm} = \{\sigma_i \pm \rho_i \times \sqrt{v}\} \qquad 0 \leq i < N.$$

The adjusted option price is obtained by taking the binomial expectation over these two states,

$$W(t, U; \vec{\sigma}, K, T) = p_- W_0\left(t, U; \vec{\sigma}^-, K, T\right) + p_+ W_0\left(t, U; \vec{\sigma}^+, K, T\right),$$

where $p_\pm = 0.5$ and $\vec{\sigma}$ is calibrated to the liquid market prices.

Exercise 27.1: Assume the volatility term structure $\vec{\sigma}$ and shape vector $\vec{\rho}$ are constant and value the following equity call option and compute its implied volatility,

$$\begin{aligned} T &= 0.25yrs \\ S &= 50 \\ K &= 60 \\ \sigma_{ATM} &= 0.30 \\ D_T &= 0.985 \\ v &= 0.0225 \end{aligned}$$

28 Yield Dynamics

The fact the bond price is a nonlinear function of yield means that there will be profits associated with hedging the yield risk of coupon bonds just as there is with options. Since bonds have different amounts of convexity we might expect there to be a way to profit by going long a bond IP_1 with "high" convexity and short a bond IP_2 with "low" convexity in a duration neutral portfolio,

$$\mathcal{P} = IP_1(t, y_1; C_1, T_1) + \alpha IP_2(t, y_2; C_2, T_2),$$

where $IP_{1.2}$ are the invoice prices of the respective bonds,

$$IP_{1,2} = B_{1,2}(t, y_{1,2}; C_{1,2}, T_{1,2}) + AI_{1.2}.$$

Now assume their yields are perfectly correlated and obey the following log-normal dynamics,

$$\frac{\Delta y_{1,2}}{y_{1,2}} = \mu_{1,2}\Delta t + \sigma_{1,2}(t)\Delta\tilde{w}_t.$$

The principle of no-arbitrage will tell us that it is not possible to take advantage of the convexity differential because the yields of the bonds will drift with respect to each other in a way which prevents any certain profit,

$$\Delta\mathcal{P} = \Delta IP_1 + \alpha\Delta IP_2 = r\mathcal{P}\Delta t.$$

According to Ito's lemma,

$$\begin{aligned}\frac{\partial IP_1}{\partial t}\Delta t &+ \frac{\partial IP_1}{\partial y_1}\Delta y_1 + \frac{1}{2}\sigma_1^2 y_1^2 \frac{\partial^2 IP_1}{\partial y_1^2}\Delta t \\ &+ \alpha\left(\frac{\partial IP_2}{\partial t}\Delta t + \frac{\partial IP_2}{\partial y_2}\Delta y_2 + \frac{1}{2}\sigma_2^2 y_2^2 \frac{\partial^2 IP_2}{\partial y_2^2}\Delta t\right) = r\left(IP_1 + \alpha IP_2\right)\Delta t.\end{aligned}$$

Taking the derivatives,

$$\begin{aligned}\frac{\partial IP_{1,2}}{\partial t} &= \frac{\partial B_{1,2}}{\partial t} + C_{1,2}, \\ \frac{\partial IP_{1,2}}{\partial y_{1,2}} &= \frac{\partial B_{1,2}}{\partial y_{1,2}}, \\ \frac{\partial^2 IP_{1,2}}{\partial y_{1,2}^2} &= \frac{\partial^2 B_{1,2}}{\partial y_{1,2}^2},\end{aligned}$$

which leads to the equation,

$$\begin{aligned}\left(\frac{\partial B_1}{\partial t}+C_1\right)\Delta t &+ \frac{\partial B_1}{\partial y_1}\Delta y_1+\frac{1}{2}\sigma_1^2y_1^2\frac{\partial^2 B_1}{\partial y_1^2}\Delta t\\ &+ \alpha\left[\left(\frac{\partial B_2}{\partial t}+C_2\right)\Delta t+\frac{\partial B_2}{\partial y_2}\Delta y_2+\frac{1}{2}\sigma_2^2y_2^2\frac{\partial^2 B_2}{\partial y_2^2}\Delta t\right]\\ &= (RP_1\times IP_1+\alpha RP_2\times IP_2)\,\Delta t.\end{aligned}$$

We choose the hedge ratio to make the portfolio riskless,

$$\left(\frac{\partial IP_1}{\partial y_1}\sigma_1+\alpha\frac{\partial IP_2}{\partial y_2}\sigma_2\right)\Delta\tilde{w}_t=0,\Longrightarrow \alpha=-\sigma_1y_1\frac{\partial B_1}{\partial y_1}/\left(\sigma_2y_2\frac{\partial B_2}{\partial y_2}\right).$$

Substituting above and separating terms,

$$\begin{aligned}&\left(\frac{\partial B_1}{\partial t}+C_1-RP_1\times IP_1+y_1\mu_1+\frac{1}{2}\sigma_1^2y_1^2\frac{\partial^2 B_1}{\partial y_1^2}\right)/\left(\sigma_1y_1\frac{\partial B_1}{\partial y_1}\right)\\ =\;&\left(\frac{\partial B_2}{\partial t}+C_2-RP_2\times IP_2+y_2\mu_2+\frac{1}{2}\sigma_2^2y_2^2\frac{\partial^2 B_2}{\partial y_2^2}\right)/\left(\sigma_2y_2\frac{\partial B_2}{\partial y_2}\right).\end{aligned}$$

Solving for the drift constraint,

$$\begin{aligned}\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2} &= \left(\frac{\partial B_1}{\partial t}+C_1-RP_1\times IP_1+\frac{1}{2}\sigma_1^2y_1^2\frac{\partial^2 B_1}{\partial y_1^2}\right)/\left(\sigma_1y_1\frac{\partial B_1}{\partial y_1}\right)\\ &- \left(\frac{\partial B_2}{\partial t}+C_2-RP_2\times IP_2+\frac{1}{2}\sigma_2^2y_2^2\frac{\partial^2 B_2}{\partial y_2^2}\right)/\left(\sigma_2y_2\frac{\partial B_2}{\partial y_2}\right).\end{aligned}$$

It is important to point out that although this drift constraint was derived for the real-world drifts, it holds for all equivalent measures.

To derive the risk-neutral drift we apply Ito's lemma to the invoice price of the bond and set the drift equal to the repo rate times the price,

$$\frac{\Delta IP}{IP}=\left(\frac{\partial B}{\partial t}+C\right)\Delta t+\frac{\partial B}{\partial y}\left(y\hat{\mu}\Delta t+\sigma\Delta\tilde{w}_t\right)+\frac{1}{2}\sigma^2y^2\frac{\partial^2 B}{\partial y^2}\Delta t=(RP\times IP)\,\Delta t.$$

$$\Longrightarrow \hat{\mu}(t,y)=\left(\frac{\partial B}{\partial t}+C-RP\times IP+\frac{1}{2}\sigma^2y^2\frac{\partial^2 B}{\partial y^2}\right)/\left(y\frac{\partial B}{\partial y}\right).$$

Setting the carry terms to zero, we see that the risk-neutral drift is proportional to the convexity per unit duration of the bond,

$$\text{Drift} \sim \frac{\text{Convexity}}{\text{Duration}}.$$

The fact that bonds are nonlinear functions of yield means that hedging duration results in convexity profits just as one does hedging the delta of an option. However, in the option case we paid an option premium for the privledge of earning those profits, while in the bond case we only contracted to deposit our money at the forward rates. Hence, we expect the yield of each bond to drift in a way which exactly cancels these profits in a risk-neutral world.

<u>Exercise 28.1</u>:
For a prescribed change Δy_1 in the yield of B_1, derive the change Δy_2 in the yield of bond B_2 which prevents arbitrage.

We can write down the equation governing an arbitrary derivative expressed as a function of yield y,

$$\frac{\partial W}{\partial t} + \hat{\mu}(t,y)\frac{\partial W}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 W}{\partial y^2} - rW(t,y;C,T) = 0,$$

subject to the appropriate initial and boundary conditions. For example, to value an option on a bond we must satisfy the following condition at expiration,

$$\begin{aligned} W(T,y) &= \max\left[\left(B(T,y;T_{mat},C) - K\right),0\right] \quad \text{Call} \\ &= \max\left[\left(K - B(T,y;T_{mat},C)\right),0\right] \quad \text{Put} \end{aligned}$$

In the American case we must also satisfy the parity condition,

$$\begin{aligned} W(t,y) &= \max\left[W(t,y),(B(t,y;T_{mat},C) - K)\right] \quad \text{Call} \\ &= \max\left[W(t,y),(B(t,y;T_{mat},C) - K)\right] \quad \text{Put} \end{aligned}$$

29 Interest Rate Derivatives

Interest rate derivatives have the forward curve as underlying,

$$W = W(t, \vec{f})$$

We have already encountered the following interest rate derivatives:

- **Money Market Accounts**
- **Term Deposits**
- **Forward Rate Agreements**
- **Coupon Bonds**
- **Bond Options**

Here are some additional examples:

Eurodollar Futures

An **ED future** $\hat{\mathcal{F}}_n$ is an exchange traded future contract on a 3m Libor deposit for the tenor period, (τ_n, τ_{n+1}). It is traded on the International Monetary Market (IMM) at the Chicago Mercantile Exchange. The complete set of contracts out to 5 years is called the **strip**. ED futures are used primarily to hedge caps and swaps.

To make it resemble a bond its price is quoted as,

$$\hat{F}_n = 100 - \hat{f}_n,$$

where, $\hat{f}_n$ is the **future rate** which is marked to spot 3m Libor at maturity. The fact that $\hat{f}_n$ is positively correlated with the over-night rate means that the margin account of the long will positively marked when rates are low and negatively marked when rates are high. The long is compensated for this profit "leakage" by lower future prices which leads to future rates being higher than the equivalent forward rates,

$$\hat{f}_n > f_n.$$

The difference between the future and forward is referred to as the **future-forward bias**. Since the future rate is a martingale we can value it as the risk-neutral expectation of the forward rate at maturity,

$$\hat{f}_n = \hat{\mathbf{E}}\left[\tilde{f}_0(\tau_n)\right],$$

where the dynamics of the forward rate are described in the upcoming sections.

Floating Rate Notes

A **floating rate note** FRN is a bond with variable coupons $\tilde{C}$ based on a floating index. Consider the following the time line,

$$\tau_0 \leq t < \tau_1 < \tau_2 \cdots \tau_i \cdots < \tau_n,$$

where the tenor periods are given by,

$$\Delta\tau_i = \tau_{i+1} - \tau_i.$$

In the **plain vanilla** case coupon for the tenor period (τ_n, τ_{n+1}) is set by the spot deposit rate at time τ_n and paid at time τ_{n+1},

$$\tilde{C}_{n+1} = r_0(\tau_n)\Delta\tau_n.$$

We say that the coupon is **set upfront** and **paid in arrears**. Since this rule is consistent with the corresponding FRA's we replace the floating rate with the fixed forward rate,

$$\tilde{r_0}(\tau_n) \Longrightarrow f_n.$$

The invoice price of the floating rate note can now be expressed in terms of static coupons given by the forward rates,

$$\begin{aligned}(FRN)_{IP} &= D_1 r_0(t_0)\Delta\tau_0 \\ &+ \sum_{n=2}^{N} D_n f_{n-1}\Delta\tau_{n-1} + D_N.\end{aligned}$$

Since the discount bonds are given by,

$$D_n = [1 + f_0(\tau_1 - t)]^{-1}\prod_{i=1}^{n-1}(1 + f_i\Delta\tau_i)^{-1},$$

the floating rate note becomes,

$$\begin{aligned}
(FRN)_{IP} &= \frac{r_0(t_0)\Delta\tau_0}{1+f_0(\tau_1-t)} + \frac{f_1\Delta\tau_1}{[1+f_0(\tau_1-t)](1+f_1\Delta\tau_1)}, \\
&+ \cdots + \frac{f_{N-2}\Delta\tau_{N-2}}{[1+f_0(\tau_1-t)](1+f_1\Delta\tau_1)\cdots(1+f_{N-2}\Delta\tau_{N-2})}, \\
&+ \frac{1+f_{N-1}\Delta\tau_{N-1}}{[1+f_0(\tau_1-t)](1+f_1\Delta\tau_1)\cdots(1+f_{N-1}\Delta\tau_{n-1})}, \\
&= \frac{1+r_0(t_0)\Delta\tau_0}{1+f_0(\tau_1-t)}.
\end{aligned}$$

The accrued interest is given by,

$$AI = r_0(t_0) \times (t-\tau_0),$$

and the quoted price becomes,

$$(FRN)_Q = (FRN)_{IP} - AI = \frac{1+r_0(t_0)\Delta\tau_0}{1+f_0(\tau_1-t)} - r_0(t_0)(t-t_0).$$

When t is a reset date we find that,

$$(FRN)_{IP} = (FRN)_Q = 1.$$

The fact that a floating rate note is par on reset dates is due the fact that it has exactly the same payoff as the strategy of investing par at the spot Libor rate and then succesively rolling it over until maturity T. Floating rate notes which have a coupon that is **set in arrears** or which have a **mismatch** between the tenor of the index and the note will in general not be valued at a par on reset dates.

Interest Rate Swaps

An **interest rate swap** S is a over-the counter contract where party A pays party B a floating coupon $\tilde{C}$ and party B pays a fixed coupon C_0 until maturity with no exchange of principal. We call A the (fixed) **receiver** and B the fixed **payer**. Interest rate swaps are used to convert fixed (floating) cashflows into floating (fixed) cashflows. For example, a corporation needing to raise capital may wish to have floating rate obligation to match its assets, but it finds it has a comparative advantage issuing fixed rate debt. The floating liability can be constructed by issuing fixed debt and swapping it into floating via a payer swap. This strategy will presumably lead to a lower floating payment than if the corporation had tried to issued floating direct directly.

In the **plain vanilla** swap, the **floating leg** is 3m Libor set upfront and paid every 3m on an Act/360 daycount basis while the **fixed leg** is paid every 6m on a 30/360 daycount basis. Swaps in which the floating leg is set at the end of the tenor period are known as **arrears swaps**.

We can model swap S as the difference between a coupon bond $B(C_0, T)$ and a floating rate note FRN. From the reciever's perspective we have,

$$S = B\left(C_0, T\right) - FRN.$$

An **at market** interest rate swap has a value of zero,

$$S_{\text{ATM}} = B\left(C_S, T\right) - FRN = 0 \Longrightarrow B\left(C_S, T\right) = FRN,$$

where C_S is called the **swap rate**. In the case of a plain vanilla swap on a reset date the floating rate note is worth par and we have,

$$B\left(C_S, T\right) = 1,$$

so that the swap rate is simply the par coupon.

Bond Futures

The **bond future** $\hat{F}$ is a future contract at the Chicago Board of Trade (CBOT) on a hypothetical Treasury bond. The is the principal vehicle for hedging long term interest rate risk and as a consquence is the most liquid future contract. The bond future gives the short the option of delivering any Treasury bond with greater than 15 years to maturity for the invoice amount,

$$(IP)_i = k_i \times \hat{F} + (AI)_i,$$

where the **factor** k_i for bond B_i is defined by,

$$k_i = \frac{B(t, y = 0.08; C_i, t + T_i)}{B(t, y = 0.08; C = 0.08, T = t + 20)}.$$

Therefore the cost of delivering bond B_i is,

$$(Cost)_i = k_i \times \hat{F} - B_i.$$

The factors were designed to make all bonds equally attractive to deliver for a flat 8% yield curve. However, for general yield curve shapes the delivery cost will depend on the particular bond B_i. The short will always deliver the **cheapest to deliver** bond which to prevent arbitrage must lead to a zero cost of delivery,

$$Cost\,(B_{CTD}) = k_{CTD} \times \hat{F} - B_{CTD} = 0.$$

This leads to the following initial condition for the future price,

$$\hat{F} = \frac{B_{CTD}}{k_{CTD}} = \min_{i \in I} \frac{B_i}{k_i}.$$

Since the future price is a martingale we can compute it as the risk-neutral expectation of its value at maturity,

$$\hat{F}(t) = \hat{\mathbf{E}}\left[\hat{F}(T)\right] = \hat{\mathbf{E}}\left[\min_{i \in I}\left(\frac{B_i}{k_i}\right)\right].$$

We can think of the contract as a future on the CTD bond with an OTM option to switch to the other deliverable bonds.

