Assignment 6, due November 2

Corrections (check the class message board): (none yet.)

1. Suppose u_t is a smooth function of t. Show that

$$\int_0^T u_t^2 \, du_t = \frac{1}{3} u_T^3 \; .$$

Use Ito's lemma to show that this is not true for Brownian motion

$$\int_0^T W_t^2 \, dW_t \neq \frac{1}{3} W_T^3 \; .$$

With the result of Ito's lemma, write the Ito integral on the left as the sum of $\frac{1}{3}W_T^3$ and a Riemann integral.

2. Use the independent increments property and explicit Gaussian calculations to calculate

$$f(w,t) = \mathrm{E}_{x,t}\left[W_T^4\right] \; .$$

Show that your calculated f satisfies the backward equation

$$\partial_t f + \frac{1}{2}\partial_x^2 f = 0 \; ,$$

and the final condition $f(w,T) = w^4$.

3. Suppose X_t is an Ornstein Uhlenbeck process that satisfies the SDE

$$dX_t = -\gamma X_t \, dt \; + \; \sigma dW_t \; .$$

(a) Write the backward equation satisfied by

$$f(x,t) = \mathrm{E}_{x,t}[V(X_t)] \; .$$

(b) Show that this PDE has solutions of the form

$$f(x,t) = A(t) e^{-B(t)x^2} \; .$$

Find the differential equations A and B must satisfy in order that f satisfies the backward equation of part (a).

(c) Suppose $V(x) = e^{-x^2}$. Find the behavior of $A(0)$ and $B(0)$ as $T \to \infty$.

(d) Find a representation of the solution in the form

$$X_t = e^{-\gamma t} X_0 + \int_0^t M_{t,s}\, dW_s \ .$$

(e) Find the steady state probability density $\pi(x)$ that has the property that if $X_0 \sim \pi$ then $X_t \sim \pi$. Hint: π is Gaussian. You may assume that s is only slightly larger than t. Part (d) may be helpful.

(f) Calculate $\mathrm{E}_\pi\left[e^{-X^2}\right]$ and show that this is consistent with the calculations of part (c) assuming that the distribution of X_T converges to π as $T \to \infty$.

4. Let W_t be standard Brownian motion and consider

$$f(w,t) = \mathrm{Pr}_{w,t}(W_s > 0 \ \text{ for all } \ t \leq s \leq T) \ .$$

Find the boundary condition that f satisfies as $w \to 0$ and the final condition $f(w,T)$, for $w > 0$. Find the solution to this problem using the method of images. Find a final condition for $f(w,T)$ for all w that insures that f satisfies the boundary condition. Verify that this calculation is consistent (gives the same answer) with the hitting time calculations we did earlier using the forward equation.