Introduction to Mathematical Modeling, Spring 2000

Assignment 2, due February 2.

1. Find the three complex numbers, z, with $z^3 = 8i$. Hint: find the modulus and argument of $8i$.

2. Find the two solutions of the quadratic polynomial $z^2 - (1+i)z + 2 - 2i$.

3. Use the sin and cos angle sum formulae, as well as the formula $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$ (for *real* numbers x_1 and x_2), to show that $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for any complex numbers z_1 and z_2.

4. If $u = e^z$, we say that $z = \log(u)$.

a. Show that there are many numbers, z, with $e^z = u$. Hint: use question 3 to find complex numbers a so that $e^{z+a} = e^z \cdot 1 = e^z$.

b. Find a $\log(1-i)$. In math jargon, the different possible values of $\log(u)$ are called the "branches" of the log function.

5. We have the differential equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

There are four different behaviors the solutions can exhibit, depending on the value of the parameter a.

(i) Oscillatory growth: solutions oscillate and grow exponentially at the same time.

(ii) Neutral oscillation: solutions oscillate but the amplitude of the oscillation does not change in time.

(iii) Oscillatory decay: solutions oscillate and the amplitude of the oscillation decays exponentially in time.

(iv) Simple decay solutions decay monotonically to zero without oscillating.

Find the a value or range of values in which each of these behaviors chould be observed. Hint: suppose $x(t) = e^{i\omega t - \mu t}$. Find the values or range of values of ω and μ where x has these behaviors.

6. Coulomb's model of friction gives the friction force, f_{fr} as a function of speed as: $f_{fr} = -\gamma$ if $v > 0$ and $f_{fr} = \gamma$ if $v < 0$. This is interpreted to mean that if you push a mass that is initially not moving with a force, say, $\gamma/2$, then the mass will not move. If you push with force 3γ, then ma will be 2γ: the total force on the mass will be the applied force minus the friction force. Consider a harmonic oscillator with Coulomb friction with a small friction coefficient. Estimate the energy loss over one complete oscillation and use that to find a differential equation for the amplitude $A(t)$. Compare this behavior to the behavior of a harmonic oscillator with simple linear friction.