Caps and Floors

Investors with floating rate obligations purchase **caps** to protect against rates going too high, while those with floating rate assets buy **floors** to protect against rates going too low. Caps are comprised of a quarterly series of call options on forward rates called **caplets** which expire when the forwards become spot. Similarly, floors are portfolios quarterly **floorlets** which are puts on forward rates. The caplets (floorlets) are generally all struck at the same rate K. The **ATM cap** (floor) has a strike equal to the 3 month forward starting swap rate. This means that in an ATM cap some of the caplets are ITM and others are OTM.

$$Cap = \sum_{i=1}^{N} Caplet(t, f_i; K, T_i),$$

where the caplet payoff at expiration is,

$$Caplet(T_i, f_i; K, T_i) = \max\left(\frac{f_i \Delta\tau_i}{1 + f_i \Delta\tau_i}, 0\right).$$

There is an interesting analogy between caps (floors) which are comprised of caplets (floorlets) and coupon bonds which are portfolios of discount bonds. Black's formula which is used to value the individual caplets is analogous to the discount bond pricing equation. In general each caplet is valued on its own volatility, just as each coupon is discounted at its own rate. However, there exists a unique volatility called the **cap vol** which when used to value each of the caplets correctly prices the cap. It is naturally analogous to the concept of a bond yield.

ED Future Options

The International Monetary Market (IMM) also lists options on the ED future contract. These **ED future options** are American exercise and expire on the maturity of the underlying future. They are similar to caplets (floorlets) but differ from them in the following four ways. First, they are traded on an exchange instead of over-the-counter. Second, they are traded separately unlike caplets which trade as part of a cap. Third, they are American exercise and caplets are European. Fourth, their underlying is the future contract rather than the forward rate.

One feature ED future options do share with caplets is that they expire when the rate becomes spot. This means they both probe the volatility of the rate over its entire lifetime. This makes it difficult to ascertain the instantaneous volatility of the forward rates. To circumvent this problem the exchange introduced **mid curves** which are short dated options on future rates. These allows us to partially observe the instantaneous shape of the forward rate volatilities.

European Swaptions

A **European swaption** is an option to enter into an interest rate swap at expiration. Since a swap can be modelled as long a bond and short a floating rate notE worth par,

$$S = B(C_0, T) - \underbrace{FRN}_{Par=1},$$

we can value the swaption as an option on a coupon bond struck at par. We will price European swaptions using both our dynamic forward rate model which assumes that each forward rate is lognormal and a Black model which assumes that the forward swap rate is lognormal and show that they agree to leading order.

One of the major research efforts in the interest rate derivative area is to understand the relationship between caps and swaptions. Their values are ultimately connected because they are both options on the forward rate curve. The distinction between caps and swaptions can be summarized as follows. A cap is a portfolio of options (caplets) while a swaption is an option on a portfolio (discount bonds). The prospect of taking advantage of the relative mispricing of caps ans swaptions is called **cap-swaption arbitrage**.

Cancellable Swaps

Interest rate swaps where one of the counterparties has the right to cancel on cashflow dates after a **lockout** period are called **cancellable swaps**. The off-market coupon which makes the cancellable swap worth zero is called the **breakeven**. One can think of a cancellable swap as a combination of an interest rate swap and a **Bermudan swaption**. If the fixed payer owns the right to cancel he has a receiver swaption, while if the fixed reciever has the right to cancel he holds a payer swaption. Bermudan swaptions can be thought of as European swaptions expiring on the lockout date plus an option to extend.

30 Forward Rate Model

The **Forward Rate Model** (FRM) prescribes the dynamics of the discrete forward rate curve. This is in contrast to models which are based on the dynamics of bond prices, bond yields, or spot interest rates.

We assume that the forward rates follow a lognormal Brownian motion with drift to be consistent with the market convention for valuing caps. The path-dependent evolution of the forward rate curve requires us to employ perturbation theory in order to implement the model in a recombining lattice.

The derivation of the model will be carried out for the one-factor case where all the forward rates are perfectly correlated. We will then show how the model can be easily extended to two or more factors.

We assume the n^{th} forward rate obeys the following one-factor log-normal diffusion equation,

$$\frac{\Delta f_n(t)}{f_n(t)} = \mu_n(t)\Delta t + \sigma_n(t)\Delta\tilde{w}.$$

This implies perfect correlation among forward rates,

$$\rho_{m,n} = 1 \qquad 0 \le m, n < N.$$

For convenience we prefer to work with log-forward rates r_n so that the dynamical equation is normal,

$$r_n \equiv \log f_n,$$

$$\Delta r_n = \hat{\mu}_n \Delta t + \sigma_n \Delta\tilde{w},$$

$$\text{where,} \quad \hat{\mu}_n = \mu_n - \frac{1}{2}\sigma_n^2.$$

The dynamics of r_n are therefore completely specified in terms the drift $\mu_n(t)$ and the volatility $\sigma_n(t)$.

Unlike the spot rate model, all the inputs to FRM have a simple physical interpretation. However, they are not directly observable and must be calibrated to the market. The forward rate curve $\vec{f}$ is estimated by matching the market prices of interest rate instruments such as CD's, Eurodollar futures, and swaps in as smooth a way as possible.

The **volatility matrix** σ is defined by,

$$\sigma_{k,n} \equiv \sigma_n(t_k) \qquad \forall k, n$$

Hence, $\sigma_{k,n}$ gives the volatility of the n^{th} forward rate at time t_k. In the "Volatility Calibration" section we describe how the matrix is chosen to match market prices.

Once the volatility matrix has been specified the no-arbitrage condition requires the drifts between successive forward rates to satisfy a constraint relation.

$$\mu_n(t_k) - \frac{\sigma_n(t_k)}{\sigma_{n-1}(t_k)}\mu_{n-1}(t_k) = h_n(t_k, f_n)$$

The absolute scale of the drifts is then set by choosing the risk-neutral measure. This procedure is described in **Sections 32** and **34** entitled "Drift Constraint" and "Risk-Neutral Measure" respectively.

Finally, we need to specify the correlation between all pairs of forward rates f_i and f_j expressed by the **correlation matrix**,

$$\rho = \{\rho_{m,n}\} \;\; \forall m, n$$

where,

$$\rho_{m,n} \equiv \frac{\mathbf{E}\left[\left(f_m - \bar{f}_m\right)\left(f_n - \bar{f}_n\right)\right]}{\sqrt{v_m}\sqrt{v_n}}.$$

We will estimate the correlations from historical data. In the 2-factor model we will specify the correlation $\rho_{0,n}$ between the spot rate f_0 and each of the forward rates f_n.

31 Propagation Equation

In this section we derive how a shock in the spot rate propagates along the forward curve. This will enable us to evolve the forward curve given the path $\mathcal{P}$ followed by the spot rate.

The change in r_{n-1} is given by,

$$\Delta r_{n-1} = \hat{\mu}_{n-1}\Delta t + \sigma_{n-1}\Delta\tilde{w}.$$

Solving for the stochastic shock $\Delta\tilde{w}$,

$$\Delta\tilde{w} = \frac{\Delta r_{n-1} - \hat{\mu}_{n-1}\Delta t}{\sigma_{n-1}}.$$

Similarly the change in r_n is given by,

$$\Delta r_n = \hat{\mu}_n\Delta t + \sigma_n\Delta\tilde{w}.$$

Substituting the expression for $\Delta\tilde{w}$ into this equation,

$$\Delta r_n = \hat{\mu}_n\Delta t + \sigma_n\left(\frac{\Delta r_{n-1} - \hat{\mu}_{n-1}\Delta t}{\sigma_{n-1}}\right).$$

Collecting terms leads to the following relationship between the changes in adjacent forward rates,

$$\Delta r_n = \frac{\sigma_n}{\sigma_{n-1}}\Delta r_{n-1} + \left(\hat{\mu}_n - \frac{\sigma_n}{\sigma_{n-1}}\hat{\mu}_{n-1}\right)\Delta t.$$

Now define the **drift constraint** function h_n by,

$$h_n \equiv \hat{\mu}_n - \frac{\sigma_n}{\sigma_{n-1}}\hat{\mu}_{n-1}.$$

(In **Section 32** we show that in the lognormal case h_n depends only on the n^{th} log-forward rate r_n).

The local evolution equation can now be written in the form,

$$\Delta r_n = \frac{\sigma_n}{\sigma_{n-1}}\Delta r_{n-1} + h_n(f_n)\Delta t.$$

This equation has a nice physical interpretation. The first term is $O\left(\sqrt{\Delta t}\right)$ and represents the change in shape of the curve caused by the forward volatility term structure. The second term is only $O\left(\Delta t\right)$ and is due to the relative drift constraint required to prevent arbitrage between adjacent forward rate agreements (FRA's).

We can now proceed inductively to derive the global evolution equation relating the change in r_n to the change in the first rate r_1,

$$\Delta r_n = \underbrace{\frac{\sigma_n}{\sigma_1}\Delta r_1}_{O\left(\sqrt{\Delta t}\right)} + \underbrace{\sum_{j=2}^{n} \frac{\sigma_n}{\sigma_j} h_j(r_j)\Delta t}_{O(\Delta t)}.$$

Now define separately the change due to the shape and drift terms,

$$\begin{aligned} \Delta^{\sigma} &= \frac{\sigma_n}{\sigma_1}\Delta r_1, \\ \Delta^{\mu} &= \sum_{j=2}^{n} \frac{\sigma_n}{\sigma_j} h_j(r_j)\Delta t. \end{aligned}$$

32 Drift Constraint

Construct a riskless self-financing portfolio $\mathcal{P}$,

$$\mathcal{P} = \underbrace{\mathcal{F}_{n-1} + \alpha \mathcal{F}_n}_{\text{Adjacent FRA's}} + \underbrace{\beta D_n}_{\text{Financing Bond}} = 0,$$

where the adjacent FRA's $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$ are given by,

$$\begin{aligned} \mathcal{F}_{n-1} &= D_n \left(K_{n-1} - e^{r_{n-1}}\right) \Delta\tau_{n-1}, \\ \mathcal{F}_n &= D_n \frac{\left(K_n - e^{r_n}\right) \Delta\tau_n}{1 + e^{r_n} \Delta\tau_n}. \end{aligned}$$

Since the portfolio is both riskless and self-financing an arbitrage opportunity would be created unless its valued remained locally unchanged,

$$\Delta \mathcal{P} = 0.$$

To eliminate the dependence of the portfolio $\mathcal{P}$ on all forward rates except r_{n-1} and r_n perform a change of numeraire to value the portfolio in terms of the n period discount bond D_n,

$$\mathcal{P}^* \equiv \frac{\mathcal{P}}{D_n} = \mathcal{F}^*_{n-1} + \alpha \mathcal{F}^*_n + \beta = 0.$$

Since the choice of numeraire is arbitrary, no arbitrage requires that the value of $\mathcal{P}^*$ also remain unchanged,

$$\Delta \mathcal{P}^* = 0.$$

Exercise 32.1:
Apply Ito's lemma to prove that the local stationarity of the portfolios $\mathcal{P}$ and $\mathcal{P}^*$ are equivalent,

$$\Delta \mathcal{P} = 0 \iff \Delta \mathcal{P}^* = 0.$$

Recall that the log-forward rate r_n obeys the normal diffusion equation,

$$\Delta r_n = \hat{\mu}_n \Delta t + \sigma_n \Delta \tilde{w}.$$

The no-arbitrage condition becomes,

$$\text{No-Arbitrage} \Longrightarrow \Delta\mathcal{P}^* = \Delta\mathcal{F}^*_{n-1} + \alpha\Delta\mathcal{F}^*_n = 0.$$

Now apply Ito's Lemma,

$$\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}}\Delta r_{n-1} + \frac{1}{2}\sigma^2_{n-1}\frac{\partial^2\mathcal{F}^*_{n-1}}{\partial r^2_{n-1}}\Delta t + \alpha\Big[\frac{\partial\mathcal{F}^*_n}{\partial r_n}\Delta r_n + \frac{1}{2}\sigma^2_n\frac{\partial^2\mathcal{F}^*_n}{\partial r^2_n}\Delta t\Big] = 0.$$

Apply the riskless conditon to solve for the hedge ratio α,

$$\left(\sigma_{n-1}\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}} + \alpha\sigma_n\frac{\partial\mathcal{F}^*_n}{\partial r_n}\right)\Delta\tilde{w} = 0$$

$$\Longrightarrow \alpha = -\sigma_{n-1}\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}}/\sigma_n\frac{\partial\mathcal{F}^*_n}{\partial r_n}.$$

Substituting above,

$$\left(\sigma_{n-1}\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}}\right)^{-1}\left[\hat{\mu}_{n-1}\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}} + \frac{1}{2}\sigma^2_{n-1}\frac{\partial^2 F^*_{n-1}}{\partial r^2_{n-1}}\right] =$$
$$\left(\sigma_n\frac{\partial\mathcal{F}^*_n}{\partial r_n}\right)^{-1}\left[\hat{\mu}_n\frac{\partial\mathcal{F}^*_n}{\partial r_n} + \frac{1}{2}\sigma^2_{n+1}\frac{\partial^2 F^*_n}{\partial r^2_n}\right],$$

$$\frac{\hat{\mu}_{n-1}}{\sigma_{n-1}} + \frac{1}{2}\sigma_{n-1}\frac{\partial^2\mathcal{F}^*_{n-1}}{\partial r^2_{n-1}}/\frac{\partial^2\mathcal{F}^*_{n-1}}{\partial r^2_{n-1}} = \frac{\hat{\mu}_n}{\sigma_n} + \frac{1}{2}\sigma_n\frac{\partial^2\mathcal{F}^*_n}{\partial r^2_n}/\frac{\partial\mathcal{F}^*_n}{\partial r_n}.$$

Computing the FRA derivatives,

$$\begin{aligned}
\mathcal{F}^*_{n-1} &= \left(K_{n-1} - e^{r_{n-1}}\right)\Delta\tau_{n-1}, \\
\frac{\partial\mathcal{F}^*_{n-1}}{\partial r_{n-1}} &= -e^{r_{n-1}}\Delta\tau_{n-1}, \\
\frac{\partial^2\mathcal{F}^*_{n-1}}{\partial r^2_{n-1}} &= -e^{r_{n-1}}\Delta\tau_{n-1}, \\
\mathcal{F}^*_n &= \frac{\left(K_n - e^{r_n}\right)\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}, \\
\frac{\partial\mathcal{F}^*_n}{\partial r_n} &= -\frac{\left(1 + K_n\Delta\tau_n\right)e^{r_n}\Delta\tau_n}{\left(1 + e^{r_n}\Delta\tau_n\right)^2}, \\
\frac{\partial^2\mathcal{F}^*_n}{\partial r^2_n} &= \frac{(1 + K_n\Delta\tau_n)e^{r_n}\Delta\tau_n}{\left(1 + e^{r_n}\Delta\tau_n\right)^2}\left[\frac{2e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n} - 1\right],
\end{aligned}$$

leads to the following result for the derivative ratios,

$$\begin{aligned}\frac{\partial^2 \mathcal{F}^*_{n-1}}{\partial r^2_{n-1}} / \frac{\partial \mathcal{F}^*_{n-1}}{\partial r_{n-1}} &= 1, \\ \frac{\partial^2 \mathcal{F}^*_n}{\partial r^2_n} / \frac{\partial \mathcal{F}^*_n}{\partial r_n} &= \frac{1 - e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}.\end{aligned}$$

Substituting above leads to the following equation for the drifts,

$$\frac{\hat{\mu}_{n-1}}{\sigma_{n-1}} + \frac{1}{2}\sigma_{n-1} = \frac{\hat{\mu}_n}{\sigma_n} + \frac{1}{2}\frac{1 - e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}\sigma_n.$$

The drift constraint function h_n,

$$h_n \equiv \hat{\mu}_n - \frac{\sigma_n}{\sigma_{n-1}}\hat{\mu}_{n-1},$$

is then given by,

$$h_n(r_n) = \frac{1}{2}\sigma_{n-1}\sigma_n - \frac{1}{2}\frac{1 - e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}\sigma^2_n.$$

Exercise 32.2:
Derive the drift constraint function for normal dynamics.

