Assignment 2, due September 24

Corrections: [none yet]

1. Suppose times $T_1, \ldots, T_M$ are chosen randomly, independently, and with a uniform density in the interval $[0, R]$. Suppose $\lambda > 0$ is a *rate* parameter and we choose $M = \lambda R$ and take the limit $R \to \infty$ (or $R = M/\lambda$ with $M \to \infty$). Define $T_{[1]} = \min T_k$, $T_{[2]}$ as the smallest $T_k > T_{[1]}$, and so on. Then $\{T_1, \ldots, T_N\} = \left\{T_{[1]}, \ldots, T_{[N]}\right\}$ and $T_{[1]} < T_{[2]} < \cdots$. (The probability of two times being equal is zero.) As $R \to \infty$, the increasing sequence $T_{[k]}$ converges to a Poisson process with rate λ. The number of arrivals up to time t is
$$N_t = \#\left\{T_k < t\right\} \ .$$
   (a) Show that in the limit $R \to \infty$, we have
   $$\Pr(N_t = n) = \frac{t^n e^{-\lambda t}}{n!} \ .$$
   This is called the *Poisson* random variable.

   (b) Show that the probability density of $T_{[1]}$ converges to $u_1(t) = \lambda e^{-\lambda t}$. Hint: it may be easier to calculate the limit as $R \to \infty$ of $\Pr(N_t = 0)$.

   (c) Show that (in the Poisson limit $R \to \infty$), and $p = 1,\ 2,\ 3,\ 4$,
   $$\mathrm{E}[\,(\Delta N)^p] = \lambda \Delta t + O(\Delta t^2) \ .$$
   Here, $\Delta N = N_{t+\Delta t} - N_t$. Conclude that N_t is not a diffusion.

2. Consider the Ornstein Uhlenbeck problem (5). The transition density is[1]
$$X_{t_0+t} \sim \mathcal{N}\left(e^{-\gamma t} X_{t_0},\, \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)\right) \ .$$
If you know X_{t_0}, the conditional mean at time $t_0 + t$ is $X_{t_0} e^{-\gamma t}$ and the conditional variance is $\frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)$. The conditional distribution is Gaussian with those parameters.

   (a) Show that this is consistent in the following sense. Suppose you start at X at time $t_0 = 0$ (to simplify the notation only) and go to Y at time t using the transition distribution given. Then you start at Y and go to Z time $t + s$ using the transition distribution for time s

[1] We will derive this easily in a future lesson. For now, please just accept that it is true.

starting at Y. The result (if it's consistent) is that Z comes from X with the transition distribution given for time $t+s$. Hint: one way to do this is to calculate integrals using Gaussian probability density formulas. Another way is to use the following trick: if $Y \sim \mathcal{N}(\mu, s^2)$, then Y may be represented as $Y = \mu + s\xi$, where $\xi \sim \mathcal{N}(0,1)$. In the Ornstein Uhlenbeck case, this becomes

$$Y = e^{-\gamma t} X + \left[\frac{\sigma^2}{2\gamma} \left(1 - e^{-2\gamma t}\right) \right]^{\frac{1}{2}} \xi \ ,$$

$$Z = e^{-\gamma s} Y + \left[\frac{\sigma^2}{2\gamma} \left(1 - e^{-2\gamma s}\right) \right]^{\frac{1}{2}} \eta \ .$$

(b) Show that a process with this transition density has the infinitesimal mean and infinitesimal variance of Ornstein Uhlenbeck ($a = -\gamma x$, and $v = \sigma^2$).

(c) Show that the fourth moment satisfies

$$\mathrm{E}\Big[(\Delta X)^4 \, | \mathcal{F}_t \Big] = O\left(\Delta t^2\right) \ .$$

(d) Find the following value function directly

$$f(x,t) = \mathrm{E}_{x,t}\left[X_T^2 \right]$$

(hint: you can get this from the transition distribution). Show that the this f satisfies the backward equation for Ornstein Uhlenbeck and that it satisfies the final condition $f(x,T) = x^2$.

3. Let W_t be standard Brownian motion and consider the formula

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t} \ .$$

Show by direct calculation (not by the Ito calculus, if you know it) that the infinitesimal mean and variance are those of the geometric Brownian motion (6). That is

$$\mathrm{E}[\, \Delta S \mid \mathcal{F}_t] = \mu S_t \Delta t + O(\Delta t^2) \ ,$$

and

$$\mathrm{E}\Big[\, (\Delta S)^2 \mid \mathcal{F}_t \Big] = \sigma^2 S_t^2 \Delta t + O(\Delta t^2) \ ,$$

and

$$\mathrm{E}\Big[\, (\Delta S)^4 \mid \mathcal{F}_t \Big] = O(\Delta t^4) \ .$$

Conclude that this formula defines a geometric Brownian motion. You may use the formula

$$e^{\sigma \Delta W} = 1 + \sigma \Delta W + \frac{1}{2} \sigma^2 \Delta W^2 + \frac{1}{6} \sigma^2 \Delta W^3 + O(\Delta W^4) \ .$$

This is not actually true, but it's off only by stuff so technical that only a really pure mathematician would object. If you feel like doing the calculation actually correctly, fine.