Week 8
Diffusion processes, part 2

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1 Integration and Ito's lemma for dX_t

`sec:il`

Outline of this section:

1. Ways to define new processes using old ones:
   (a) An Ito integral with respect to an old process, $Y_t = \int_0^t a_s dX_s$
   (b) A function of an old process, $Y_t = f(X_t, t)$
2. Figure out stuff about the new processes
   (a) Ito's lemma
3. It is more useful when it is more general
   (a) General Ito process, not just diffusions
   (b) Multi-component processes

1.1 Technical overview

A *diffusion* is a process that satisfies an SDE, which makes it a Markov process. The operations listed above produce processes that have Ito differentials but are not Markov. We call X_t an *Ito process* if (in these formulas, $\Delta t > 0$ and $\Delta X = X_{t+\Delta t} - X_t$):

$$\mathrm{E}[\,\Delta X \mid \mathcal{F}_t] = a_t \Delta t + o(\Delta t) \tag{1}$$ `eq:im`

$$\mathrm{E}\Big[\,(\Delta X)^2 \mid \mathcal{F}_t\Big] = \mu_t \Delta t + o(\Delta t) \quad \text{(one component)} \tag{2}$$ `eq:iv`

$$\mathrm{E}\Big[\,(\Delta X)\,(\Delta X)^t \mid \mathcal{F}_t\Big] = \mu_t \Delta t + o(\Delta t) \quad \text{(multi-component)} \tag{3}$$ `eq:ivm`

$$\mathrm{E}\Big[\,|\Delta X|^4 \mid \mathcal{F}_t\Big] \le C \Delta t^2 \tag{4}$$ `eq:m4`

The *little oh* notation $A = o(\Delta t)$ means that A is smaller than Δt. More precisely, $A/\Delta t \to 0$ as $\Delta t \to 0$. It is usually correct to think $O(\Delta t^2)$ whenever you see $o(\Delta t)$. That's what it would be if it were a Taylor series expansion. But it is convenient to make do with an error estimate that is weaker than $O(\Delta t^2)$. That way you don't waste time proving that the error is very small, when it only matters that the error is $o(\Delta t)$. If X_t is a Markov process (i.e., a diffusion), then the infinitesimal mean and variance are functions of X_t. We may write $a_t = a(X_t, t)$, and $\mu_t = \mu(X_t, t)$ in that case.

A diffusion process must be Markov and satisfy an SDE. An Ito process just needs to satisfy (1), (2) or (3), and (1). For example, suppose $(X_{1,t}, X_{2,t})$ is a two component diffusion process and $Y_t = X_{1,t} + X_{2,t}$. Then Y_t is an Ito process but not a diffusion, because the infinitesimal mean and variance of Y_t are not functions of Y_t alone, but they depend on $X_{1,t}$ and $X_{2,t}$ separately.

There is one thing diffusions do but non-Markov Ito processes do not – satisfy backward equations. The backward equation of last week has $a(x,t)$ and $\mu(x,t)$, not just a_t and μ_t.

The Ito calculus for Ito process is almost the same as it is for Brownian motion, but there is one new technical thing. The general treatment here is a little more complicated, though not much harder, because general Ito processes are not martingales. A general Ito process may be separated into a martingale part, which looks like Brownian motion for our purposes here, and a "smoother" part that can be integrated in the ordinary way.

Ito's lemma for general Ito process has a natural version of the informal rule $(dW_t)^2$ "=" dt. This rule makes sense because $\mathrm{E}\left[\, (dW_t)^2 \mid \mathcal{F}_t \right] = dt$. The general rule is

$$(dX_t)^2 \;\; \text{“="} \;\; \mathrm{E}\left[\, (dX_t)^2 \mid \mathcal{F}_t \right] = \mu(X_t, t) dt \; .$$

This fake "=" is not a true equality because the standard deviation of $(dX_t)^2$ is also of order dt. We can substitute the right side for the left side in an Ito integral because the total effect of the "fluctuations" in $(dX_t)^2$ vanishes in the limit $\Delta t \to 0$.

1.2 Martingale diffusions

By definition, a stochastic process is a *martingale* if, for any $s > 0$,

$$\mathrm{E}[\, X_{t+s} \mid \mathcal{F}_t] = X_t \; . \tag{5}$$

eq:m

This definition applies to general processes, which do not have to have continuous paths or be Markov processes. But for Ito processes, the martingale property (5) is equivalent to $a = 0$, the zero drift property, in (1). This equivalence means that zero drift implies (5), and (5) implies zero drift. The second statement is "obvious", as (5) implies that $\mathrm{E}[\, \Delta X \mid \mathcal{F}_t] = 0$ for any $\Delta t > 0$. The first statement is also "obvious", so much so that we defer the discussion to a later subsection.