Exercise 32.3:
Derive the drift constraint function for square-root dynamics.

We computed the drift constraint between r_{n-1} and r_n by valuing the associated FRA's in terms of the numeraire D_n. Recall that the forward measure Q_n makes the prices of assets measured relative to the discount bond D_n martingales. In particular, the relative price of the $(n-1)^{\text{st}}$ FRA is a martingale in forward measure,

$$\mathcal{F}^*_{n-1} = \mathbf{E}_{n-1}\left[\mathcal{F}^*_0\right]$$

Since the relative price of the FRA is given by,

$$\mathcal{F}^*_{n-1} = (K - f_{n-1})\,\Delta t$$

the above equation becomes,

$$(K - f_{n-1})\,\Delta t = \mathbf{E}_{n-1}\left[(K - f_0)\,\Delta t\right]$$

$$\Longrightarrow f_{n-1} = \mathbf{E}_{n-1}\left[f_0\right]$$

Hence, the forward rate f_{n-1} is a martingale in the forward measure. We will use this fact later to show that Black's formula provides an exact analytic expression for a caplet.

33 Path-Dependence

To propagate the term structure multiple time steps we shall see that the resulting forward rates depend in general on the path $\mathcal{P}$ followed by the spot rate. To be more specific, assume that we propagate our initial term structure to a state at epoch k where we prescribe the spot rate to be $r_0 = \hat{r}_0(t_k)$. Since the forward rate $r_k(t_0)$ will be the spot rate at time k we need to propagate it along a k-step path $\mathcal{P}_k$ that produces a total change $\hat{r}_0(t_k) - r_k(t_0)$ in its value.

$$\sum_{i=0}^{k-1} \Delta r_{k-i}(t_i) = \hat{r}_0(t_k) - r_k(t_0).$$

In this case the evolution equation becomes,

$$\Delta r_{n-i}(t_i) = \underbrace{\frac{\sigma_{n-i}(t_i)}{\sigma_{k-i}(t_i)} \Delta r_{k-i}(t_i)}_{\text{Volatility Term}} + \underbrace{\sum_{j=k+1-i}^{n-i} \frac{\sigma_{n-i}(t_i)}{\sigma_{j-i}(t_i)} h_{j-i}(r_{j-i}(t_i)) \Delta t_i.}_{\text{Drift Term}}$$

At time t_k the term structure is then given by,

$$r_{n-k}(t_k) = r_n(t_0) + \sum_{i=0}^{k-1} \Delta r_{n-i}(t_i).$$

Even though all admissable paths $\mathcal{P}_k$ lead to the same spot rate $\hat{r}_0(t_k)$, the forward rates $r_{n-k}(t_k)$, $n > k$ will depend explicitly on the details of the path $\mathcal{P}_k$. This path dependence is caused in general by both the drift and volatility terms in the evolution equation.

The **drift term** clearly produces path dependence when the constraint function h_n depends on the level of rates. As we saw in the previous section, in the case of lognormal dynamics h_n depends on the rate r_n via the equation,

$$h_n(r_n) = \frac{1}{2}\sigma_{n-1}\sigma_n - \frac{1}{2}\frac{1 - e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}\sigma_n^2.$$

However, one can show that h_n is only weakly dependent on the rate r_n for the case of normal dynamics,

$$h_n(r_n) = \frac{\sigma_n^2 \Delta\tau_n}{1 + f_n \Delta\tau_n}.$$

To see the contribution of the **volatility term** to the path dependence we neglect the drift term and consider a pair of two-step paths leading to the same spot rate. Along the first path $\mathcal{P}^{(1)}$ the rate r_2 changes by $\Delta r^{(1)}$ over the first time step and by $\Delta r^{(2)}$ over the second step. The forward curve becomes,

$$\begin{aligned}
r_0 &= r_2 + \Delta r^{(1)} + \Delta r^{(2)}, \\
r_1 &= r_3 + \frac{\sigma_{k,3}}{\sigma_{k,2}}\Delta r^{(1)} + \frac{\sigma_{k+1,2}}{\sigma_{k+1,1}}\Delta r^{(2)}, \\
&\vdots \\
r_n &= r_{n+1} + \frac{\sigma_{k,n+1}}{\sigma_{k,n}}\Delta r^{(1)} + \frac{\sigma_{k+1,n}}{\sigma_{k+1,n-1}}\Delta r^{(2)} \\
&\vdots \\
r_{N-3} &= r_{N-1} + \frac{\sigma_{k,N-1}}{\sigma_{k,N-2}}\Delta r^{(1)} + \frac{\sigma_{k+1,N-2}}{\sigma_{k+1,N-3}}\Delta r^{(2)}.
\end{aligned}$$

Along the second path $\mathcal{P}^{(2)}$ the sequence is reversed so that rate r_2 changes by $\Delta r^{(1)}$ over the first time step and by $\Delta r^{(1)}$ over the second step. In this case the forward curve becomes,

$$\begin{aligned}
r_0 &= r_2 + \Delta r^{(2)} + \Delta r^{(1)}, \\
r_1 &= r_3 + \frac{\sigma_{k,3}}{\sigma_{k,2}}\Delta r^{(2)} + \frac{\sigma_{k+1,2}}{\sigma_{k+1,1}}\Delta r^{(1)}, \\
&\vdots \\
r_n &= r_{n+1} + \frac{\sigma_{k,n+1}}{\sigma_{k,n}}\Delta r^{(2)} + \frac{\sigma_{k+1,n}}{\sigma_{k+1,n-1}}\Delta r^{(1)} \\
&\vdots \\
r_{N-3} &= r_{N-1} + \frac{\sigma_{k,N-1}}{\sigma_{k,N-2}}\Delta r^{(2)} + \frac{\sigma_{k+1,N-2}}{\sigma_{k+1,N-3}}\Delta r^{(1)}.
\end{aligned}$$

We are now interested in deriving a condition on the vol matrix which leads to the same term structure for all $\Delta r^{(1)}$ and $\Delta r^{(2)}$. Since the shocks are arbitrary we must can set $\Delta r^{(2)} = 0$. For this case the term structures agree only when the vol ratio is invariant under time translations,

$$\frac{\sigma_{k,n+1}}{\sigma_{k,n}} = \frac{\sigma_{k+1,n}}{\sigma_{k+1,n-1}} \quad \forall n, k.$$

This proves that the condition is unique. The fact that it also leads to path-indendence for arbitrary $\Delta r^{(1)}$ and $\Delta r^{(2)}$ proves its existence.

To construct a path-independent volatility matrix, consider an arbitrary initial volatility shape,

$$\{\sigma_{0,n}\} \quad 0 \leq n < N,$$

we can translate it in time to produce a path-independent volatility matrix,

$$\sigma_{k,n-k} = A_k \times \sigma_{0,n} \quad \forall n > k,$$

where A_k is a dilation constant that sets the scale of volatility at each time k.

Exercise 33.1:
Prove that this translated volatility matrix satisfies the path-independence condition.

The exponential is the only function which is shape invariant under time translations,

$$\sigma_{k,n} = \sigma_{0,k} e^{-\alpha \tau_n}$$

Exercise 33.2:
Show that the exponential volatility automatically satisfies the path-independence condition.

Exercise 33.3:
Compute the dilation vector A_k that generates the above exponential volatility under time translations.

In the normal case this volatility matrix leads to an Ornstein-Uhlenbeck process for the spot rate,

$$\Delta r_0 = \alpha \left(\bar{r}_0 - r_0 \right) \Delta t + \sigma_{0,k} \Delta \tilde{w}.$$

Therefore, we can conclude that a normal model with exponential volatility is approximately Markovian in the spot rate. This class of models can be shown to be equivalent to the extended Vasicek or Hull-White models.

Other Markovian models can be generated by prescribing dynamics for the spot rate and requiring the bond prices to be functions of only it and time. Examples of these spot rate models include Vasicek, Black-Derman-Toy (B-DT), Black-Karasinski (BK), and Cox-Ingersoll-Ross (CIR).

Spot rate models generally allow one to input a subset of the total volatility matrix and the rest of the volatilities are internally determined by the model in a way which preserves the Markovian dynamics. For example, the model may enable the user to input the volatility of the spot rate for all time and then the volatilities of the rest of the forward rates are outputs of the model.

In the more general model we will be able to prescribe the entire volatility matrix $\mathbf{\Sigma}$. This will enable us to calibrate to a broader range of market instruments. However, in this case we must deal with the complications caused by the path dependence. Fortunately, we have developed a perturbation technique which uses analytic approximations to correct the errors due to the path dependence.

34 Risk-Neutral Measure

The drift constraint derived above enables us to determine the drifts $\hat{\mu}_i$ of all the forward rates r_i in terms of the first drift $\hat{\mu}_1$. Therefore, the choice of $\hat{\mu}_1$ can be said to set the scale for all the forward rate drifts. In this section we show that any derivative W can be valued as the discounted risk-neutral expectation of its future cashflows so that a convenient choice for $\hat{\mu}_1$ that consistent with the risk-neutral measure. Therefore, we will compute $\hat{\mu}_1$ as the drift which correctly prices the simplest non-trivial interest rate derivative which in this case is the 2-period discount bond D_2.

We begin by constructing a riskless portfolio $\mathcal{P}$ consisting of a derivative W and α units of a discount bond D,

$$\mathcal{P} = W + \alpha D.$$

In a binomial world the spot-rate r_1 is subject to the following up and down stochastic shocks over the time interval Δt,

$$\Delta r_1 = \pm \sigma_1 \sqrt{\Delta t}.$$

The corresponding values of the portfolio $\mathcal{P}$ are,

$$\mathcal{P}_{\pm} = W_{\pm} + \alpha D_{\pm}.$$

The riskless conditon requires the portfolio $\mathcal{P}$ to have the same value in either the up or down state,

$$\mathcal{P}_{+} = \mathcal{P}_{-} \quad \Longrightarrow \quad W_{+} + \alpha D_{+} = W_{-} + \alpha D_{-}.$$

Solving for the hedge ratio α,

$$\alpha = -\frac{W_{+} - W_{-}}{D_{+} - D_{-}}.$$

Since the portfolio $\mathcal{P}$ is riskless it must grow at risk-free rate to prevent arbitrage,

$$\mathcal{P}_{+} = \mathcal{P}_{-} = \left(1 + e^{r_0} \Delta t\right) \mathcal{P}.$$

Substituting the above expression for $\mathcal{P}_+$,

$$W_+ + \alpha D_+ = (1 + e^{r_0}\Delta t)\, W + \alpha D^f,$$

where we have used the fact that the one-period forward bond price D^f is given by,

$$D^f = (1 + e^{r_0}\Delta t)\, D.$$

Substituting for the hedge ratio α and solving for W,

$$W = -\,(1 + e^{r_0}\Delta t)^{-1}\, \frac{D^f - D_+}{D_+ - D_-} W_- + \frac{D^f - D_-}{D_+ - D_-} W_+.$$

Define the coefficients,

$$p_\mp \;\; \equiv \;\; -\frac{D^f - D_\pm}{D_+ - D_-}.$$

This allows us to write the derivative W in the form,

$$W = (1 + e^{r_0}\Delta t)^{-1} \left(p_- W_- + p_+ W_+\right).$$

Since p_- and p_+ satisfy the constraint,

$$p_- + p_+ = \frac{D_+ - D_-}{D_+ - D_-} = 1,$$

we can interpret them as risk-neutral binomial probabilities and express the derivative W has a discounted expected value,

$$W(t) = (1 + e^{r_0}\Delta t)^{-1} \hat{\mathbf{E}}_{t+\Delta t}\Big[W\Big].$$

Computing the risk-neutral expectation of D,

$$\begin{aligned} \hat{\mathbf{E}}_{t+\Delta t}\left[D\right] &= p_- D_- + p_+ D_+ \\ &= -\frac{D^f - D_+}{D_+ - D_-} D_- + \frac{D^f - D_-}{D_+ - D_-} D_+ = D^f. \end{aligned}$$

Therefore, the forward price of the bond D^f is simply its risk-neutral expectation,

$$D^f(t) = \hat{\mathbf{E}}_{t+\Delta t}\left[D\right].$$

Alternatively, we can write this result in the form,

$$D_{n+1}(t) = (1 + e^{r_0}\Delta t)^{-1}\hat{\mathbf{E}}_{t+\Delta t}\Big[D_n\Big].$$

The drift constraint ensures that this condition is met provided we satisfy the case $n = 1$,

$$\begin{aligned} D_2(t) &= (1 + e^{r_0}\Delta t)^{-1}\,\hat{\mathbf{E}}_{t+\Delta t}\Big[D_1\Big] \\ &= (1 + e^{r_0}\Delta t)^{-1}\,(p_- D_- + p_+ D_+)\,. \end{aligned}$$

The down probability is given by,

$$p_- = -\frac{(1 + e^{r_1}\Delta t)^{-1} - \left(1 + e^{r_1+\Delta r}\Delta t\right)^{-1}}{(1 + e^{r_1+\Delta r}\Delta t)^{-1} - (1 + e^{r_1-\Delta r}\Delta t)^{-1}}.$$

Taking the expectation of the r_1 which becomes spot r_0 at $t + \Delta t$,

$$\begin{aligned} \hat{\mathbf{E}}_{t+\Delta t}[r_0] &= p_-\,(r_1 - \Delta r) + p_+\,(r_1 + \Delta r) \\ &= r_1 + (1 - 2p_-)\,\Delta r. \end{aligned}$$

Computing the drift of r_1,

$$\begin{aligned} \hat{\mu}_1 &\equiv \frac{\hat{E}_{t+\Delta t}\,[r_0] - r_1}{\Delta t}, \\ &= \frac{(1 - 2p_-)\,\Delta r}{\Delta t}, \\ &= (1 - 2p_-)\,\frac{\sigma_1}{\sqrt{\Delta t}}. \end{aligned}$$

35 Analytic Caplet Model

We shall show that the value of caplets is given exactly by Black's formula within the framework of our discrete lognormal model. We begin by constructing a riskless self-financing portfolio,

$$\mathcal{P} = C_n + \alpha \mathcal{F}_n + \beta D_{n+1}.$$

Now choose then discount bond maturing at the end of the caplet tenor period τ_{n+1} as the numeraire,

$$\mathcal{P}^* \equiv \frac{\mathcal{P}}{D_{n+1}} = C_n^* + \alpha \mathcal{F}_n^* + \beta,$$

$$\text{where,} \quad \begin{aligned} C_n^* &= \frac{C_n}{D_{n+1}} \\ \mathcal{F}_n^* &= \frac{\mathcal{F}_n}{D_{n+1}} = (K - f_n)\ \Delta t. \end{aligned}$$

Because the portfolio $\mathcal{P}$ is self-financing the no-arbitrage condition is given by,

$$\Delta \mathcal{P}^* = \Delta C_n^* + \alpha \Delta \mathcal{F}_n^* = 0.$$

Expand using Ito's Lemma,

$$\frac{\partial C_n^*}{\partial t} \Delta t + \frac{\partial C_n^*}{\partial f_n} \Delta f_n + \frac{1}{2} \frac{\partial^2 C_n^*}{\partial f_n^2} \Delta f_n^2 + \alpha \Big[\frac{\partial \mathcal{F}_n^*}{\partial f_n} \Delta f_n + \frac{1}{2} \frac{\partial^2 \mathcal{F}_n^*}{\partial f_n^2} \Delta f_n^2 \Big] = 0.$$

To make the portfolio riskless the coefficent of the stochastic Brownian motion term must be zero,

$$\left(\frac{\partial C_n^*}{\partial f_n} + \alpha \frac{\partial \mathcal{F}_n^*}{\partial f_n} \right)\ \Delta f_n = 0.$$

Solving for the hedge ratio α,

$$\alpha = -\frac{\partial C_n^*}{\partial f_n} \Big/ \frac{\partial \mathcal{F}_n^*}{\partial f_n}.$$

