Assignment 1, due September 17

Corrections: [none yet]

1. Let X_t be the standard Brownian motion.
   (a) Calculate $\mathrm{E}\big[X_t^4\big]$.
   (b) The formula $\mathrm{E}\big[X_t^2\big] = t$ is thought of as representing the fact that $|X_t|$ is on the order of $\sqrt{t}$. Can you get a similar picture from the fourth moment calculation of part a?

2. Integrate the formula (4) for u_2 over the variable x_1 to see that the u_2 formula is consistent with $u_1(x_2, t_2) = (2\pi t_2)^{-1/2}\, e^{-\frac{x_2^2}{2t_2}}$.

3. Repeat the fourth moment calculation to calculate (assume $0 \le t < T$)
$$f(x,t) = \mathrm{E}\big[\, X_T^4 \mid X_t = x \big] \ .$$
   Calculate the appropriate partial derivatives of f to see that it satisfies the backward equation (11).

4. Repeat exercise 3 for the function
$$f(x,t) = \mathrm{E}\big[\, e^{aX_T} \mid X_t = x \big] \ .$$
   Show that this value function f also satisfies the backward equation (11).

5. Show that $\mathrm{E}[\, \tau_a] = \infty$ by showing that the integral that defines the expectation diverges.

6. The hitting probability is $\int_0^\infty v_a(t)\, dt$. An event happens *almost surely* if its probability is equal to one. Show that a Brownian motion particle hits $x = a$ almost surely, for any $a > 0$. Hint: calculate the integral using the change of variables $t = s^{-\frac{1}{2}}$.

7. The *stopped process* Y_t is defined by
$$Y_t = \begin{cases} X_t & \text{if } t < \tau_a \\ a & \text{if } t \ge \tau_a \end{cases}$$
   The stopped process is a Brownian motion that "sticks" the first time it touches $x = a$.
   (a) Show that $Y_t \to a$ as $t \to \infty$ almost surely

(b) Show that $\mathrm{E}[Y_t] = 0$ for all $t > 0$. Hint: Show that $\frac{d}{dt}E[Y_t] = av_a(t) + \int_{-\infty}^{a} x\partial_t u(x,t)dx$.

These two calculations show that the limit of $E[Y_t]$ is not equal to the expected value of the limit of Y_t.

8. Here is an example related to the phenomenon of Problem 7. A random variable S is *log-normal* if $X = \log(S)$ is normal. For any *volatility* parameter, σ, there is an m so that $S = e^{\sigma Z - m}$ has $\mathrm{E}[S] = 1$. Here, and many times in this class, we use Z to represent the *standard* normal random variable. That means $Z \sim \mathcal{N}(0,1)$, which is the Gaussian distribution with mean zero and variance one.

   (a) Find this relation

   (b) Suppose $\epsilon > 0$. Show that $\Pr(S > \epsilon) \to 0$ as $\sigma \to \infty$. Hint: Show that $S > \epsilon$ is equivalent to $Z > r$, where r is some number that depends on σ and ϵ. Show that $r \to \infty$ as $\sigma \to \infty$.

   Thus, the random variables S_σ all have expected value equal to one even though the probability mass is concentrating at zero. The probability density of S_σ must be forming long tails on the positive side to balance the mass near zero.