Assignment 3, due October 2

Corrections: [none yet]

1. Integration by parts is generally different for Ito integrals. But only if the integrand is "rough", like a diffusion path. Show that if F_t is differentiable in the sense that
$$F_{t+\Delta t} - F_t = \frac{dF_t}{dt}\Delta t + O(\Delta t^2)$$

2. Suppose X_t is a function of t, random or not. The *quadratic variation* of X up to time T is
$$Q_T(X) = \lim_{n\to\infty} \sum_{t_k<T} \left(X_{t_{k+1}} - X_{t_k}\right)^2 \ .$$
If the limit does not exist the quadratic variation is not defined. In this limit, $h_n = 2^{-n}$ and $t_k = kn_n$, as in class and the notes. Assume that X_t is a Lipschitz continuous function of t, which means that there is a C so that $|X_{t+\Delta t} - X_t| \leq C\,|\Delta t|$ as long as $t \geq 0$ and $t + \Delta t \geq 0$. Show that the quadratic variation of a Lipschitz continuous function is zero.

3. Let W_t be a standard Brownian motion. This means that future increments $W_{t+s} - W_t$ are independent of $W_{[0,t]}$, and that they are Gaussian with mean zero and variance s (assume $t \geq 0$ and $s > 0$). This exercise gives a proof that $Q_T(W) = T$. This includes a proof that the limit exists. Define
$$V_k = \left(W_{t_{k+1}} - W_{t_k}\right)^2 - h_n \ .$$
Define
$$Q_T^{(n)}(X) = \sum_{t_k<T} \left(X_{t_{k+1}} - X_{t_k}\right)^2 \ .$$
Then $Q_T(W) = T$ is the same as showing that $Q_T^{(n)}(W) - T \to 0$ as $n \to \infty$. This is the same as showing
$$\left(\sum_{t_k<T} V_k\right) \to 0 \ , \quad \text{as } n \to \infty \ .$$
   (a) Prove a version of the Borel Cantelli lemma for this. Suppose $R^{(n)} \geq 0$ is a family of random variables and
$$\sum_{n=1}^{\infty} \mathrm{E}\Big[\, R^{(n)} \Big] < \infty \ ,$$

Then $R_n \to 0$ as $n \to \infty$ almost surely. (Hint: if $R^{(n)} \geq 0$ and $\sum R^{(n)} < \infty$, then $R^{(n)} \to 0$ as $n \to \infty$.)

(b) Take $R^{(n)} = \left(Q^{(n)} - T\right)^2$. Show that

$$\mathrm{E}\Big[\, R^{(n)} \Big] = \sum_{t_j < T} \sum_{t_k < T} \mathrm{E}[\, V_j V_k] \ .$$

This uses some properties of Brownian motion. Show that if $j \neq k$ then $\mathrm{E}[\, V_j V_k] = 0$.

(c) Show that $\mathrm{E}\big[\, V_k^2 \big] = 2h_n^2$.

(d) Show that the sum in part(a) is finite and finish the argument.

4. Repeat the steps of problem 3 to show that if X_t is a diffusion, then

$$Q_t(X) = \int_0^T v(X_t)\, dt \ .$$

5. Show that

$$\int_0^T W_t dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T \ .$$

Hint: in the approximation, use the identity

$$W_k \left(W_{k+1} - W_k\right) = \frac{1}{2}\left(W_{k+1}^2 - W_k^2\right) - \frac{1}{2}\left(W_{k+1} - W_k\right)^2 \ .$$

Then use problem 3. Show by explicit calculation that this Ito integral is a martingale $(W_t^2 - t)$.

6. Consider the incorrect approximation

$$S_T^{(n)} = \sum_{t_n < T} W_{t_{k+1}} \left(W_{t_{k+1}} - W_{t_k}\right) \ .$$

Show that $S_T^{(n)}$ converges to $W_T + \frac{1}{2}T$ as $n \to \infty$. Hint: find a version of the trick of problem 5.

7. (Read the last section of Lesson 3 before working on this). The SDE $dS = \sigma S dW_t$ has solution $S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$.

(a) Show that $\mathrm{E}[S_t] = S_0$ for all $t > 0$.

(b) Show that for any $\epsilon > 0$,

$$\Pr(S_t > \epsilon) \to 0 \quad \text{as} \quad t \to \infty \ .$$

This is called convergence to zero *in probability.*

(c) Let s be a positive "speed" and let A_n be the event $\tau_{sn} \geq n$. Here τ_a is the first hitting time for a, that is, $\tau_a = \min\{t \text{ so that } W_t = a\}$ Show that if $\tau_{sn} > n$ for all $n > M$, then $W_t < st$ for all $t > M$.

(d) Use the hitting time PDF formula from an earlier homework to show that

$$\sum_n \Pr\left(\tau_{sn} < n\right) < \infty \ .$$

(e) Show that $S_t \to 0$ as $t \to \infty$ almost surely.

(f) (not an action item – nothing to hand in for this) Note that it is harder and more subtle to prove almost sure convergence than convergence in probability. In statistics (for example, hypothesis testing), there are dangerous examples of sequences of indicator functions of events (hypothesis tests) that go to zero in probability but not almost surely. Even though $\Pr(A_n) \to 0$ as $n \to \infty$, still A_n happens infinitely often.