Substituting for α above leads to the following governing equation for caplets,

$$\frac{\partial C_n^*}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 C_n^*}{\partial f_n^2}\ \Delta f_n^2 = 0,$$

where we have used the fact that,

$$\frac{\partial^2 \mathcal{F}_n^*}{\partial f_n^2} = 0.$$

For lognormal forward rate dynamics we,

$$\Delta f_n^2 = \sigma_n^2 f_n^2 \Delta t + O\left(\Delta t^2\right).$$

Substituting above yields **Black's equation** for C_n^*,

$$\frac{\partial C_n^*}{\partial t} + \frac{1}{2}\sigma_n^2 f_n^2 \frac{\partial^2 C_n^*}{\partial f_n^2} = 0.$$

At the expiration of the caplet we must impose the initial condition,

$$C_0 = (1 + f_0\Delta t)^{-1} \max\left[f_0 - K, 0\right]\Delta t.$$

Since the absolute price is given by,

$$C_0 = \frac{C_0^*}{1 + f_0\Delta t},$$

the IC can be written in terms of C_0^* as,

$$C_0^* = \max\left[f_0 - K, 0\right]\Delta t.$$

To summarize, we solve the following equation for C_n^*,

$$\frac{\partial C_n^*}{\partial t} + + \frac{1}{2}\sigma_n^2 f_n^2 \frac{\partial^2 C_n^*}{\partial f_n^2} = 0,$$

subject to the IC,

$$C_0^* = \max\left[f_0 - K, 0\right]\Delta t.$$

$$\Longrightarrow C_n = D_{n+1} \times C_n^*$$

According to the Feynman-Kac formula,

$$C_n = D_{n+1}\mathbf{E}_n\left[\max\left(f_0 - K, 0\right)\right]$$

where the forward rate obeys the lognormal process,

$$\frac{\Delta f_n}{f_n} = \sigma_n \Delta\tilde{w}$$

The solution is called **Black's formula** which is given by,

$$C_n = D_{n+1}\left[f_n N(d_1) - K N(d_2)\right],$$

where the cumulative norm arguments are defined by,

$$\begin{aligned} d_1 &= \frac{\log(f_n/K) + v/2}{\sqrt{v}}, \\ d_2 &= d_1 - \sqrt{v}. \end{aligned}$$

and the total variance is given by,

$$v = \sum_{k=0}^{n-1} \sigma^2_{k,n-k}\Delta t,$$

which is the sum of the squares of volatilities along the diagonal of the volatility matrix.

We will now derive this result using the concept of forward measure. Recall that under the forward measure Q_n all assets prices relative to the numeraire D_n are martingales. In particular, the relative price of a caplet can be written as the following expectation,

$$C_n^* \equiv \frac{C_n}{D_{n+1}} = \mathbf{E}_n\left[(1 + f_0\Delta t)C_0\right].$$

Recall that the initial condition for caplets is,

$$C_0 = \frac{\max\left(f_0 - K, 0\right)\Delta t}{1 + f_0\Delta t}.$$

Substituting above leads to the solution,

$$C_n = D_{n+1}\mathbf{E}_n \left[(f_0 - K, 0) \, \Delta t \right],$$

where under forward measure the forward rate f_n is a martingale and obeys the following lognormal process,

$$\frac{\Delta f_n}{f_n} = \sigma_n \Delta \tilde{w}.$$

and therefore, according to Feynman-Kac satisfies Black's equation.

36 Analytic Swaption Model

We will now show that the Black-Scholes equation can also be used to value swaptions when we assume tht the swap rate obeys lognormal dynamics. This is inconsistent with FRM which precribes lognormal dynamics for the individual forward rates. We can map FRM into the Black-Scholes swaption model by using the volatily matrix to compute the swap rate variance. This ensures that we match the first three moments (mean, variance, and skew) of the distribution. This leads to very good agreement except in the case of deep out-of-the-money swaptions which probe the kurtosis of the distribution.

To derive the analytic swaption model we need to first introduce the concept of an **annuity** which makes periodic fixed payments over a prescribed period of time. Since an annuity is linear in the amplitude of the payment wlog we can confine our attention to the **unit** case which pays \$1 each tenor period. The value of a unit annuity $A^{(1)}$ with tenor Δt over the period,

$$t_n < t_{n+1} \cdots t_i \cdots < t_N$$

is easily seen to be,

$$A^{(1)}_{n,N} = \sum_{i=n+1}^{N} D_i$$

A forward starting swap $S_{n,N}$ with fixed coupon C can now be written as the product of a unit annuity and the difference between its coupon and the forward swap rate $F_{n,N}$,

$$\begin{aligned} S_{n,N} &= A^{(1)}_{n,N} \times (C - F_{n,N}) \quad \text{Receiver} \\ &= A^{(1)}_{n,N} \times (F_{n,N} - C) \quad \text{Payer} \end{aligned}$$

To derive the analytic swaption formula we construct a riskless, self-financing portfolio consisting of the swaption, the underlying forward starting swap and the unit annuity as a financing bond,

$$\mathcal{P} = \underbrace{W}_{\text{Swaption}} + \underbrace{\alpha S}_{\text{Underlying}} + \underbrace{\beta A^{(1)}}_{\text{Financing Bond}}$$

Now choose the numeraire to be the forward starting unit annuity,

$$\mathcal{P}^* \equiv \frac{\mathcal{P}}{A^{(1)}} = W^* + \alpha S^* + \beta$$

where the relative prices are,

$$\begin{aligned} W^*(t, C^f) &\equiv \frac{W}{A^{(1)}} \\ S^*(t, C^f) &\equiv \frac{S}{A^{(1)}} = C - F \end{aligned}$$

Now assume that the forward swap rate F is lognormal,

$$\frac{\Delta F}{F} = \mu_F \Delta t + \sigma_F(t) \Delta \tilde{w}$$

The no arbitrage condition requires that,

$$\Delta \mathcal{P}^* = \Delta W^* + \alpha \Delta S^* = 0.$$

Next, we apply Ito's lemma,

$$\frac{\partial W^*}{\partial t} \Delta t + \frac{\partial W^*}{\partial F} \Delta F + \frac{1}{2} \sigma_F^2 F^2 \frac{\partial^2 W^*}{\partial F^2} \Delta t + \alpha \left(\frac{\partial S^*}{\partial F} \Delta F + \frac{1}{2} \sigma_F^2 F^2 \frac{\partial^2 S^*}{\partial F^2} \Delta t \right) = 0$$

Computing the swap derivatives,

$$\begin{aligned} \frac{\partial S^*}{\partial F} &= \pm 1 \\ \frac{\partial^2 S^*}{\partial F^2} &= 0 \end{aligned}$$

and substituting above.

$$\frac{\partial W^*}{\partial t} \Delta t + \frac{\partial W^*}{\partial F} \Delta F + \frac{1}{2} \sigma_F^2 F^2 \frac{\partial^2 W^*}{\partial F^2} \Delta t \pm \alpha \Delta F = 0.$$

Solving for the hedge ratio α which makes the portfolio riskless,

$$\left(\frac{\partial W^*}{\partial F} \pm \alpha \right) \Delta F = 0 \Longrightarrow \alpha = \mp \frac{\partial W^*}{\partial F}$$

Substituting above for α shows that W^* satisfies the heat equation,

$$\frac{\partial W^*}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 W^*}{\partial F^2} = 0$$

subject to the following initial condition,

$$\begin{aligned} W^*(t_n, F) &= \max[C - F, 0] \quad \text{Receiver} \\ &= \max[F - C, 0] \quad \text{Payer} \end{aligned}$$

We can again apply the Feynman-Kac formula,

$$\begin{aligned} W(t, F) &= A^{(1)} \times \mathbf{E}_n [\max(C - F, 0)] \quad \text{Reciever} \\ &= A^{(1)} \times \mathbf{E}_n [\max(S - F, 0)] \quad \text{Payer} \end{aligned}$$

where the expectation is taken with respect to **swap forward measure** under which the swap rate is a martingale,

$$\frac{\Delta F}{F} = \sigma_F \Delta \tilde{w}$$

The solution is given by,

$$\begin{aligned} W &= A^{(1)} \times [CN(-d_2) - FN(-d_1)] \quad \text{Receiver} \\ &= A^{(1)} \times [FN(d_1) - CN(d_2)] \quad \text{Payer} \end{aligned}$$

where d_1 and d_2 are defined by,

$$\begin{aligned} d_1 &= \frac{\log(F/C) + V/2}{\sqrt{V}}, \\ d_2 &= d_1 - \sqrt{V}. \end{aligned}$$

and V is the total variance of the log-swap rate which is computed explicitly in the next section.

Exercise 36.1:
Derive the above equation for European swaptions using the concept of swap forward measure.

37 Swap-Rate Variance

In this section we will derive an expression for the variance of the forward swap rate F from t_n to t_N over the option period $0 < t < t_n$,

$$0 < t_1 \cdots < t_i \cdots < t_n < t_j < t_N$$

The forward swap rate at time t_i can be written as a function of the forward rates,

$$\begin{aligned} F_{n-i,N-i}(t_i) &= F_{n-i,N-i}\left[(r_{n-i}(t_i), \cdots, r_{N-1-i}(t_i)\right] \\ &= \left[1 - D_{n-i,N-i}(t_i)\right] / \sum_{j=n+1-i}^{N-i} D_{n-i,j}(t_i) \end{aligned}$$

The total variance of the forward swap rate F can be written as the sum of variances over the epoch periods Δt_i, $\;0 \leq i < n$,

$$V_{n,N}^F = \sum_{i=0}^{n-1} \left[\Delta F_{n-i,N-i}(t_i)\right]^2$$

Since each of forward rates is stochastic we don't know their values for times $t_i > t_0$. Therefore, in order to compute the total variance of F we need to make the simplifying assumption that forward curve is static. The forward swap rate is then given for all times by,

$$\begin{aligned} F_{n-i,N-i}(t_i) &= F_{n-i,N-i}\left[r_n(t_0), r_{n+1}(t_0), \cdots, r_{N-1}(t_0)\right] \\ &= F_{n,N}(t_0) = \left[1 - D_{n,N}(t_0)\right] / \sum_{j=n+1}^{N} D_{n,j}(t_0) \end{aligned}$$

where the forward discount bond prices $D_{n,i}(t_0)$ are given by,

$$D_{n,i} = \prod_{j=n}^{i-1} (1 + e^{r_j}\Delta t)^{-1}$$

Applying Ito's Lemma to compute the leading order change in the forward swap rate $F_{n-i,N-i}$ at time t_i,

$$\Delta F(t_i) = \sum_{j=n}^{N-1} \sigma_{i,j-i} \frac{\partial F}{\partial r_j}(t_0) \Delta \tilde{w}_{j-i} + O(\Delta t_i)$$

The total variance of the forward swap rate can now be written as,

$$
\begin{aligned}
V_{n,N}^{F} &= \sum_{i=0}^{n-1} [\Delta F(t_i)]^2 \\
&= \sum_{i=0}^{n-1} \left[\sum_{j=n}^{N-1} \sum_{k=n}^{N-1} \rho_{j-i,k-i} \; \sigma_{i,j-i} \frac{\partial F_{n,N}}{\partial r_j} \; \sigma_{i,k-i} \frac{\partial F_{n,N}}{\partial r_k} \right] \Delta t_i
\end{aligned}
$$

where we have used the fact that,

$$\Delta \tilde{w}_{j-i} \Delta \tilde{w}_{k-i} = \rho_{j-i,k-i} \Delta t_i$$

Computing the partial derivatives of the swap rates,

$$\frac{\partial F}{\partial r_j} = - \left[\frac{\partial D_{n,N}}{\partial r_j} \sum_{i=n+1}^{N} D_{n,i} + (1 - D_{n,N}) \sum_{i=n+1}^{N} \frac{\partial D_{n,i}}{\partial r_j} \right] / \left[\sum_{i=n+1}^{N} D_{n,i} \right]^2$$

where the bond derivatives are given by,

$$\frac{\partial D_{n,i}}{\partial r_j} = - \left(1 + e^{r_j} \Delta t\right)^{-1} D_{n,i} e^{r_j} \Delta t$$

The Black-Scholes formula actually requires the variance of $X \equiv \log(F)$ which we can compute using Ito's lemma,

$$\Delta X = \frac{\partial X}{F} \Delta F + \frac{\partial^2 X}{\partial F^2} \Delta t = \frac{\Delta F}{F} + O(\Delta t)$$

The variance rate v^X of X is given by,

$$v^X = \frac{(\Delta X)^2}{\Delta t} = \frac{v^F}{F^2},$$

so that the total variance of X becomes,

$$V^X = \frac{V^F}{F^2}$$

38 Two-Factor Dynamics

In the two-factor model the forward rates satisfy the following diffusion e-quation,

$$\frac{\Delta f_n(t)}{f_n(t)} = \mu_n(t) + \sigma_n^1(t)\Delta\tilde{z}_1 + \sigma_n^2(t)\Delta\tilde{z}_2,$$

where we have assumed that the two factors are independent,

$$\Delta\tilde{z}_1\Delta\tilde{z}_2 = 0.$$

In terms of log-forward rates,

$$\Delta r_n(t) = \hat{\mu}_n(t) + \sigma_n^1(t)\Delta\tilde{z}_1 + \sigma_n^2(t)\Delta\tilde{z}_2,$$

where the modified drift is defined by,

$$\hat{\mu}_n(t) = \mu_n(t) - \frac{1}{2}(\sigma_n^1)^2 - \frac{1}{2}(\sigma_n^2)^2.$$

Make the change of variables,

$$\begin{aligned} \sigma_n^1 &= \sigma_n \times \kappa_i, \\ \sigma_n^2 &= \sigma_n \times \sqrt{1-\kappa_i^2}. \end{aligned}$$

Dividing the first equation by the second,

$$\frac{\kappa_n}{\sqrt{1-\kappa_n^2}} = \frac{\sigma_n^1}{\sigma_n^2}.$$

Solving for κ_n,

$$\kappa_n = \frac{\sigma_n^1/\sigma_n^2}{\sqrt{1+(\sigma_n^1/\sigma_n^2)^2}} < 1.$$

Substituting to find σ_n,

$$\sigma_n = \frac{\sigma^1}{\kappa_n} = \sigma^2 \times \sqrt{1+(\sigma_n^1/\sigma_n^2)^2}.$$

The dynamics become,

$$\Delta r_n = \hat{\mu}_n(t) + \sigma_n(t)\left[\kappa_n \Delta\tilde{z}_1 + \sqrt{1-\kappa_n^2}\,\Delta\tilde{z}_2\right],$$

where the vector $\vec{\kappa}$ is referred to as the **factor structure**.

The correlation between Δr_m and Δr_n is given by,

$$\rho_{m,n} \equiv \frac{\hat{\mathbf{E}}\left[\Delta r_m \Delta r_n\right]}{\sqrt{\Delta v_m}\sqrt{\Delta v_n}} = \kappa_m\kappa_n + \sqrt{1-\kappa_m^2}\sqrt{1-\kappa_n^2}.$$

This demonstrates that the dynamics naturally factorize into a amplitude term given by $\vec{\sigma}$ and a correlation term characterized by the factor structure $\vec{\kappa}$.

