Introduction to Mathematical Modeling, Spring 2000

Assignment 4, due February 16.

1. The eigenvectors of a matrix, A, are called $u^{(k)}$ and satisfy

$$Au^{(k)} = \lambda_k u^{(k)} \ , \tag{1}$$

where the numbers λ_k are the eigenvalues. It is a theorem that if the eigenvalues of A are distince, then the eigenvectors are linearly independent. Since any set of n linearly independent vectors forms a basis in n dimensional space, the eigenvectors form a basis. This means that any vector, y, may be written as a linear combination of the $u^{(k)}$:

$$y = \sum_{k=1}^{n} c_k u^{(k)} \ . \tag{2}$$

To do this in a practical way, we make a matrix, U, out of the eigenvectors $u^{(k)}$ by taking the k^{th} column of U to be $u^{(k)}$. This is written

$$U = \begin{pmatrix} | & & | \\ u^{(1)}, & \cdots, & u^{(n)} \\ | & & | \end{pmatrix} \ .$$

To name the components of $u^{(k)}$, write

$$u^{(k)} = \begin{pmatrix} u_1^{(k)} \\ \vdots \\ u_n^{(k)} \end{pmatrix} \ .$$

Then, the (j,k) entry of U is $U_{jk} = u_j^{(k)}$. In this notation, equation (2) takes the form

$$y_j = \sum_k U_{jk} c_k \qquad \text{for} j = 1, \cdots, n.$$

In matrix notation, this is simply

$$x = Uc \ .$$

The solution is $c = U^{-1}x$. All this says that we can find the "expansion coefficients" c_k by solving a system of equations (or inverting a matrix) involving the eigenvectors.

For the simple harmonic oscillator with linear friction, the differential equation in matrix form is

$$\dot{y} = Ay \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-\gamma}{m} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \ . \tag{3}$$

Find the eigenvalues and eigenvectors of A. Find the matrices U and U^{-1}. Notes: (*i*) It will be much easier if you express your answer in terms of ω and μ rather than k, m, and γ. (*ii*) The answer is not unique. If u is an eigenvector then $2u$ is also an eigenvector.

2. We want to understand the solution of

$$\dot{y} = Ay \tag{4}$$

in terms ofeigenvalues and eigenvectors. For each t we can use (2. This just leads to

$$y(t) = \sum_k c_k(t) u^{(k)} . \tag{5}$$

The "expansion coefficients", $c_k(t)$, can change with time but the eigenvectors of A do not. Differentiate the expression (5) with respect to t and use the relations (4) and (1) to write differential equations for each of the $c_k(t)$. Write the solution to these equations in terms of the (possibly complex) exponential and the values of $c_k(0)$.

3. In the case of (3) from part 1, use the method of part 2 to

a. Write $c_1(0)$ and $c_2(0)$ in terms of $y_1(0) = x(0)$ $y_2(0) = \dot{x}(0)$.

b. Write $c_1(t)$ and $c_2(t)$ in terms of $c_1(0)$ and $c_2(0)$.

c. Write the components of $y(t)$ in terms of $c_1(t)$ and $c_2(t)$.

d. Put this together to write $x(t)$ and $\dot{x}(t)$ in terms of $x(0)$ and $\dot{x}(0)$.

In this way, construct a matrix, $M(t)$ so that

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = M(t) \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} .$$

If everything goes right, the matrix M will have only real entries even though the intermediate quantities U, c, and λ are not real.

4. The Fibonacci numbers (named after an early Renaissence Italian monk and amateur mathematician), f_n are given as follows

$$f_0 = 1 \; , \;\; f_1 = 1 \; , \quad f_{n+1} = f_n + f_{n-1} \;\text{ for } n > 1. \tag{6}$$

The first few are 2, 3, 5, 8.

a. Write a Matlab program to compute and plot the first, say, 30 Fibonacci numbers. Plot them on a log scale and try to estimate the slope of the line that the points are tending to.

b. Let $y^{(n)}$ be defined in terms of the Fibonacci numbers as follows:

$$y^{(n)} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} .$$

Find the matrix, A so that (6) is equivalent to

$$y^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad y^{(n+1)} = Ay^{(n)} \; .$$

c. Find the eigenvalues and eigenvectors of A.

d. Use this information to find a formula for f_n. From this formula, what can you say about $\log(f_n)/n$ when n is large? Is this consistent with your answer to part a?