Assignment 5, due November 2

Corrections (check the class message board): (none yet.)

1. Let X_t be a standard Brownian motion. Assume $X_0 = 0$. The maximal function is
$$M_t = \max_{0 \le s \le t} X_s \ .$$
Let $f(m,t)$ be the PDF of M_t. For any $T > 0$, you can define a *time re-scaled* Brownian motion path $Y_t = T^p X_{t/T}$.
   (a) Find the power p that makes the re-scaled process Y_t a standard Brownian motion.
   (b) Use the scaling law for Y and an appropriate T to find a formula for the function $f(m,t)$ in terms of $f(m,1)$. Do this without finding a formula for $f(m,1)$ of $f(m,t)$.
   (c) Use the result of part (b) to show that f has the *scaling* form
$$f(m,t) = \frac{1}{\sqrt{t}} h(\sqrt{t}\, m) \ .$$
for some function $h(m)$. Do not find this function. Hint: the overall scaling factor $\frac{1}{\sqrt{t}}$ is necessary because $f(m,t)$ is a PDF as a function of m for each t.
   (d) Find the explicit formula for $f(m,t)$. Show that it has the *scaling* form of part (c).

2. Suppose X_t is standard Brownian motion and
$$Y_t = \int_0^t X_s \, dx \ .$$
Calculate $\text{cov}(Y_t, Y_s)$. Hint: if $t > s$, you can write $Y_t = Y_s + (s-t)X_s + **??*$, the last part being independent of $X_{[0,s]}$.

3. (*computing*) One reason to use a PDE for the PDF of X_t is that we can find the solution of a PDF to high accuracy without using Monte Carlo or simulation. This exercise explores using a finite difference method to solve the heat equation
$$\partial_t u = \frac{1}{2} \partial_x^2 u \ .$$
We suppose this is supplemented with boundary conditions that correspond to absorbing boundaries at $x = 0$ and $x = L$.

(a) Show that if $u(x,t)$ is a 4 times differentiable function of x, and if $\Delta x > 0$, then

$$\partial_x^2 u(x,t) = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} + O(\Delta x^2) \ .$$

Show that if $u(x,t)$ is a twice differentiable function of t, and if $\Delta t > 0$, then

$$\partial_t u(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t) \ .$$

(b) Use this to show that if u satisfies the heat equation, and if u is sufficiently differentiable, then

$$u(x,t+\Delta t) = a\,u(x,t-\Delta a) + b\,u(x,t) + c\,u(x+\Delta x,t) + \Delta t O(\Delta x^2 + \Delta t) \ . \tag{1}$$

Find formulas for a, b, and c as functions of Δt and Δx. Find the inequality $\Delta t \leq ?**?(\Delta x)$ that is equivalent to $a \geq 0$, $b \geq 0$, and $c \geq 0$.

(c) Consider a *lattice* (or *grid*) of points in space time of the form $(x_j, t_k) = j\Delta x, k\Delta t)$, where $x_n = L$. Define approximate values

$$u(j,k) \approx u(x_j, t_k) \ .$$

Define satisfy the initial conditions

$$u(j,0) = u(x_j, 0) \ ,$$

and the absorbing boundary conditions

$$u_{0,k} = u(0,t_k) = u_{n,k} = u(L,t_k) = 0 \ .$$

Write a formula for $u_{j,k+1}$ in terms of a, b, and c, and the values $u_{j-1,k}$, $u_{j,k}$, and $u_{j+1,k}$, which are derived from part (b) by neglecting the error term $\Delta t O(\Delta x^2 + \Delta t)$. If u_k is the vector in $\mathbb{R}^n$ with components u_{jk}, show that these equations determine u_{k+1} from u_k.

(d) Show that the heat equation with these boundary conditions has an exact solution of the form $u(x,t) = A(t)\sin(\frac{\pi x}{L})$.

(e) Write a script in `R` that evaluates the difference equations of part (c) starting from initial conditions $u(x,0) = \sin(\frac{\pi x}{L})$. Choose a sequence of n values that start small and get larger. Use the relation between Δt and Δx found in part (b). Plot the solutions at time $t = 1$ along with the exact solution from part (d). These plots should demonstrate that the finite difference approximations converge to the exact solution as $n \to \infty$.

(f) Apply the script from part (f) to the initial conditions $u(x,0) = 1$ if $0 < x < 1$, and $u(x,0) = 0$ if $1 \leq x < L$. Choose a reasonably large value of n and one or two interesting values of L. Make some plots to illustrate the following facts about the solution.

i. The initial discontinuities at $x = 0$ and $x = 1$ are quickly transformed to rapid transitions at positive t.

ii. The solution converges to zero at $t \to \infty$ at the same exponential rate as the solution of part (d).

iii. The "spatial structure" of the solution converges to $\sin(\frac{\pi x}{L})$ (scaled) as $t \to \infty$.