It is interesting to note that the constant factor structure case,

$$\kappa_n = 1, \qquad 0 \le n < N,$$

leads to the one-factor model with perfect correlation,

$$\rho_{m,n} = 1, \qquad 0 \le m, n < N.$$

We can parametrize the factor structure in the form,

$$\kappa_n = \omega e^{-\gamma\tau_n}.$$

In this case the correlation matrix becomes,

$$\rho_{m,n} = \omega^2\left[e^{-\gamma(\tau_m+\tau_n)} + \sqrt{1-\omega^2 e^{-2\gamma\tau_m}}\sqrt{1-\omega^2 e^{-2\gamma\tau_n}}\right].$$

Looking at the asymptotic correlation with the spot-rate,

$$\rho_{0,\infty} = \lim_{n\to\infty}\rho_{0,n} = \omega.$$

In a two-factor model we do not have enough degrees of freedom to match the entire correlation matrix because it can be completely characterized by the factor structure $\vec{\kappa}$ which is only a vector quantity. Instead, we choose to match the correlations $\rho_{0.n}$ between each forward rate r_n and the spot rate r_0. This yields the following set of nonlinear equations for $\kappa_n, \quad 0 < n < N$,

$$\rho_{0.n} = \kappa_0 \kappa_n + \sqrt{1 - \kappa_0^2} \sqrt{1 - \kappa_n^2}.$$

Rewriting this equation in the form,

$$\sqrt{1 - \kappa_n^2} = \frac{\rho_{0.n} - \kappa_0 \kappa_n}{\sqrt{1 - \kappa_0^2}}.$$

Next, we square both sides,

$$1 - \kappa_n^2 = \left(\frac{\rho_{0.n} - \kappa_0 \kappa_n}{\sqrt{1 - \kappa_0^2}} \right)^2.$$

and rearrange terms to obtain the quadratic equation,

$$\kappa_n^2 - 2\rho_{0,n} \kappa_0 \kappa_n + \rho_{0,n}^2 - (1 - \kappa_0^2) = 0,$$

which subject to the constraint,

$$\rho_{0,0} = 1.$$

If we choose the case $\kappa_0 = 1$ the equation reduces to,

$$\kappa_n^2 - 2\rho_{0,n} \kappa_n + \rho_{0,n}^2 = 0.$$

The solution is given by the quadratic formula,

$$\kappa_n = \frac{2\rho_{0,n} \pm \sqrt{4\rho_{0,n}^2 - 4\rho_{0,n}^2}}{2} = \rho_{0,n}.$$

Therefore, in this case the factor structure is simply equal to the correlation between the corresponding forward rate and spot. However, since we must satisfy the condition $\rho_{0,0} = 1$ we also have $\kappa_0 = 1$ which leads to singular second factor volatility,

$$\sigma_{k,0}^{(2)} = \sigma_{k,0} \sqrt{1 - \kappa_0^2} = 0.$$

This singularity is indicative of the fact that in a two-factor model we cannot freely specify the correlation. In fact, Riccardo Rebonato has shown that in a general two-factor model the correlation satisfies the following constraint,

$$\lim_{n \to 0} \frac{\rho_{0,n+1} - \rho_{0,n}}{(n+1) - n} = 0,$$

or equivalently, the correlation is locally flat at zero. To compute the factor structure we set $\kappa_0 > 0$, prescribe a spot correlation $\rho_{0,n}$ satisfying the locally flat condition and then solve the resulting series of quadratic equations for the positive roots.

39 General Volatility Calibration

We need to solve for the forward rate volatility matrix $\mathbf{\Sigma}$ subject to the set of constraints that we match the market prices of liquid instruments such as ATM caps and swaptions. The constraints are non-linear, but because the options are ATM we can linearize them to leading order. In addition, in order to obtain a unique solution we need to optimize an objective functions $\mathcal{O}$. For example, we may either choose to maximize smoothness or minimize the day-to-day change in the matrix. This type of problem can either be solved using the classical non-linear programming or a genetic algorithm. In the former apporach we will probably need to linearize the constraints in order to make the problem tractable.

- **Market Data**
  - CD's, ED Futures, and Swaps
  - Historical Correlations $\rho_{i,j}$
  - Caplet Volatilities: Σ_n^C
  - Swaption Volatilities: $\Sigma_{n,N}^S$
- **Inputs**
  - Tenor Period: $\Delta t = 0.25$ yrs
  - Forward Rates: $\mathbf{f} = \{f_i\} \;\; \forall i$
  - Factor Structure: $\vec{\kappa} = \{\kappa_i\} \;\; \forall i$
  - Caplet Variances: $V_n^C = \left(\Sigma_n^C\right)^2 \; (n\Delta t)$
  - Swaption Variances: $V_{n,N}^S = \left(\Sigma_{n,N}^S\right)^2 \; (n\Delta t)$
- **Output**
  - Forward Rate Volatility Matrix: $\mathbf{\Sigma} = \{\Sigma_{i,j}\} \;\; \forall i,j$

- **Objective Function**

  We wish to minimize the change in the day-to-day change in the volatility grid. Assume that we can write the current volatility $\sigma_{i,j}$ as a perturbation of the previous volatility $\sigma^0_{i,j}$

$$\sigma_{i,j} = \sigma^0_{i,j} + \epsilon_{i,j}$$

  We wish satisfy the following minimization problem,

$$\mathcal{O} = \min \sum_{i=0}^{N} \sum_{j=0}^{N} \epsilon^2_{i,j}$$

- **Caplet Constraints**

  Let V^C_n be the total variance of the forward rate $r_n(t_0)$ implied by the market price of an n-period caplet price through Black's formula. Since the total variance is simply the sum of the variances at each time discrete time $i, \quad 0 \le i \le n-1$, we have the following set of constraints on the volatility matrix,

$$V^C_n = \sum_{i=0}^{n-1} \sigma^2_{n-i,i} \Delta t \quad 1 \le n \le N.$$

- **Swaption Constraints**

  European swaption prices imply swap rate variances $V^S_{n,N-n}$ through the Black-Scholes equation and lead to the following set of constraints (see Section : Swap-Rate Variance),

$$V^S_{n,N} = \sum_{i=0}^{n-1} \Big[\sum_{j=n}^{N-1} \sum_{k=n}^{N-1} \rho_{j-i,k-i} \; \sigma_{i,j-i} \frac{\partial F_{n,N}}{\partial r_j} \; \sigma_{i,k-i} \frac{\partial F_{n,N}}{\partial r_k} \Big] \Delta t \quad \forall N, n.$$

Instead of trying to match both caplets and swaptions we can be less ambitious and assume a shape along each row and use the spot rate volatility to match either the caplets or a linear set of swaptions. What we choose to match will in general depend on the derivative we are trying to price. For example, if we are pricing an **Index Amortizing Swap** whose index is 3M Libor then we would choose to match the caplets. On the other hand, if we are pricing a 10NC1 Bermudan swaption then we would choose to match the underlying diagonal swaptions, $1 \times 9, 2 \times 8, \cdots 9 \times 1$.

40 Monte Carlo Simulation

In order to value path dependent derivatives we need to know the entire history of the forward rate curve at all points in time. The path dependence can either arise because of the definition of the derivative contract as in the case of an IAS, or because the evolution of the forward curve itself is path dependent. A **Monte Carlo Simulation** naturally gives us the past history of rates by randomly propagating the forward curve through time according to the evolution equation,

$$\Delta r_n = \frac{\sigma_n}{\sigma_1}\Delta r_1 + \sum_{j=2}^{n} \frac{\sigma_n}{\sigma_j} h_j(r_j)\Delta t$$

where the constraint function is given by,

$$h_n(r_n) = \frac{1}{2}\sigma_{n-1}\sigma_n - \frac{1}{2}\frac{1 - e^{r_n}\Delta\tau_n}{1 + e^{r_n}\Delta\tau_n}\sigma_n^2.$$

Notice that the evolution of the entire curve is known once the change in the first rate Δr_1 is prescribed. Recall that r_1 obeys the following dynamics,

$$\Delta r_1 = \hat{\mu}_1\Delta t + \sigma_{k,1}\Delta\tilde{w},$$

where the risk neutral drift is given by,

$$\hat{\mu}_1 = (1 - 2p_-)\,\frac{\sigma_1}{\sqrt{\Delta t}}.$$

and the down probability is,

$$p_- = -\frac{(1 + e^{r_1}\Delta t)^{-1} - \left(1 + e^{r_1 + \Delta r}\Delta t\right)^{-1}}{(1 + e^{r_1 + \Delta r}\Delta t)^{-1} - (1 + e^{r_1 - \Delta r}\Delta t)^{-1}}.$$

All that remains is to simulate the Brownian shock $\Delta\tilde{w}$. The first approach is to assume the shock is **binomial**. This yields the following distribution,

$$\Delta\tilde{w} = \pm\sqrt{\Delta t} \qquad p_\pm = \frac{1}{2}.$$

This can be simulated by selecting from a **uniform distribution** on the interval $q \in (0,1)$ and choosing

$$\begin{aligned}\Delta\tilde{w} &= -\sqrt{\Delta t} \quad q \le 0.5 \\ &= +\sqrt{\Delta t} \quad q > 0.5\end{aligned}$$

The second approach is to allow the shock to be **normal**,

$$\Delta\tilde{w} \sim \mathcal{N}(0, \Delta t).$$

This distribution is generated by again choosing $q \in (0,1)$ from the uniform distribution and setting,

$$\Delta\tilde{w} = \sqrt{\Delta t} \times N^{-1}(q).$$

Finally, the third approach to modeling the shock is **stratified sampling** which involves partitioning the cumulative normal into equal areas to create a set of deterministic shocks which are one-to-one with the number of paths. This can be acomplished by uniformly partioning the unit interval (0.1),

$$0 < q_1 < q_2 < \cdots < q_i \cdots < q_p < 1 \quad \text{where,} \ \ q_i = \frac{i}{p+1}$$

The shocks are now deterministic and equal to,

$$\Delta w_i = \sqrt{\Delta t} \times N^{-1}(q_i).$$

Then for each step we need to **scramble** the shocks prior to mapping them to the paths.

The the derivative is valued in the Monte-Carlo simulation by computing the cashflows along each path,

$$CF = CF(t_k, \vec{r})$$

and then discounting them back along the path by the actual spot rates,

$$W = \hat{\mathbf{E}}\left[\sum_{k=1}^{N}\prod_{i=0}^{k-1}(1+\tilde{r}_0(t_i))^{-1} \times \tilde{CF}(t_k, \vec{r})\right]$$

Project II

Build a Monte-Carlo simulator to value ATM forward 2×5 European receiver and payer swaptions and compare result with Black-Scholes formula.

Assume the following parameters,

$$\begin{aligned} t_0 &= 19990215 \\ T_{exp} &= 20010215 \\ T_{mat} &= 20060215 \\ C &= \text{ATM Forward} \end{aligned}$$

where the spot correlations $\rho_{0,n}$ are given by,

19990215	1.00
19990815	0.98
20000215	0.95
20000815	0.90
20010215	0.82
20010815	0.75
20020215	0.65
20020815	0.60
20030215	0.56
20030815	0.53
20040215	0.50
20040815	0.48

Assume a flat volatility shape along each row and calibrate to the diagonal European swaptions. Compare results and convergence for all three simulation types. Check put-call parity.

41 Ambient Nodal Term Structures

In the trinomial lattice the node $\mathcal{N}_{k,j}$ represents a spot rate $r_0^j(t_k)$ at epoch k. We must build a forward curve $\hat{r}_i,\ 0 \le i < N$ at each node $\mathcal{N}_{k,j}$ such that $\hat{r}_0 = r_0^j(t_k)$. However, because of the path dependent evolution of interest rates the forward curve at the node will not be unique. Instead, it will depend on the nature of the path $\mathcal{P}_{k,j}$ followed by the spot rate from the origen out to the node $\mathcal{N}_{k,j}$. In reality there exists a family of forward rate curves at node $\mathcal{N}_{k,j}$ weighted by the probability distribution of associated paths $\mathcal{P}_{k,j}$.

Therefore, to construct a unique forward curve at the node $\mathcal{N}_{k,j}$ we must specify not only the spot rate $r_0^j(t_k)$ but the path $\mathcal{P}_{k,j}$ that it propagates along out to the node. We will choose a **straight-line** path $\bar{\mathcal{P}}_{k,j}$ which represents an "average" over the set of all possible paths. This will produce a unique curve at each node which is approximately the mean of the distribution of forward rate curves. Later we will correct for the path dependence by incorporating the deviations from this average behavior.

We define a straight-line path,

$$\bar{\mathcal{P}}_{k,j} = \{\Delta r_{k-i}(t_i)\}$$

such that the spot rate moves in equal increments Δr from its initial value $r_k(0)$ to its final value $r_0^j(t_k)$ at the node $\mathcal{N}_{k,j}$,

$$\Delta r_{k-i}(t_i) = \Delta r = \frac{r_0^j(t_k) - r_k(t_0)}{k}.$$

According to the evolution equation the changes in the forward curve along the straight-line path $\bar{\mathcal{P}}_{k,j}$ are given by,

$$\Delta r_{n-i}(t_i) = \frac{\sigma_{n-i}(t_i)}{\sigma_{k-i}(t_i)}\Delta r + \sum_{j=k+1-i}^{n-i} \frac{\sigma_{n-i}(t_i)}{\sigma_{j-i}(t_i)} h_{j-i}(r_{j-i}(t_i))\Delta t_i \qquad n \ge k.$$

This propagation leads to the following unique forward rate term structure at node $\mathcal{N}_{k,j}$,

$$\bar{r}_{n-k}(t_k) = r_n(t_0) + \sum_{i=0}^{k-1} \Delta r_{n-i}(t_i).$$

which we call the **ambient curve**.

42 Trinomial Lattice Probabilities

In a trinomial lattice each node $\mathcal{N}_{k,j}$ at epoch k can evolve to three contiguous nodes at epoch $k+1$. The mean m of the distribution is given by,

$$\begin{aligned} m &\equiv \hat{\mathbf{E}}_{t_{k+1}}[\tilde{r}_0] \\ &= \bar{r}_1(t_k) + \mu_1(t)\Delta t, \end{aligned}$$

where the drift $\mu_1(t)$ was found in **Section 34** to be,

$$\mu_1(t) = (1 - 2p_-)\,\frac{\sigma_1}{\sqrt{\Delta t}}.$$

Recall that the **center** of the distribution

$$x_0 \equiv r_0^{\iota}(t_{k+1}),$$

is defined to be the node at epoch $k+1$ closest to the mean,

$$|x_0 - m| = \min_{0 \le i \le 2k} |r_0^i(t_{k+1}) - m|.$$

The **minus** state x_- and **plus** state x_+ are then defined by,

$$\begin{aligned} x_- &= r_0^{\iota-1}(t_{k+1}), \\ x_+ &= r_0^{\iota+1}(t_{k+1}). \end{aligned}$$

The variance v of the spot rate can be written as,

$$\begin{aligned} v &\equiv \hat{\mathbf{E}}_{t_{k+1}}\left[(\tilde{r}_0 - m)^2\right] \\ &= \sigma_1(t)^2\Delta t. \end{aligned}$$

We must solve the following three linear equations for the trinomial probabilities,

$$\begin{aligned} 1 &= p_- + p_0 + p_+, \\ m &= p_- x_- + p_0 x_0 + p_+ x_+, \\ v &= p_- x_-^2 + p_0 x_0^2 + p_+ x_+^2 - m^2. \end{aligned}$$

This can be solved quite easily using Cramer's rule which yields,

$$
\begin{aligned}
p_- &= \frac{(m - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(v + m^2 - x_0^2)}{(x_- - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(x_-^2 - x_0^2)}, \\
p_+ &= \frac{(x_- - x_0)(v + m^2 - x_0^2) - (m - x_0)(x_-^2 - x_0^2)}{(x_- - x_0)(x_+^2 - x_0^2) - (x_+ - x_0)(x_-^2 - x_0^2)}, \\
p_0 &= 1 - p_- - p_+
\end{aligned}
$$

43 Path-Dependent Corrections

In the absence of path dependence an option $W_j(t_k)$ can be valued at node $\mathcal{N}_{k,j}$ using the one-step risk-neutral pricing condition,

$$W_j(t_k) = \left(1 + e^{r_0^j(t_k)}\Delta t\right)^{-1} (p_- W_- + p_0 W_0 + p_+ W_+) .$$

However, when there is path dependence the states $\mathcal{N}_{k,j}$ and $\mathcal{N}_{k+1,l}$ for $l = \iota - 1, \iota, \iota + 1$ are inconsistent. This is because the two stage straight-line paths $\bar{P}^{'}_{k+1,l}$ from the origen to the node $\mathcal{N}_{k,j}$ and then on to the three nodes $\mathcal{N}_{k+1,l}$ are different then those single stage straight-line paths $\bar{\mathcal{P}}_{k+1,l}$ from the origen directly to the nodes $\mathcal{N}_{k+1,l}$. Fortunately, the difference between the resulting term structures $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}^{'}$ is in general small because $\bar{\mathcal{P}}_{k+1,l}$ and $\bar{\mathcal{P}}^{'}_{k+1,l}$ are neighboring paths. This allows us to use perturbation theory to correct the inconsistencies caused by the path-dependent evolution of the forward curve.