If an Ito process is not a martingale, we can separate out the martingale part using an ordinary integral

$$Y_t = X_t - \int_0^t a_s \, ds \; .$$

The increment of Y is $\Delta Y = \Delta X - a_t \Delta t +$ (smaller), so Y_t satisfies the infinitesimal drift equation (1) eq:im with drift coefficient equal to zero. Therefore, we have

$$X_t = Y_t + Z_t \ , \tag{6}$$

eq:md

where Y_t is a martingale and

$$Z_t = \int_0^t a_s \, ds \ . \tag{7}$$

eq:cp

In this decomposition, the drift part of X_t is Z_t, and the quadratic variation of X_t is in Y_t. We can see this by calculating

$$\begin{aligned} \mathrm{E}\Big[(\Delta Y)^2 \mid \mathcal{F}_t \Big] &= \mathrm{E}\Big[(\Delta X - \Delta Z)^2 \mid \mathcal{F}_t \Big] \\ &= \mathrm{E}\Big[(\Delta X)^2 \mid \mathcal{F}_t \Big] + 2\mathrm{E}[\, \Delta X \Delta Z \mid \mathcal{F}_t] + \mathrm{E}\Big[(\Delta Z)^2 \mid \mathcal{F}_t \Big] \\ &= \mu_t \Delta t + 2\mathrm{E}[\, \Delta X \Delta Z \mid \mathcal{F}_t] + a_t^2 \Delta t^2 + O(\Delta t^2) \ . \end{aligned}$$

Cauchy Schwarz handles the middle term on the right:

$$\begin{aligned} |\mathrm{E}[\, \Delta X \Delta Z \mid \mathcal{F}_t]| &\leq \sqrt{\mathrm{E}\Big[(\Delta X)^2 \mid \mathcal{F}_t \Big]} \sqrt{\mathrm{E}\Big[(\Delta Z)^2 \mid \mathcal{F}_t \Big]} \\ &\leq C \sqrt{\Delta t} \sqrt{\Delta t^2} \\ &\leq C \Delta t^{3/2} \ . \end{aligned}$$

The power $3/2$ shows the convenience of writing $o(\Delta t)$ instead of $O(\Delta t^2)$. With some more work, we could get $C\Delta t^2$ instead of $C\Delta t^{3/2}$. But this is unnecessary for the present purpose.

To summarize,

$$\begin{aligned} \mathrm{E}[\, \Delta Y \mid \mathcal{F}_t] &= 0 \\ \mathrm{E}[\, \Delta Z \mid \mathcal{F}_t] &= a_t \Delta t + O(\Delta t^2) \\ \mathrm{E}\Big[(\Delta Y)^2 \mid \mathcal{F}_t \Big] &= \mu_t \Delta t + O(\Delta t^2) \\ \mathrm{E}\Big[(\Delta Z)^2 \mid \mathcal{F}_t \Big] &= O(\Delta t^2) \\ \mathrm{E}\Big[|\Delta Y|^4 \mid \mathcal{F}_t \Big] &\leq C \Delta t^2 \\ \mathrm{E}\Big[|\Delta Z|^4 \mid \mathcal{F}_t \Big] &\leq C \Delta t^4 \end{aligned}$$

We divided X into two pieces Y and Z. The martingale piece is as irregular as X but is a martingale. The regular piece is continuous enough to be treated with ordinary calculus. The definition of the integral is

$$\int_0^t f_s dX_s = \int_0^t f_s dY_t + \int_0^t f_s a_s ds \ . \tag{8}$$

eq:ci

Only the first term on the right needs a definition.

1.3 Ito integral for martingale Ito processes

sec:ii

This subsection defines

$$U_t = \int_0^t f_s dY_s \ ,$$

where Y_s is an Ito process and a martingale. In the notation from the past few weeks:

$$U_t = \lim_{m\to\infty} U_t^{(m)} = \lim_{m\to\infty} \sum_{t_j<t} f_{t_j} \left(Y_{t_{j+1}} - Y_{t_j} \right) \ .$$

Recall that $\Delta t = 2^{-m}$ and $t_j = j\Delta t$. Our Borel Cantelli type lemma implies that the limit exists if

$$\sum_{m=1}^{\infty} \mathrm{E}\left[\left| U_t^{(m+1)} - U_t^{(m)} \right| \right] < \infty \ .$$

