Stochastic Calculus, Courant Institute, Fall 2018

http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2018/StochCalc.html

Always check the `classes` message board before doing any work on the assignment.

Assignment 4, due October 9

Corrections: [none yet]

1. Suppose f_t is a continuous and deterministic function of t and W_t is Brownian motion Consider the random variable

$$X_T = \int_0^T f_t dW_t \ .$$

Show that X_T is Gaussian and find its mean and variance. (Hint: the approximations $X_T^{(n)} = \sum f_{t_k} \Delta W_k$ are Gaussian (why?). If a family of Gaussians converges almost surely to something, that thing is Gaussian (why?)) Philosophy: in finite dimensions, any linear function of a Gaussian is Gaussian. You can think of Brownian motion as an infinite dimensional (i.e., fancy) Gaussian and the integral is a linear function of it.

2. Consider the ordinary differential equation

$$\frac{d}{dt} x_x = -\gamma x_t + g_t$$

This may be solved using the method of integrating factors:

$$\begin{aligned} &e^{\gamma t} \frac{d}{dt} x_t + \gamma e^{\gamma t} x_t = e^{\gamma t} g_t \\ &\frac{d}{dt} \left(e^{\gamma t} x_t \right) = e^{\gamma t} g_t \\ &\qquad \text{integrate from 0 to } T, \text{ multiply by } e^{-\gamma T} \\ &x_T = e^{-\gamma T} x_0 + \int_0^T e^{-\gamma (T-t)} g_t \, dt \ . \end{aligned}$$

(a) Use a related calculation with Ito's lemma to find a formula for X_T, which satisfies the OU stochastic differential equation $dX_t = -\gamma X_t + \sigma dW_t$. Your formula should express X_T as a function of X_0 and an integral involving $W_{[0,T]}$.

(b) Use the result of part (a) and Problem (1) to characterize the probability density of X_T. Show explicitly from this that the density of X_T converges to a limit as $T \to \infty$. Find the limit and show that it has variance $\sigma^2 / 2\gamma$.

(c) Calculate $C(s) = \text{cov}_\infty(X_{t+s}, X_t)$. The subscript cov_∞ means the covariance assuming both X_t and X_{t+s} have the limiting distribution, which (this Problem shows) has mean zero and variance $\sigma^2/2\gamma$. Hint: if $s > t$ you can write $X_{t+s} = a(s)X_t + \textit{noise}$, where the noise part is a stochastic integral that is independent of X_t. What about the case $s < t$? Is it different?

(d) Assume that $X_0 = 0$. Consider the Riemann integral

$$Y_T = \int_0^T X_t \, dt \ .$$

Find a formula for D in terms of γ and σ^2 so that for large T

$$\text{E}\left[Y_T^2 \right] \approx DT \ .$$

This is a famous formula of Einstein, D is the *Einstein diffusion coefficient*. We will see why D is a diffusion coefficient in a later Lesson. Hint: you can express the answer as a double integral, just as we expressed the expected square of a sum as a double sum. Take the expectation inside the double integral and use the formula from part (c).

3. This is an exercise in using the Ito calculus to find *cancellation* in an expectation. Define

$$M_t = \text{E}[\cos(kW_t)] \ .$$

The parameter k is the *wave number*. Large k makes $\cos(kw)$ a rapidly oscillating function of w.

(a) Use the Ito calculus to find a differential equation of the form

$$\frac{d}{dt} M_t = \text{something involving } k \text{ and } M_t.$$

(b) Use this to show that M_t is an exponentially decreasing function of t, and that the exponential decay rate is faster when k is larger.

(c) We know that $W_t \sim \mathcal{N}(0, t)$. Write the corresponding integral formula for M_t, calculate the integral, and see that it agrees with your answer from part (b). To calculate the integral, note that

$$\cos(y) = \frac{e^{iy} + e^{-iy}}{2}$$

and that (completing the square in the exponent)

$$\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} e^{ibx} \, dx = C(a,b) \int_{-\infty}^{\infty} e^{-\frac{a(x-z)^2}{2}} \, dx \ .$$

Why is the integral on the right independent of z even when z is complex? (If $X \sim f$ is a random variable with density f, then

$\chi(k) = \mathrm{E}\left[e^{ikx} \right] = \int f(x)e^{ikx}\, dx$ is the *characteristic function* of X. The integral is a *Fourier transform*. The Fourier transform of a Gaussian turns out to look a lot like a Gaussian function of k.)

(d) An expectation has *cancellation* when the expectation is much smaller than typical values of the random variable. This happens when the "positive parts" of X are roughly as likely (in a weighted sense) as the negative parts. Give an informal explanation of the smallness of M_t for large t or large k, possibly using a graph.

4. Let $X_T = \int_0^T W_t dW_t$. Compute $\mathrm{E}\left[X_T^2 \right]$ directly from the Ito isometry formula. Also compute $\mathrm{E}\left[\left(\frac{1}{2}W_T^2 - \frac{1}{2}T \right)^2 \right]$. If you do both calculations correctly, the results should be the same.