We begin by computing the perturbed term structure $\hat{\mathbf{r}}^{'} = \{r_i^{'}\}$ by propagating the existing forward curve at node $\mathcal{N}k, j$ one-step to the nodes $\mathcal{N}_{k+1,l}$. The next step is to compute the perturbed derivative value $W_l^{'}(t_{k+1})$ given the ambient value $W_l(t_{k+1})$ and the shift in the forward curve. We assume to leading order that the perturbed option value can be written as,

$$W^{'} \approx W + W_a^{'} - W_a,$$

where W_a and $W_a^{'}$ are analytic approximations for the ambient and perturbed option values respectively. For European swaptions we can use the analytic Black-Scholes result we derived in **Section 36**. We will derive lat a similar result for the early exercise value of a American swaption based on a paper by Farshid Jamshidian.

We can now write the value $W_{k,j}$ of the option at node (k, j) as the discounted expectation of the perturbed option values,

$$W_j(t_k) = \left(1 + e^{r_0^j(t_k)}\Delta t\right)^{-1} \left(p_- W_-^{'} + p_0 W_0^{'} + p_+ W_+^{'}\right) .$$

44 Bucket Hedging

We partition the forward curve into **buckets**,

$$T_0^b \le T_1^b < T_2^b \cdots \le T_n^b.$$

where the bucket boundaries correspond to the maturities of the liquid at-market swaps,

$$S_j = S(t, \vec{f}; C_S, T_j^b) \quad 1 \le j \le n$$

Consider a constant shock to the forward rates in the k^{th} bucket,

$$\begin{aligned} f_i' &= f_i & 0 < t_i \le T_{k-1}^b \\ &= f_i + \Delta f & T_{k-1}^b < t_i \le T_k^b \\ &= f_i & t_i > T_k^b \end{aligned}$$

The k^{th} **bucket delta** of an arbitrary derivative W is defined by,

$$\begin{aligned} \Delta_k^b(W) &\equiv \lim_{\Delta f \to 0} \frac{W(t, \vec{f}\,') - W(t, \vec{f})}{\Delta f} \\ &= \sum_{T_{k-1}^b \le t_i \le T_k^b} \frac{\partial W}{\partial f_i} \end{aligned}$$

We can hedge these bucket deltas with the at-market swaps $S_j \;\; 1 \le j \le n$ whose bucket deltas are given by,

$$\begin{aligned} \Delta_k^b(S_j) &= \sum_{T_{k-1}^b \le t_i \le T_k^b} \frac{\partial W}{\partial f_i} & j \ge k \\ &= 0 & j < k \end{aligned}$$

Our hedged portfolio initially consists of only the derivative W and hence its bucket deltas are given by,

$$\Delta_k^b(\mathcal{P}) = \Delta_k^b(W) \quad 1 \le k \le n$$

We begin by doing N_n swaps S_n to eliminate the n^{th} bucket delta for the hedged portfolio $\mathcal{P}$,

$$\Delta_n^b(\mathcal{P}) + N_n \times \Delta_n^b(S_n) = 0 \Longrightarrow N_n = -\frac{\Delta_n^b(W)}{\Delta_n^b(S_n)}$$

The other bucket deltas for the hedged portfolio now become,

$$\Delta^b_k(\mathcal{P}) = \Delta^b_k(W) + N_n \times \Delta^b_k(S_n) \quad 1 \le k < n$$

We continue by doing N_{n-1} swaps S_{n-1} to eliminate the $n-1^{th}$ portfolio bucket delta,

$$\Delta^b_{n-1}(\mathcal{P}) + N_{n-1} \times \Delta^b_{n-1}(S_{n-1}) = 0 \Longrightarrow N_{n-1} = -\frac{\Delta^b_{n-1}(\mathcal{P})}{\Delta^b_{n-1}(S_{n-1})}$$

The remaining hedged portfolio bucket deltas become,

$$\Delta^b_k(\mathcal{P}) = \Delta^b_k(W) + N_n \times \Delta^b_k(S_n) + N_{n-1} \times \Delta^b_k(S_{n-1})$$

Proceeding inductively we see that,

$$N_k = -\frac{\Delta^b_k(\mathcal{P})}{\Delta^b_k(S_k)} \quad 1 \le k < n$$

where the k^{th} bucket delta for the hedged portfolio is given by,

$$\Delta^b_k(\mathcal{P}) = \Delta^b_k(W) + \sum_{j=k+1}^{n} N_j \times \Delta^b_k(S_j)$$

We partition the volatility matrix into **strips**,

$$T^s_0 \le T^s_1 < T^s_2 \cdots \le T^s_n.$$

The k^{th} strip comprises the forward volatilities,

$$\sigma_{k,i} \quad T^s_{k-1} < t_j < T^s_k, \ \ \forall\ i$$

where the strip boundaries occur in calendar time and correspond to the expirations of the liquid ATM European swaptions,

$$W_j = W(t, \vec{\sigma}; K, T^s_j) \quad 1 \le j \le n$$

Consider a constant shock to the forward volatilities in the k^{th} strip $\forall\ i$,

$$\begin{aligned} \sigma'_{j,i} &= \sigma_{j,i} & 0 < t_j \le T^s_{k-1} \\ &= \sigma_{j,i} + \Delta\sigma & T^s_{k-1} < t_j \le T^s_k \\ &= \sigma_{j,i} & T^s_k < t_j \le T^s_n \end{aligned}$$

The k^{th} **strip vega** of a derivative W is defined by.

$$\begin{aligned} \mathcal{V}_k^s(W) &\equiv \lim_{\Delta\sigma \to 0} \frac{W(t, \vec{\sigma}\,') - W(t, \vec{\sigma})}{\Delta\sigma} \\ &= \sum_{T_{k-1}^s \le t_j \le T_k^s} \sum_{0 \le i < N-j} \frac{\partial W}{\partial \sigma_{j,i}} \end{aligned}$$

We will hedge the vega exposure of W with the ATM European swaptions whose strip vegas are given by,

$$\begin{aligned} \mathcal{V}_k^s(W_j) &= \sum_{t_{k-1}^s \le t_l \le t_k^s} \sum_{0 \le i < N-l} \frac{\partial W_j}{\partial \sigma_{l,i}} \qquad j \ge k \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad j < k \end{aligned}$$

The hedges in terms of the ATM European swaptions are computed just as the interest rate swap hedges above.

45 Convexity Adjustments

To value a floating rate note we found we could replace the stochastic spot rate $\tilde{L}(t_i)$ with its forward rate f_i. This **static** valuation was possible because there exists a contract called an FRA which allows us to lock in the forward rate for the deposit period (t_i, t_{i+1}). Equivalently, the linearity of the FRA after changing the numeraire to B_{i+1} means that f_i is a martingale under forward measure.

When the tenor of the floating index is **mismatched** with the tenor of the deposit period the FRA becomes a nonlinear function under the appropriate change of numeraire so that so the f_i is no longer a martingale. This manifests itself physically by the fact that we are not able to lock in a fixed rate if the floating index was 6m Libor and the tenor the note was 3m or if the index was 3m Libor and the tenor was 6m. Another example where we are prevented from locking in a fixed cashflow is when when the floating payment is set in arrears. In these cases to get an exact result we would have to take the risk-neutral expectation of the stochastic cashflows.

An alternative approach is to convert to forward measure and compute the drift of the f_i caused by the convexity of the FRA. We can then make a **convexity adjustment** to the forward rate equal to the integral of drift and then obtain an approximate result by applying either the static valuation method or the Black-Scholes equation. We will now illustrate how this procedure can be generalized to other products by analyzing CMS/CMT swaps and caps.

The **Constant Maturity Swap** (CMS) rate C_T is the coupon of a forward starting fixed term swap which can be expressed as,

$$C = \left(1 - D^f_{T,N}\right) / \sum_{i=1}^{N} D^f_{T,i} \Delta t$$

To illustrate how to make convexity adjustments we will look at derivatives which have the CMS rate as their underlying. for simplicity, we will assume that the rate is both set and paid in arrears. Two examples of CMS derivatives are **CMS swaps** which exchange floating CMS for a fixed rate,

$$\tilde{C}_T \Longleftrightarrow C_0$$

and **CMS caps** $\mathcal{W}$ which are portfolios of caplets,

$$\mathcal{W} = \sum_{i=1}^{N} W_i$$

that are call options on the CMS rate,

$$W_i(t_i, C_i) = \max\left[\tilde{C}_i(t_i) - K, 0\right]$$

Unfortunately, there are no traded assets which depend directly on the CMS rate. Hence, there is no way to lock-in the cashflows or convert to forward measure. However, since the CMS rate can be expressed in terms of the forward curve we can view CMS based products as derivatives on the forward rates. This is the most rigorous approach to valuation but it requires a Monte Carlo simulation which is slow to converge. Alternatively, we can obtain a very good approximation by treating CMS products as derivatives on the yield of forward bond which is currently priced at par.

Recall that we can model a forward starting interest rate swap S_T^f as the discounted value of a forward starting coupon bond B_T^f minus par,

$$S_T^f = D_T \times (B_T^f - 1)$$

The forward yield $y \equiv y_T^f$ of the bond B^f is by definition equal to the CMS rate at inception when swap has zero value,

$$B_T^f(y) = 1 \Longrightarrow y = CMS$$

However, as the forward curve moves the bond B_T^f is no longer equal to par and its yield deviates from the CMS rate. Fortunately, to leading order it has the same convexity adjustment as the CMS rate. The difference involves a convexity correction to the convexity adjustment which is clearly higher order. Hence model CMS based products as derivatives on the forward yield of the current par bond.

Since both CMS derivatives are European we can simplify the analysis by converting to forward measure Q_T. We begin by constructing a riskless self-financing portfolio,

$$\mathcal{P} = W + \alpha \mathcal{F}_T + \beta D_T$$

where $\mathcal{F}_T$ is a forward contract on the underlying bond,

$$\mathcal{F}_T = D_T \times \left[B^f_T(t, y; T_{mat}, C) - K \right]$$

Choose the numeraire to be the discount bond D_T which matures at the expiration of the option,

$$\mathcal{P}^* \equiv \frac{\mathcal{P}}{D_T} = W^* + \alpha \mathcal{F}^*_T + \beta$$

The no-arbitrage condition becomes,

$$\Delta \mathcal{P}^* = \Delta W^* + \alpha \Delta \mathcal{F}^*_T = 0$$

Assume the forward yield y is lognormal,

$$\frac{\Delta y}{y} = \mu \Delta + \sigma(t) \Delta \tilde{w}$$

and apply Ito's Lemma,

$$\frac{\partial W^*}{\partial t} \Delta t + \frac{\partial W^*}{\partial y} \Delta y + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 W}{\partial y^2} \Delta t + \alpha \left(\frac{\partial \mathcal{F}^*_T}{\partial t} \Delta t + \frac{\partial \mathcal{F}^*_T}{\partial y} \Delta y_f + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \mathcal{F}^*_T}{\partial y_f^2} \Delta t \right) = 0$$

The riskless condition requires,

$$\left(\frac{\partial W^*}{\partial y} + \alpha \frac{\partial \mathcal{F}^*_T}{\partial t} \right) \Delta \tilde{w} = 0,$$

and leads to the following hedge ratio,

$$\alpha = - \frac{\partial W^*}{\partial y} / \frac{\partial \mathcal{F}^*_T}{\partial y}.$$

In addition, the derivatives of the forward contract are,

$$\begin{aligned}
\frac{\partial \mathcal{F}_T^*}{\partial t} &= \frac{\partial B_T^f}{\partial t}, \\
\frac{\partial \mathcal{F}_T^*}{\partial y} &= \frac{\partial B_T^f}{\partial y}. \\
\frac{\partial^2 \mathcal{F}_T^*}{\partial y^2} &= \frac{\partial^2 B_T^f}{\partial y^2}.
\end{aligned}$$

Substituting above yields the following equation for W^*,

$$\frac{\partial W^*}{\partial t} - \left(\frac{\partial B_T^f}{\partial y}\right)^{-1} \left(\frac{\partial B_T^f}{\partial t} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 B_T^f}{\partial y^2}\right) \frac{\partial W^*}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 W}{\partial y^2} = 0,$$

subject to the following initial conditions,

$$\begin{aligned}
W(T, y) &= y \times tenor && \text{CMS Swap} \\
&= \max(y - K, 0) \times tenor && \text{CMS Caplet}
\end{aligned}$$

Notice that this equation for the European option does not explicitly involve the repo financing of the underlying bond. In this case the financing can be imbedded in the forward price of the bond because the exercise date is known with certainty. The equation is solved for the relative price W^* and the option value is simply given by,

$$W = D_T \times W^*$$

An approximate solution can be obtained by noticing the risk-neutral process for the yield is given by,

$$\frac{\Delta y}{y} = \hat{\mu}\Delta t + \sigma(t)\Delta\tilde{w}$$

where the risk-neutral drift is,

$$\hat{\mu} = \frac{\partial B_T^f}{\partial t} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 B_T^f}{\partial y^2}.$$

Since the drift depends on the yield $\tilde{y}$ it is in general stochastic. However, for our purposes we can statically value it along the forward curve to a high degree of accuracy. In this case it is a constant $\hat{\mu}_0$ equal to its value today. The convexity adjustment to the yield is then given by,

$$y' = e^{\hat{\mu}_0 T} y$$

The floating payment on the swap then becomes,

$$W = D_T \left(y' \times tenor \right)$$

while the caplet is gen by the Black-Scholes formula with the forward price given by y',

$$W = D_T \times [y' N(d_1) - K N(d_2)]$$

where the cum norm arguments are,

$$\begin{aligned} d_1 &= \frac{\log y'/K + v/2}{\sqrt{v}}, \\ d_2 &= d_1 - \sqrt{v}. \end{aligned}$$

and v is the total variance,

$$v = \int_0^T \sigma^2(t')dt'$$

The CMS rate is often replaced by the **Constant Maturity Treasury** (CMT) rate for those whose are interested in hedging mortgage prepayment risk. It is computed as a weighted average of all US Treasury debt maturing in the vicinity of the term of the index. Since this includes both on and off the run bonds it is difficult to compute from a single term structure. Instead, CMT derivatives are valued by adding a **CMS-CMT basis swap** or **spread lock** to the corresponding CMS derivative. In the basis swap floating CMT and CMS rates are exchanged,

$$C\tilde{M}S \Longleftrightarrow C\tilde{M}T$$

Exercise 45.1:
Derive an analytic exprestion for the convexity adjustment of a 6m tenor floating rate note which pays 3m Libor set upfront.

46 Dynamic Stochastic Volatility Model

We will build a dynamic stochastic volatility model that exploits the following analogy between forward interest rates and forward implied volatilities:

Interest Rate		**Implied Volatility**
Forward Interest Rates	$\Longleftrightarrow$	Forward Implied Volatilities
Over-Night Rate	$\Longleftrightarrow$	Instantaneous Volatility
Discount Bond Prices	$\Longleftrightarrow$	Black-Scholes Option Prices
No Analogy	$\Longleftrightarrow$	Strike Structure
Forward Rate Agreement	$\Longleftrightarrow$	No Analogy
Positive Convexity	$\Longleftrightarrow$	Positive Convexity
Positive Drift	$\Longleftrightarrow$	Negative Drift
ED-Future Martingale	$\Longleftrightarrow$	No Analogy
Cash Deposit	$\Longleftrightarrow$	No Analogy

Consider a set of **test** options at $t_0 = 0$ expiring on successive tenor dates $T_n = n\Delta t$ with fixed strikes K_n. Now define an initial term structure of **forward implied volatility**,

$$\mathbf{\Sigma}(t_0) = \{\Sigma_i(t_0)\}, \quad 0 \leq i < N,$$

such that our initial set of options is priced correctly by the Black-Scholes equation,

$$W_n(t_0) = BS\left(S_0, \Sigma_0(t_0), \cdots, \Sigma_{n-1}(t_0); K_n, T_n\right)$$

Since we know the initial option prices $W_n(t_0)$ we can solve recursively for the initial implied volatilities using a Newton-Raphson root finder.

