Assignment 6, due October 22

Corrections: [none yet]

Two of the most important things you can get from this class are the ability to create mathematical models (equations) from verbal descriptions intuition about stochastic processes, the intuition coming from solutions in specific cases. These exercises address those goals.

1. Consider a random process involving a particle that either is moving randomly or is stuck. If it is moving, it sticks in time interval $(t, t + dt)$ with probability $\lambda(x)dt$. If it is stuck, it comes unstuck in time interval $(t, t + dt)$ with probability $\mu(x)dt$. If it is moving, it satisfies $dX = a(X)dt + b(X)dW_t$. There are two densities, $u(x, t)$ and $v(x, t)$. The first is the density of the particle if it is not stuck. The second is the density of the particle if it is stuck. These satisfy

$$\Pr(\, X_t \text{ is stuck }) = \int_{-\infty}^{\infty} v(x,t)\, dx$$
$$\Pr(\, X_t \text{ is not stuck }) = \int_{-\infty}^{\infty} u(x,t)\, dx \; .$$

   Write an equation system for $\partial_t u(x, t)$ and $\partial_t v(x, t)$. These are *coupled* in the sense that the u equation involves v and the v equation involves u. Show that your equation system has the property that

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty} u(x,t)\, dx + \int_{-\infty}^{\infty} v(x,t)\, dx \right) = 0 \; .$$

2. Consider a random process involving a particle that either moves randomly with $X_t < a$ or sticks at $x = a$. If X_t touches $x = a$ it either sticks or is "reflected" back into the region $x < a$ where it does a random motion. Let $v(t) = \Pr(\, X_t = a)$. You may assume (and it's true) that the particle is at $x = a$ only if it is stuck, almost surely. Let $u(x, t)$, for $x < a$, be the probability density for the particle if it isn't stuck at $x = a$. Write a differential equation system for $u(x, t)$ and $v(t)$. You will find that v and $\partial_t v = \frac{d}{dt} v$ occurs only in the boundary condition for u at $x = a$. Verify that your equation system has the property that

$$\frac{d}{dt}\left(v(t) + \int_{-\infty}^{\infty} u(x,t)\, dx \right) = 0 \; .$$

   You need to use two parameters, λ and μ, where λ governs the rate of sticking if a moving particle touches $x = a$ and μ governs the rate at which stuck particles become unstuck.

3. Suppose X_t is a Brownian motion with drift rate zero and variance $\sigma^2 t$. Consider a "linear" interest rate rX_t, so that the value at time T starting from x at time t is

$$f(x,t) = \mathrm{E}_{x,t}\left[e^{\int_t^T rX_s ds}\right] .$$

Write the backward equation for this f. Find the solution using an ansatz of the *affine* (more properly *exponential affine*) form

$$f(x,t) = e^{a(t)x+b(t)} .$$

For this, you need to find the differential equations $a(t)$ and $b(t)$ satisfy together with the final conditions for these quantities at $t = T$. That will determine a and b completely.

4. Let X_t be a Brownian motion with standard variance and fixed constant drift rate r. This means that r is a constant, $X_0 = 0$, and $dX = rdt + dW$. This exercise explores the hitting time $\tau_a = \min\{t | X_t = a\}$ with $a > 0$, depending on whether $r > 0$ or $r < 0$. This exercise takes the PDE approach.

   (a) Write the forward equation and the boundary condition at a for $u(x,t)$, which is the probability density at time t of a particle with $\tau_a > t$.

   (b) Write a formula for $f_a(t)$, the PDF of τ_a, in terms of u and $\partial_x u$ at $x = a$.

   (c) Show that this equation and boundary condition from part (a) can be transformed into the standard heat equation with the same boundary condition at $x = a$. To do this, define $v(x,t) = e^{\mu x}e^{\lambda t}u(x,t)$ and find μ and λ so that v satisfies $\partial_t v = \frac{1}{2}\partial_x^2 v$.

   (d) Use the method of images solution for v to find u and from there get $f_a(t)$.

   (e) Show that when $r > 0$, (the drift is toward the boundary $x = a$) $\tau_a < \infty$ almost surely and $\mathrm{E}[\tau_a] < \infty$.

   (f) Show that when $r < 0$ (drift away from the boundary), $\Pr(\tau_a < \infty) < 1$. That is, there is a positive chance that X_t never touches the boundary.

   (g) Write the differential equation for $g(y) = \Pr(\tau_a < \infty \mid X_0 = y)$. Find the boundary condition g satisfies at $x = a$. Show that this equation has a solution that is a decreasing exponential as $x \to -\infty$ if $r < 0$.

   (h) Check that your solution to part (g) has the feature that $g(y) \to 1$ as $r \to 0$ with $r < 0$. How is this consistent with what you know about Brownian motion without drift?

   (i) Write a differential equation for $h(y) = \mathrm{E}[\tau_a | X_0 = y]$. Find the boundary condition at $y = a$. Show that this equation has a solution that grown linearly as $x \to -\infty$.

(j) Check that your solution to part (i) has the feature that $h(y) \to \infty$ as $r \to 0$ with $r > 0$. How is this consistent with what you know about Brownian motion without drift?