Assignment 4, due October 26

Corrections (check the class message board): (none yet.)

1. For Brownian motion, the distribution of X_{t_s} conditional on $\mathcal{F}_t$ is $\mathcal{N}(X_t, s-t)$. From this it is possible to show that if $X_t \sim u(x,t)$, then u satisfies the heat equation. The *Ornstein Uhlenbeck* process is defined as the limit as $\delta \to 0$ (in the sense of weak convergence of distributions) of the approximations
$$X^{(\delta)}_{t_{k+1}} = (1-\delta)X^{(\delta)}_{t_k} + \sqrt{\delta} Y_k \ .$$
Here, as before, $t_k = k\delta$ and the Y_k are iid $\mathcal{N}(0,1)$.

   (a) Suppose that $X^{(\delta)}_0 = 0$ and show that $X^{(\delta)}_{t_k}$ is Gaussian.

   (b) Assume that $X^{(\delta)}_t \xrightarrow{d} X_t$. Show that the distribution of X_{t_s} conditional on $\mathcal{F}_t$ is $\mathcal{N}(e^{-t}X_t, \sigma^2(s-t))$. Find the formula for $\sigma^2(s-t)$

   (c) Use this to write a formula of the form
$$X_s = e^{-(s-t)}X_t + \text{ independent Gaussian} \ .$$

   (d) Use that to discover the PDE that the PDF of X_t satisfies. It has $\frac{1}{2}\partial_x^2 u$ and another term.

2. (*computing*) Write Brownian motion simulations in `R` to verify the two formulas we did in class for Brownian motion hitting and survival. Suppose X_t is a standard Brownian motion with $X_0 = 0$ and $\tau_a = \min\{t | X_t = a\}$. The PDF of τ is
$$f(t) = \frac{1}{\sqrt{2\pi t^3}}\, e^{-\frac{a^2}{2t}} \ .$$
The PDF of surviving particles (those with $\tau_a > t$) is
$$u_a(x,t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{x^2}{2t}} - e^{-\frac{(2a-x)^2}{2t}} \right) \ .$$
Do simulations to make a large number of sample paths and make histograms of the hitting times and of the density of surviving particles and plot them against the theoretical values. To simulate a Brownian motion path, choose a small δ and use the scaling formula
$$X^{(\delta)}{}_t = \sqrt{\delta} \sum_{k\delta < t} Y_k \ .$$

In practice, you can just get values at the times $t_k = k\delta$. Assuming $a < 0$, the simulated hitting time can be

$$\tau_a^{(\delta)} = \min \left\{ t_k | X_{t_k}^{(\delta)} < a \right\} \ .$$

It should not matter what the distribution of the Y_k is, but an easy one is $Y_k \sim \mathcal{N}(0,1)$. To get good agreement with theory, you may have to take a rather small δ. You probably can take $a = -1$, but you may want to try other values. There is no template program to download, but you have done all the things here before in some form.