We now require that the forward volatilities correctly price the test options for all future times t_k,

$$W_{n-k}(t_k) = BS\left(S, \Sigma_0(t_k), \cdots, \Sigma_{n-k-1}(t_k); K_n, T_n\right).$$

Assume that both S and Σ_i are lognormal and obey the following risk-neutral dynamics,

$$\begin{aligned} \frac{\Delta S}{S} &= \mu \Delta t + \Sigma_0 \Delta \tilde{z}_1 \quad \text{where,} \quad \mu = f_0, \\ \frac{\Delta \Sigma_i}{\Sigma_i} &= \lambda_i \Delta t, + \kappa_i \Delta \tilde{z}_2 \end{aligned}$$

$$\Delta \tilde{z}_1 \Delta \tilde{z}_2 = \rho \Delta t.$$

Make the following change of variables to convert from lognormal to normal dynamics,

$$\begin{aligned} s &\equiv \log S, \\ \sigma_i &\equiv \log \Sigma_i. \end{aligned}$$

The dynamical equations for s and σ_i are,

$$\begin{aligned} \Delta s &= \hat{\mu} \Delta t + \Sigma_0 \Delta \tilde{z}_1, \\ \Delta \sigma_i &= \hat{\lambda}_i \Delta t + \kappa_i \Delta \tilde{z}_2. \end{aligned}$$

where the modified drifts are given by,

$$\begin{aligned} \hat{\mu} &= \mu - \frac{1}{2}\Sigma_0^2 = f_0 - \frac{1}{2}\Sigma_0^2, \\ \hat{\lambda}_i &= \lambda_i - \frac{1}{2}\kappa_i^2. \end{aligned}$$

To correctly price our initial set of "test" options we must satisfy the following one-step condition $\forall n, k$,

$$W_{n-k}\left(t_k\right) = \left(1 + f_0 \Delta t\right)^{-1} \hat{\mathbf{E}}_{t_{k+1}} \left[\tilde{W}_{n-k-1}\right],$$

where $\hat{\mathbf{E}}$ is the risk-neutral expectation operator.

This set of equations for $k+2 \leq n < N$ can be used to solve directly for the risk-neutral drifts,

$$\hat{\lambda}_i \quad 1 \leq i < N-k.$$

Alternatively, we will use the $n = k+2$ equation to solve for the drift of $\hat{\lambda}_1$ and then apply the no-arbitrage condition to derive a constraint relating the succesive drifts to all the previous ones.

$$\hat{\lambda}_i = h_i(\hat{\lambda}_1, , \cdots, \hat{\lambda}_{i-1}).$$

We now formulate our stochastic volatility model within the discrete binomial framework.

The binomial stochastic shocks are given by,

$$\Delta s = \Sigma_0 \sqrt{\Delta t}, \qquad \Delta \sigma_1 = \kappa_1 \sqrt{\Delta t}.$$

The risk-neutral binomial probabilities give rise to the risk-neutral expectation operator,

$$\begin{aligned} \hat{\mathbf{E}}_{t+\Delta t}\left[\tilde{s}\right] &= \hat{p}_d(s - \Delta s) + \hat{p}_u(s + \Delta s), \\ \hat{\mathbf{E}}_{t+\Delta t}\left[\tilde{\sigma}_1\right] &= \hat{q}_d(\sigma_1 - \Delta \sigma_1) + \hat{q}_u(\sigma_1 + \Delta \sigma_1). \end{aligned}$$

The modified risk-neutral drifts are defined by,

$$\begin{aligned} \hat{\mu}(t) &= \frac{\hat{\mathbf{E}}_{t+\Delta t}\left[\tilde{s}\right] - s}{\Delta t} = \frac{(1 - 2\hat{p}_d)\Delta s}{\Delta t}, \\ \hat{\lambda}_1(t) &= \frac{\hat{\mathbf{E}}_{t+\Delta t}\left[\tilde{\sigma}_1\right] - \sigma_1}{\Delta t} = \frac{(1 - 2\hat{q}_d)\Delta \sigma_1}{\Delta t}. \end{aligned}$$

The 2-period option W_2 must satisfy the 1-step pricing equation,

$$W_2(t, s, \sigma_0, \sigma_1) = (1 + f_0 \Delta t)^{-1} \hat{\mathbf{E}}_{t+\Delta t}\left[\tilde{W}_1\right].$$

Taking the risk-neutral expectation within the binomial framework under the assumption of zero correlation between the stock and implied vololatility,

$$\begin{aligned}
W_2(t,s,\sigma_0,\sigma_1) &= (1+f_0\Delta t)^{-1}\left[\hat{p}_d\hat{q}_d W_1(t+\Delta t, s-\Delta s, \sigma_1-\Delta\sigma_1)\right.\\
&+ \hat{p}_d(1-\hat{q}_d)W_1(t+\Delta t, s-\Delta s, \sigma_1+\Delta\sigma_1)\\
&+ (1-\hat{p}_d)\hat{q}_d W_1(t+\Delta t, s+\Delta s, \sigma_1-\Delta\sigma_1)\\
&+ \left.(1-\hat{p}_d)(1-\hat{q}_d)W_1(t+\Delta t, s+\Delta s, \sigma_1+\Delta\sigma_1)\right].
\end{aligned}$$

Taylor expanding the rhs and collecting terms,

$$\begin{aligned}
W_2(t,s,\sigma_0,\sigma_1) &= (1+f_0\Delta t)^{-1}\left[W_1 + \frac{\partial W_1}{\partial t}\Delta t + (1-2\hat{p}_d)\frac{\partial W_1}{\partial s}\Delta s + \frac{1}{2}\frac{\partial^2 W_1}{\partial s^2}(\Delta s)^2\right.\\
&+ \left.(1-2\hat{q}_d)\frac{\partial W_1}{\partial \sigma_1}\Delta\sigma_1 + \frac{1}{2}\frac{\partial^2 W_1}{\partial \sigma_1^2}(\Delta\sigma_1)^2 + O\left(\Delta t^2\right)\right].
\end{aligned}$$

Recall that the adjusted stock drift is given by,

$$\hat{\mu} = \frac{(1-2\hat{p}_d)\Delta s}{\Delta t},$$

$$\Longrightarrow (1-2\hat{p}_d)\Delta s = \hat{\mu}\Delta t,$$

In addition, make the following approximation,

$$(1+f_0\Delta t)^{-1} \approx 1 - f_0\Delta t + O(\Delta t^2).$$

Upon substitution we have to leading order,

$$\begin{aligned}
0 &= \left[\frac{\partial W_1}{\partial t} + \hat{\mu}\frac{\partial W_1}{\partial s} + \frac{1}{2}\Sigma_0^2\frac{\partial^2 W_1}{\partial s^2} - f_0 W_2\right]\Delta t\\
&+ (1-2\hat{q}_d)\frac{\partial W_1}{\partial \sigma_1}\Delta\sigma_1 + \frac{1}{2}\kappa_1^2\frac{\partial^2 W_1}{\partial \sigma_1^2}\Delta t.
\end{aligned}$$

Since W_1 satisfies Black-Scholes,

$$\frac{\partial W_1}{\partial t} + \hat{\mu}\frac{\partial W_1}{\partial s} + \frac{1}{2}\Sigma_0^2\frac{\partial^2 W_1}{\partial s^2} - f_0 W_1 = 0,$$

the equation for q_d simplifies to,

$$0 = (1 - 2\hat{q}_d)\frac{\partial W_1}{\partial \sigma_1}\Delta\sigma_1 + \frac{1}{2}\kappa_1^2\frac{\partial^2 W_1}{\partial \sigma_1^2}\Delta t.$$

Solving for the risk-neutral $\hat{\lambda}_1$ drift in the $\rho = 0$ case,

$$\hat{\lambda}_1 = \frac{(1-2\hat{q}_d)\Delta\sigma_1}{\Delta t} = -\frac{1}{2}\kappa_1^2\frac{\partial^2 W_1}{\partial \sigma_1^2}/\frac{\partial W_1}{\partial \sigma_1}.$$

The requirement that movements in implied forward volatility not create arbitrage opportunities leads to a constraint equation which must be satisfied by the drifts $\hat{\lambda}_i$. To derive this constraint create a riskless portfolio **P** consisting of a 2-period option W_2, α $(n+1)$-period options W_{n+1}, and β shares of stock S,

$$\mathcal{P} = W_2(t, S, \sigma_0, \sigma_1) + \alpha W_{n+1}(t, S, \sigma_0, \sigma_1, ..., \sigma_n) + \beta S.$$

The no-arbitrage condition is given by,

$$\Delta\mathcal{P} = \Delta W_2 + \alpha\Delta W_{n+1} + \beta\Delta S = f_0\mathcal{P}\Delta t.$$

Expand using a discrete version of Ito's Lemma,

$$\begin{aligned}
\frac{\partial W_1}{\partial t}\Delta t &+ \frac{\partial W_1}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 W_1}{\partial S^2}(\Delta S)^2 + \frac{\partial W_1}{\partial \sigma_1}\Delta\sigma_1 + \frac{1}{2}\frac{\partial^2 W_1}{\partial \sigma_1^2}(\Delta\sigma_1)^2 + \frac{\partial^2 W_1}{\partial S\partial\sigma_1}\Delta S\Delta\sigma_1 \\
&+ \alpha\left[\frac{\partial W_n}{\partial t}\Delta t + \frac{\partial W_n}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 W_n}{\partial S^2}(\Delta S)^2 + \sum_{i=1}^{n}\frac{\partial W_n}{\partial \sigma_i}\Delta\sigma_i\right. \\
&+ \left.\frac{1}{2}\sum_{i,j=1}^{n}\frac{1}{2}\frac{\partial^2 W_n}{\partial\sigma_i\partial\sigma_j}\Delta\sigma_i\Delta\sigma_j + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 W_n}{\partial S\partial\sigma_i}\Delta S\Delta\sigma_i\right] + \beta\Delta S \\
&= f_0(W_2 + \alpha W_n + \beta S)\Delta t.
\end{aligned}$$

Choose α and β to make the portfolio $\mathcal{P}$ riskless,

$$\frac{\partial W_1}{\partial S} + \alpha\frac{\partial W_n}{\partial S} + \beta = 0,$$

$$\Longrightarrow \beta = -\left(\frac{\partial W_1}{\partial S} + \alpha\frac{\partial W_n}{\partial S}\right).$$

$$\kappa_1 \frac{\partial W_1}{\partial \sigma_1} + \alpha \sum_{i=1}^{n} \kappa_i \frac{\partial W_n}{\partial \sigma_1} = 0,$$

$$\Longrightarrow \alpha = -\kappa_1 \frac{\partial W_1}{\partial \sigma_1} / \sum_{i=1}^{n} \kappa_i \frac{\partial W_n}{\partial \sigma_i}.$$

Substituting for α and β above,

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \kappa_i \frac{\partial W_n}{\partial \sigma_i}\right)^{-1} \left[\frac{\partial W_n}{\partial t} + \frac{1}{2}\Sigma_0^2 S^2 \frac{\partial^2 W_n}{\partial S^2} + f_0 S \frac{\partial W_n}{\partial S} - f_0 W_n \right. \\
&\left. + \sum_{i=1}^{n} \hat{\lambda}_i \frac{\partial W_n}{\partial \sigma_i} + \frac{1}{2} \sum_{i,j=1}^{n} \kappa_i \kappa_j \frac{\partial^2 W_n}{\partial \sigma_i \partial \sigma_j} + \frac{1}{2} \sum_{i=1}^{n} \rho S \Sigma_0 \kappa_i \frac{\partial^2 W_n}{\partial S \partial \sigma_i}\right] \\
&= \left(\kappa_1 \frac{\partial W_1}{\partial \sigma_1}\right)^{-1} \left[\frac{\partial W_1}{\partial t} + \frac{1}{2}\Sigma_0^2 S^2 \frac{\partial^2 W_1}{\partial S^2} + f_0 S \frac{\partial W_1}{\partial S} - f_0 W_1 \right. \\
&\qquad \left. + \hat{\lambda}_1 \frac{\partial W_1}{\partial \sigma_1} + \frac{1}{2}\kappa_1^2 \frac{\partial^2 W_1}{\partial \sigma_1^2} + \frac{1}{2}\rho S \Sigma_0 \kappa_1 \frac{\partial^2 W_1}{\partial S \partial \sigma_1}\right].
\end{aligned}$$

Since both W_1 and W_n satisfy the Black-Scholes equation this equation simplifies to,

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \kappa_i \frac{\partial W_n}{\partial \sigma_i}\right)^{-1} \left[\sum_{i=1}^{n} \hat{\lambda}_i \frac{\partial W_n}{\partial \sigma_i} + \frac{1}{2} \sum_{i,j=1}^{n} \kappa_i \kappa_j \frac{\partial^2 W_n}{\partial \sigma_i \partial \sigma_j} + \frac{1}{2} \sum_{i=1}^{n} \rho S \Sigma_0 \kappa_i \frac{\partial^2 W_n}{\partial S \partial \sigma_i}\right] \\
&\qquad = \left(\kappa_1 \frac{\partial W_1}{\partial \sigma_1}\right)^{-1} \left[\hat{\lambda}_1 \frac{\partial W_1}{\partial \sigma_1} + \frac{1}{2}\kappa_1^2 \frac{\partial^2 W_1}{\partial \sigma_1^2} + \frac{1}{2}\rho S \Sigma_0 \kappa_1 \frac{\partial^2 W_1}{\partial S \partial \sigma_1}\right].
\end{aligned}$$

Recall that for the zero correlation case $\hat{\lambda}_1$ is given by,

$$\hat{\lambda}_1 = -\frac{1}{2}\kappa_1^2 \frac{\partial^2 W_1}{\partial \sigma_1^2} / \frac{\partial W_1}{\partial \sigma_1},$$

so that the constraint equation reduces to,

$$\sum_{i=1}^{n} \hat{\lambda}_i \frac{\partial W_n}{\partial \sigma_i} + \frac{1}{2} \sum_{i,j=1}^{n} \kappa_i \kappa_j \frac{\partial^2 W_n}{\partial \sigma_i \partial \sigma_j} = 0.$$

Solving for the n^{th} drift $\hat{\lambda}_n$,

$$\hat{\lambda}_n = -\left[\sum_{i=1}^{n-1} \hat{\lambda}_i \frac{\partial W_n}{\partial \sigma_i} + \frac{1}{2}\sum_{i,j=1}^{n} \kappa_i \kappa_j \frac{\partial^2 W_n}{\partial \sigma_i \partial \sigma_j}\right] / \frac{\partial W_n}{\partial \sigma_n}.$$

We must now evaluate the derivatives that appear in the drift constraint formula. Define the term implied volatility,

$$\Omega \equiv \sqrt{\frac{1}{\tau}\sum_{i=1}^{N} \Sigma_i^2 \Delta t}.$$

The total kappa of both puts and calls is given by (Hull pp.315),

$$\frac{\partial W}{\partial \Omega} = \frac{S\tau}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^{+2}},$$

where $d_1^{\pm}$ is defined by,

$$d_1^{\pm} \equiv \frac{\log(S/K) + (R \pm \frac{1}{2}\Omega^2)\tau}{\Omega\sqrt{\tau}}.$$

Now apply the chain rule to compute the partial kappa,

$$\frac{\partial W}{\partial \sigma_i} = \frac{\partial W}{\partial \Omega}\frac{\partial \Omega}{\partial \Sigma_i}\frac{\partial \Sigma_i}{\partial \sigma_i},$$

where the last two derivatives are given by,

$$\begin{aligned}
\frac{\partial \Omega}{\partial \Sigma_i} &= \frac{\partial}{\partial \Sigma_i}\sqrt{\frac{1}{\tau}\sum_{i=1}^{N} \Sigma_i^2 \Delta t} = \frac{\Sigma_i \Delta t}{\Omega \tau}, \\
\frac{\partial \Sigma_i}{\partial \sigma_i} &= \frac{\partial}{\partial \sigma_i}\left[e^{\sigma_i}\right] = \Sigma_i.
\end{aligned}$$

