Stochastic Calculus, Courant Institute, Fall 2018
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2018/StochCalc.html
Always check the `classes` message board before doing any work on the assignment.

Assignment 5, due October 15

Corrections: [none yet]

1. Suppose X_t satisfies the Ornstein Uhlenbeck SDE $dX = -\gamma X dt + \sigma dW$. Suppose $X_t = y$ is known. We saw in Assignment 4 that conditional on this, X_{t+s} is normal with mean $e^{-\gamma s}y$ and variance

$$\frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma s}\right) \ .$$

   (a) Use this information to write a formula for the Green's function (transition density) $G(x, y, s)$.

   (b) Verify by direct calculation that G satisfies the forward equation as a function of x and s for each y.

   (c) Verify by direct calculation that G satisfies the backward equation as a function of y and s for each x.

2. (This exercise shows that the forward and backward equations, at least for Brownian motion, have a "gain of regularity". The solution at $t > 0$ or $t < T$ is more regular (differentiable) than the data $u_0(x)$ or $V(x)$. This implies that the forward equation cannot be run backwards and the backward equation cannot be run forwards.) The Green's function for Brownian motion is

$$G(x, y, s) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}} \ .$$

   (a) Show that, for all y,

$$\int_{-\infty}^{\infty} \left|\partial_x G(x, y, s)\right| \, dx = \frac{C}{s^{\frac{1}{2}}} \ .$$

   Find C. Hint: First explain why the answer does not depend on y, then set $y = 0$. Second, do a *scaling* argument to show why the answer has $s^{-\frac{1}{2}}$ (substitute $\frac{x^2}{s} = z^2$) and set $s = 1$. The answer is $2\int_0^\infty \cdots dz$.

   (b) This inequality is "obvious". For "any" two functions g and h,

$$\left|\int g(y)h(y)dy\right| \leq \left(\int |g(y)| \, dy\right)\left(\max_y |h(y)|\right) \ .$$

Use this and part (a) to show that if u satisfies the forward equation for Brownian motion, then

$$|\partial_x u(x,t+s)| \leq \frac{C}{s^{\frac{1}{2}}} \max_y |u(y,t)|$$

(c) Show that if f satisfies the backward equation for Brownian motion, then

$$\max_x |\partial_x f(x,t-s)| \leq \frac{C}{s^{\frac{1}{2}}} \max_x |f(x,t)| \ .$$

Use this to show that there is a bounded but discontinuous payout function $V(x)$ leads to a differentiable (and therefore continuous) value function $f(x,t)$ for $t < T$. Therefore, not every bounded function can be the value function for a bounded payout function.

3. Consider a linear diffusion process with a control

$$dX_t = aX_t dt + \sigma dW_t + Bu_t dt \ .$$

The control u_t must be chosen to be known at time t in terms of $W_{[0,t]}$ and/or $X_{[0,t]}$. *Optimal stochastic control* is the problem of choosing u to optimize (maximize or minimize) something. A *linear feedback* controller has the form

$$u_t = KX_t \ .$$

It is linear because u is a linear function of X. It is feedback because u_t is a function of X_t at the same time t. The constant K is for Kalman, who developed linear stochastic control theory. If $a > 0$ then the system without a control ($u = 0$) is unstable in the sense that $|X_t| \to \infty$ exponentially (almost surely) as $t \to \infty$ – not good.

(a) Show that if K is chosen properly, then the system is stabilized in the sense that the controlled system has a limiting variance and mean converging to zero as $t \to \infty$. This says that the system is stabilizable.

(b) Let $\mathrm{E}_{\mathrm{ss}}[\cdot]$ represent the expectation in this steady state. Find a formula for K in terms of the other parameters (a, $\sigma \neq 0$, $B \neq 0$, $Q > 0$, and $R > 0$) to minimize the steady state cost function

$$C = Q\mathrm{E}_{\mathrm{ss}}\left[X^2\right] + R\mathrm{E}_{\mathrm{ss}}\left[u^2\right] \ .$$

The parameter Q represents the cost of X deviating from its resting value $X = 0$. The parameter R represents the cost of the controller. Show that the cost goes to zero as the control becomes free ($R \to 0$).

4. Consider a discrete time controlled linear stochastic system

$$X_{n+1} = AX_n + \xi_n + Bu_n \ .$$

Here $|A| < 1$ makes the uncontrolled system stable. Suppose the noise process ξ_n are independent normals with mean zero and variance σ^2. Suppose that the control u_n is chosen based on a "noisy measurement" of X_n, which means

$$u_n = KY_n \ , \ \ Y_n = X_n + \eta_n \ ,$$

where the measurement errors η_n are independent mean zero gaussians with mean zero and variance r^2. Find the optimal feedback parameter K to minimize

$$C = \mathrm{E}_{\mathrm{ss}}\left[X^2\right] \ .$$

Note that we cannot achieve $C = 0$ even if the control is free.

5. Suppose that $\vec{y}_t \in \mathbb{R}^n$ is some time dependent vector. A *backward process* for the $n \times n$ matrix A (assume A is non-singular) is an $\vec{x}_t \in \mathbb{R}^n$ that satisfies $\frac{d}{dt}\vec{x} = A\vec{x}$. Suppose that $\langle y_t, x_t \rangle$ is independent of time for every backward process $\vec{x}$. Show that y is a *forward process* in the sense that $\frac{d}{dt}\vec{y} = A^*\vec{y}$. Here, A^* is the adjoint of A in the sense that $\langle \vec{y}, A\vec{x} \rangle = \langle A^*\vec{y}, \vec{x} \rangle$ for every pair $\vec{y}$ and $\vec{x}$. Use only properties of the inner product (symmetry, bi-linear), and not a specific inner product.