Assignment 7, due November 12

Corrections: [none yet]

Coding standards. Programmers (almost all) are more productive (achieve the same amount of code development in less time) if they keep code quality standards. These seem to slow you down, but they (in the long run) speed you up. The grader will subtract points from codes that do not follow these standards. Please hand in printouts of your code and printouts of the plots and other output if necessary.

- The comments at the top must say what the code does, who wrote it, contact information (email), when, and why.
- All printed output must be formatted into neat columns that are easy to read.
- Printed output must be planned to be informative and concise. Avoid very long printouts where interesting information is hard to find.
- The code must be well commented with variable names that are not obvious explained.
- The code must use white space (spaces and blank lines) to make it more readable.
- There must be no "hard wired" constants. `for k in range(10):` is not allowed. Instead, use `n = 10` (or whatever the range is called) followed by `for k in range(n):`
- The plotting parameters (number of bins, ranges of variables, axis scalings, etc.) must be chosen to make plots easy to read, and to clearly convey the information in the plot.
- The title of a plot must explain the plot and contain parameters of the run. Use subtitles or other text in the plot if it doesn't all fit in the title. You should be able to look at a printout of the plot and know from the information there what the run was.
- Plotting must be automated. The titles and axis scalings, etc., must not be hand typed into the plot. You should be able to make another `.pdf` file with the same plot with one push of the "enter" key.
- If the purpose of the plot is to compare different things (e.g., theory and simulation results), then both curves must be in the same plot.

Computational exercises: hand in printouts of code along with printed output, graphs, and written explanations.

1. The "standard" *Cauchy* distribution has PDF $u(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. This is a *fat tailed* distribution, so fat tailed that $\mathrm{E}[|X|] = \infty$. The law of large numbers doesn't apply to the Cauchy distribution. In fact if X_k are independent standard Cauchy random variables, the sample mean

$$S_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

has the same Cauchy distribution (same parameters, not just same distribution family). Demonstrate this computationally for $n = 2$ and (if you're motivated) for larger n. Print a histogram of S_n to demonstrate it's standard Cauchy.

2. Consider the geometric Brownian motion example of Section 3 of Lesson 6. Write a Monte Carlo code (starting with posted code) that estimates $\mathrm{E}[S_t] = 1$ with various values of μ in the importance sampling algorithm. You can generate $Z \sim \mathcal{N}(\mu, \sigma^2)$ by taking a standard normal, multiplying by σ and adding μ. Estimate the sample variance as described in Section 2.1 for each experiment. With $t = 36$, do a loop over μ values, printing the estimated mean and variance. Comment on the μ value that gives the best variance. How much work does it save, over direct ($\mu = 0$) simulation if you want 1% accuracy (the error bar to be 1% of the answer. How would this change for larger or smaller values of t?

Traditional exercises

1. Suppose $\mathcal{F}$ is a $\sigma-$algebra generated by sets $\mathcal{A}$, and $\mathcal{G}$ is a $\sigma-$algebra generated by sets $\mathcal{B}$. You can show that $\mathcal{F} = \mathcal{G}$ by showing that each event $C \in \mathcal{A}$ is in $\mathcal{G}$ and each event $C \in \mathcal{B}$ is in $\mathcal{F}$.

   (a) Explain the $\mathcal{F}$ and $\mathcal{G}$ thing.

   (b) In $\Omega = \mathbb{R}^2$, let $\mathcal{F}$ be the $\sigma-$algebra generated by "balls" (sets of the form $(x-a)^2 + (y-b)^2 < r^2$, which would be balls in 3D but are called balls in any dimension by mathematicians). Let $\mathcal{G}$ be the $\sigma-$algebra generated by "boxes" (sets of the form $a \leq x \leq b$ and $c \leq y \leq d$). Show $\mathcal{F} = \mathcal{G}$.

2. In path space $C([0,T])$ a "ball" is a set of paths $X_{[0,T]}$ that are within r of a given continuous path $y_{[0,T]}$ for all $t \in [0,T]$. The event generated by $y_{[0,T]}$ and r is

$$X_{[0,T]} \in A \;\text{ if }\; |X_t - y_t| \leq r \;, \quad \text{for } 0 \leq t \leq T \;.$$

The $\sigma-$algebra generated by events like this is called the *Borel sets*. Show that this $\sigma-$algebra is generated by "one time" events of the form

$$X_{[0,T]} \in A(a,b,t) \;\text{ if }\; a \leq X_t \leq b \;.$$

You may use the following fact: (1) If $X_q \leq y_q + r$ for every rational number $q \in [0, T]$, then $X_t \leq y_t + r$ for all $t \in [0, T]$.