Substituting into the expression for partial kappa yields,

$$\frac{\partial W}{\partial \sigma_i} = \frac{S}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^{+2}} \frac{\Sigma_i^2 \Delta t}{\Omega}$$

We must now compute the second partial kappa,

$$\begin{aligned}
\frac{\partial^2 W}{\partial \sigma_i^2} &= \frac{\partial}{\partial \Sigma_i}\frac{\partial W}{\partial \sigma_i}\frac{\partial \Sigma_i}{\partial \sigma_i} \\
&= \Sigma_i \frac{\partial}{\partial \Sigma_i}\left(\frac{S}{\sqrt{2\pi}\Omega} e^{-\frac{1}{2}d_1^{+2}}\Sigma_i^2 \Delta t\right) \\
&= \frac{S\Delta t}{\sqrt{2\pi}\Omega}\Sigma_i^2 e^{-\frac{1}{2}d_1^{+2}}\left[2 - \Sigma_i d_1^+ \frac{\partial d_1^+}{\partial \Omega}\frac{\partial \Omega}{\partial \Sigma_i}\right],
\end{aligned}$$

where the derivative of d_1 wrt Ω is given by,

$$\frac{\partial d_1}{\partial \Omega} = -\frac{1}{\Omega}d1^-.$$

Substituting into the expression for the pure 2^{nd} partial kappa,

$$\frac{\partial^2 W}{\partial \sigma_i^2} = \frac{S\Delta t}{\sqrt{2\pi}\Omega}\Sigma_i^2 e^{-\frac{1}{2}d_1^{+2}}\left[2 + d_1^+ d_1^- \frac{\Sigma_i^2 \Delta t}{\Omega^2 \tau}\right].$$

Expressing the result in terms of the first partial kappa leads to,

$$\frac{\partial^2 W}{\partial \sigma_i^2} = \left[2 + d_1^+ d_1^- \frac{\Sigma_i}{\Omega} - \frac{\Sigma_i^2 \Delta t}{\Omega^2 \tau}\right]\frac{\partial W}{\partial \sigma_i}.$$

Next we compute the mixed 2^{nd} partial kappa ($i \neq j$),

$$\begin{aligned}
\frac{\partial^2 W}{\partial \sigma_i \partial \sigma_j} &= \frac{\partial}{\partial \Sigma_j}\frac{\partial W}{\partial \sigma_i}\frac{\partial \Sigma_i}{\partial \sigma_i} \\
&= \frac{\partial}{\partial \Sigma_j}\left[\frac{S}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^{+2}}\frac{\Sigma_i^2 \Delta t}{\Omega}\right]\frac{\partial \Sigma_j}{\partial \sigma_j} \\
&= \frac{S}{\sqrt{2\pi}}\Sigma_i^2 \Sigma_j \Delta t \frac{\partial}{\partial \Omega}\left[\frac{e^{-\frac{1}{2}d_1^{+2}}}{\Omega}\right]\frac{\partial \Omega}{\partial \Sigma_j}.
\end{aligned}$$

Substituting for the derivatives we get,

$$\frac{\partial^2 W}{\partial \sigma_i \partial \sigma_j} = \frac{S\Sigma_i^2 \Sigma_j^2 \Delta t^2}{\sqrt{2\pi}\Omega^3 \tau} e^{-\frac{1}{2}d_1^{+2}}\left[d_1^+ d_1^- - 1\right].$$

In terms of the first partial derivative,

$$\frac{\partial^2 W}{\partial \sigma_i \partial \sigma_j} = \frac{\Sigma_j^2 \Delta t}{\Omega^2 \tau}\left[d_1^+ d_1^- - 1\right]\frac{\partial W}{\partial \sigma_i}.$$

47 Credit Derivatives

We define **credit derivatives** as extensions of interest rate derivatives to instruments whose cashflows are not guaranteed. Therefore, in addition to discounting cashflows we need to multiply them by the likelihood of receiving them. We will see that there is a simple analogy between discount factors and cashflow survival probabilities which will enable us to extend our interest rate model to credit derivatives. However, despite the similarites between interest rates and default probabilities there are a number of important differences that we need to be aware of. The most of important of which is that the default process leads to price discontinuities which can only be approximately hedged through diversification.

All bonds of the same level of subordination issued by are company are considered under the law to be **pari passu** and hence upon default all bond holders are entitled to recover the same fraction R of the par amount. Consider a corporate bond with the following coupon dates,

$$\tau_0 \leq t < \tau_1 < \tau_2 \cdots < \tau_i < \cdots < \tau_N.$$

If we denote the probability of a bond surviving to time τ_i by P_i^s, the bond price is given by,

$$B = \sum_{i=1}^{N} D_i \left[P_i^s C \Delta\tau_{i-1} + \underbrace{\left(P_{i-1}^s - P_i^s\right)}_{\text{default prob}} \times R \right] + D_N P_N^s.$$

The **survival probabilities** P_i^s implied by corporate bond prices are called the **risk-neutral** probabilities. Because the market desires a significant risk-premium for incurring the highly skewed and kurtotic distribution associated with the default process the risk-neutral probabilities tend to imply that default is 2-3 times greater than the real-world default probabilities. In valuing derivatives we will always use the risk-neutral probabilities.

We now wish to make the survival probabilities stochastic to reflect changes in perception of a company's chance of default. In order to ensure that they satisfy the constraint,

$$0 \leq P_i^s \leq 1 \quad \forall\, i \geq 0,$$

we define the **psuedo** forward spreads by,

$$s_j \equiv \left(\frac{P_j^s}{P_{j+1}^s} - 1 \right) / \Delta\tau_j,$$

and assume they satisfy the following lognormal dynamics,

$$\frac{\Delta s_j}{s_j} = \lambda_j \Delta t + \kappa_j \Delta \tilde{w}_s.$$

The above definition means that the survival probabilities are expressed in terms of the spreads by,

$$P_i^s = \prod_{j=0}^{i-1} (1 + s_j \Delta t_j)^{-1}.$$

which coupled with the positivity of s_j ensures that they satisfy the above probability constraint. The psuedo spreads correspond to the actual **forward spreads** only in the zero recovery case when $R = 0$.

In order to reprice all the corporate bonds we must preserve the survival probabilities. Hence, we must satisfy the following one-step pricing condition,

$$P_T^s(t) = \hat{\mathbf{E}}_{t+\Delta t} \left[\tilde{P}_T^s \right].$$

In the binomial world the three possible states for the spread are **minus**, **plus** and **default**. The down and up probabilites are denoted by $q_\pm$ and the instantaneous probability of default q_d is given by,

$$q_d = 1 - P_1^s = 1 - (1 + s_0 \Delta\tau_0)^{-1} = \frac{s_0 \Delta\tau_0}{1 + s_0 \Delta\tau_0}.$$

The probabilities must satisfy the constraint,

$$q_- + q_+ + q_d = 1 \Longrightarrow q_- + q_+ = 1 - q_d = (1 + s_0 \Delta\tau_0)^{-1} < 1.$$

This shows that $q_\pm$ are **defect** probabilities because they add up to less than one. The defect is accounted for by the presence of the default state. The binomial probabilities contingent upon no default are given by,

$$\hat{q}_\pm \equiv \frac{q_\pm}{(1 + s_0 \Delta t_0)^{-1}},$$

and we readily see that they add up to one,

$$\hat{q}_- + \hat{q}_+ = 1.$$

The one-step pricing condition becomes,

$$P^s(t) = q_- P^s_- + q_+ P^s_+ + q_d \times 0.$$

In terms of the contingent probabilities,

$$P^s(t) = (1 + s_0 \Delta\tau_0)^{-1} \left(\hat{q}_- P^s_- + \hat{q}_+ P^s_+ \right).$$

We recognize that is constraint has the same form as the discount bond pricing equation and hence the psuedo spreads must satisfy the same constraint equation,

$$g_n \equiv \lambda_n - \frac{\kappa_n}{\kappa_{n-1}} \lambda_{n-1} = \frac{1}{2} \kappa_{n-1} \kappa_n - \frac{1}{2} \frac{1 - s_n \Delta\tau_n}{1 + s_n \Delta\tau_n} \kappa_n^2.$$

The risk-neutral drift $\hat{\lambda}_1$ is computed just as in the interest rate case by solving the implicit equation,

$$P^s_2 = \hat{\mathbf{E}}_{t+\Delta\tau_0} \left[\tilde{P}_2 \right].$$

Any credit derivative can be written as the discounted risk-neutral expectation of its value at the next time step,

$$W(t) = (1 + f_0 \Delta\tau_0)^{-1} \hat{E}_{t+\Delta\tau_0} \left[\tilde{W} \right].$$

In the binomial case we have,

$$\begin{aligned} W(t) = (1 + f_0 \Delta\tau_0)^{-1} \Big(q_d W^d \;&+\; q_- p_- W^{-,-} + q_+ p_- W^{+,-} \\ &+\; q_- p_+ W^{-,+} + q_+ p_+ W^{+,+} \Big). \end{aligned}$$

We will now look at three examples of **static** credit derivatives which can be valued without evolving the spread curve. A **default swap** S^d is an OTC credit derivative in which Y pays X a fixed periodic fee δ and X agrees to buy the underlying bond B for par in the event of default. The value of S^d from the perspective of Y is given by,

$$S^d = \sum_{i=1}^{N} D_i \left[\left(P^s_{i-1} - P^s_i \right) \times (1 - R) - P^s_{i-1} \delta \Delta\tau_{i-1} \right],$$

and the fee δ is chosen to set the structure equal to zero at its inception. Because the bond is redeemed for par upon default rather than the value of the equivalent riskfree bond, the default swap differs from a coupon guarantee. It will be more or less valuable than the guarantee depending on whether the forward Treasury bond prices are below or above par respectively. Therefore, a corporate bond plus a default swap is not equivalent to a Treasury bond,

$$B + S^d \neq T,$$

and hence, in general, the default swap fee δ is not given by the spread of the bond over Treasuries.

In an **asset swap** S^a X sells a bond B to Y for par and then engages in a interest rate swap with Y where he receives the bond coupon C and pays $L + \delta$. The value of S^a to X is given by,

$$S^a = 1 - B + \sum_{i=1}^{N} D_i \left(f_{i-1} + \delta - C \right) \Delta\tau_{i-1}.$$

The spread δ is chosen to make the entire structure worth zero at inception and it accounts for the fact that in general the bond is not equal to par and that the coupon is off-market. In order to deliver bond B to Y, X can either purchase it outright or borrow it in the repo market. The latter provides a natural hedge for the default swap above and can be used to price the default risk in a way which takes into account the actual financing cost of the short bond position.

A third static example is a **total return swap** S^r where X pays Y the total return on an asset B (e.g. bond) and recieves $L + \delta$ floating from Y. This structure allows X to short a security B synthetically and lock in the repo financing. The spread δ is designed to account for the rolling repo risk.

Some examples of **dynamic** credit derivatives include **corporate calls**, **spread options**, **default swaptions** and **collateralized bond obligations**.

Pricing & Hedging Final Exam

Due: Wednesday, May 12 5:00pm

Do problems 1-3 and either 4A or 4B. You are not permitted to obtain help from other students in the class. If you have any question please contact Gabriel Gomez or myself. Although full credit requires numerical answers, substantial partial credit will be given for demonstrating the correct approach. If you need a numerical result from a previously unanswered question, simply choose reasonable values and proceed. Please hand the exam into to Gabriel. Assume that today's date is $t_0 = 19990315$ and use the following data to answer the examination questions:

Date	F_τ	**Caplet Vol**	**Vol Shape**	**Spot Corr**
19990315	1.0000			1.00
19990615	1.0128	8.74	1.00	0.98
19990915	1.0258	10.40	1.18	0.95
19991215	1.0392	11.75	1.30	0.90
20000315	1.0536	13.00	1.35	0.84
20000615	1.0681	13.63	1.32	0.76
20000915	1.0831		1.26	0.70
20001215	1.0983			
20010315	1.1138	14.21	1.20	0.62
20010615	1.1296			
20010915	1.1458			
20011215	1.1624			
20020315	1.1793	14.53	1.17	0.55
20020615	1.1964			
20020915	1.2137			
20021215	1.2316			
20030315	1.2496	14.36	1.13	0.52
20030615	1.2679			
20030915	1.2866			
20031215	1.3057			
20040315	1.3252	13.95	1.10	0.50

Problem 1 (25pts):

1. Compute the discount bond prices for each date.
2. Calculate the 3m spot rate for the following cases:
   (a) 30/360 simple
   (b) 30/360 continuous
3. Derive the discrete forward curve for Act/360 daycount.
4. Compute the invoice and quoted prices of a 6 semi-annual bond maturing on 20031215.
5. Compute the forward price (quoted) for delivery on 19991215 using a no-arbitrage "cash-and-carry" argument.
6. Show that the above result agrees with "sliding up" the yield curve.
7. Calculate the bond yield and duration.
8. Does the duration increase or decrease with coupon? Explain.

<u>Problem 2</u> (25pts):

1. Calibrate the forward volatility matrix to match the caplets assuming the "humped" vol shape given above.

2. Is the forward rate propagation path-dependent? Prove your answer.

3. Use a 1-factor model to compute the price of the following European receiver swaption:

T_{exp}	20010315
T_{mat}	20040315
C	6.0%

4. Calibrate the factor structure to the spot correlations.

5. Price the above European swaption using a 2-factor model.

6. Discuss the effect of correlation on swaption pricing.

Problem 3 (25pts):

Consider the S&P 500 stock index SPX:

S	1332
Div Yield	1.21%

1. Compute the price of the forward maturing $T_{mat} = 20010315$.
2. Assume the ATM European option vols on SPX are given by,

T_{exp}	σ
19990615	18.00%
19990915	18.75
19991215	19.10
20000315	19.50
20000615	19.75
20000915	20.05
20001215	20.15
20010315	20.20

   and that the correlation between interest rates and SPX is,

$$\rho_{r,SPX} = -50\%.$$

   Construct the quarterly ATM forward vol curve.

3. Discuss why the future price is a martingale.
4. Under what circumstances is the forward price a martingale?
5. Calculate the corresponding 2 year SPX future price. (Assume the 1-factor interest rate model calibrated above.)
6. Discuss the physical origen of the difference between the future and forward prices.

Problem 4A (25pts):

Consider the following equity call option:

T_{exp}	1 day
σ	30% (ATM)
S	48
K	50
r_0	6%

Over the last month (22 days) the stock has had following tick-by-tick intra-day volatilities:

Date	1-Day Vol
19990211	26%
19990212	12
19990215	18
19990216	31
19990217	112
19990218	10
19990219	41
19990222	15
19990223	24
19990224	8
19990225	13
19990226	28
19990301	20
19990302	34
19990303	76
19990304	37
19990305	53
19990308	46
19990309	62
19990310	22
19990311	17
19990312	6

1. Compute the value of the ATM call.

2. Use the ATM vol to compute the OTM call.

3. The day-to-day changes in 1-day volatility reflect its uncertainty. Discuss the discontinuities in the volatilities.

4. Compute the mean of the volatilities.

5. Should the ATM vol equal the volatility mean? Discuss.

6. Assume the volatility is lognormally distributed and compute the mean, variance and kurtosis of the log-vols.

7. Represent the log-vols as a trinomial distribution. Compute the OTM option price taking into account the uncertainty in volatility.

8. Compute the OTM implied volatility.

9. Interpret the "smile" physically.

<u>Problem 4B</u> (25pts):

Consider the following upfront receiver swap:

T_{mat}	20040315
Fixed Leg	$C = 5.5$ (semi-annual 30/360)
Floating Leg	3m Libor (quarterly 365/360)

Recall that a receiver swap S can be modelled as long a bond and short a floating rate note,

$$S = B - FRN.$$

1. Apply the concept of forward measure to prove that the floating rate note associated with the above swap is worth par.
2. Compute the value of the above reciever swap.
3. What is the par coupon?
4. Apply a convexity adjustment to value the **arrears** swap where the floating rate is set and paid in arrears.