We show that in fact the expectations on the right form a geometric series:

$$\mathrm{E}\left[\left| U_t^{(m+1)} - U_t^{(m)} \right| \right] \le C\sqrt{\Delta t} = C\left(\sqrt{2}\right)^{-m} \ .$$

We study $U_t^{(m+1)} - U_t^{(m)}$ as we did when we did integration with respect to Brownian motion. The finer approximation $U_t^{(m+1)}$ is defined using values $f_j = f_{t_j}$ with $t_j = j\Delta t$, and the half points $f_{j+\frac12} = f_{t_{j+\frac12}}$ with $t_{j+\frac12} = (j+\frac12)\Delta t$. The Y_j and $Y_{j+\frac12}$ are defined analogously. The finer approximation is

$$U_t^{(m+1)} = \sum_{t_j<t} \left[f_{j+\frac12}\left(Y_{j+1} - Y_{j+\frac12} \right) + f_j \left(Y_{j+\frac12} - Y_j \right) \right]$$

Subtracting $U_t^{(m)}$ from this gives the thing we need to estimate:

$$U_t^{(m+1)} - U_t^{(m)} = \sum_{t_j<t} R_j \ ,$$

with

$$R_j = \left(f_{j+\frac12} - f_j \right)\left(Y_{j+1} - Y_{j+\frac12} \right) \ .$$

The expected square is the sum of diagonal and off diagonal terms. For each off diagonal term with $j > k$ there is an equal term with $j < k$. A factor of 2 in the off diagonal sum takes that into account. We have

$$\mathrm{E}\left[\left(U_t^{(m+1)} - U_t^{(m)} \right)^2 \right] = \sum_{t_j<t} \mathrm{E}\left[R_j^2 \right] + 2\sum_{t_j<t} \sum_{t_j<t_k<t} \mathrm{E}\left[R_k R_j \right] \ .$$

The off diagonal terms $\mathrm{E}[R_k R_j]$ with $t_k > t_j$ vanish because

$$\mathrm{E}\left[R_k R_j \mid \mathcal{F}_{t_{j+\frac12}} \right] = \mathrm{E}\left[Y_{k+1} - Y_{k+\frac12} \mid \mathcal{F}_{t_{k+\frac12}} \right] \left(f_{k+\frac12} - f_k \right) R_j = 0 \ .$$

The first factor on the right vanishes because Y is a martingale.

The diagonal sum is also as it was in the Brownian motion case.

$$\begin{aligned}
&\mathrm{E}\left[\left(Y_{j+1}-Y_{j+\frac{1}{2}}\right)^2\left(f_{j+\frac{1}{2}}-f_j\right)^2\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\left(Y_{j+1}-Y_{j+\frac{1}{2}}\right)^2 \mid \mathcal{F}_{j+\frac{1}{2}}\right]\left(f_{j+\frac{1}{2}}-f_j\right)^2\right] \\
&\leq C\Delta t\,\mathrm{E}\left[\left(f_{j+\frac{1}{2}}-f_j\right)^2\right] \\
&\leq C\Delta t^2 \ .
\end{aligned}$$

Therefore the diagonal sum is

$$\sum_{t_j<t}\mathrm{E}\left[R_j^2\right] \leq C\Delta t\sum_{t_j<t}\Delta t = Ct\Delta t \ .$$

The Cauchy Schwarz inequality, or Jensen's inequality, then gives

$$\begin{aligned}
\mathrm{E}\left[\left|U_t^{(m+1)}-U_t^{(m)}\right|\right] &\leq \sqrt{\mathrm{E}\left[\left(U_t^{(m+1)}-U_t^{(m)}\right)^2\right]} \\
&\leq \sqrt{Ct\Delta t} \ .
\end{aligned}$$

This proves the existence of the Ito integral with respect to a martingale diffusion.

Now that the Ito integral is defined, what are its properties? If $U_t = \int_0^t f_s dX_s$, then $dU_t = f_t dX_t$. If X_t is an Ito process that satisfies (1) eq:im and (2) eq:iv, then U_t is also an Ito process U_t, and the latter has infinitesimal mean and variance

$$\begin{aligned}
\mathrm{E}[\,dU_t \mid \mathcal{F}_t] &= f_t\,\mathrm{E}[\,dX_t \mid \mathcal{F}_t] = f_t a_t dt \ , \\
\mathrm{E}\left[(dU_t)^2 \mid \mathcal{F}_t\right] &= f_t^2\,\mathrm{E}\left[(dX_t)^2 \mid \mathcal{F}_t\right] = f_t^2 \mu_t dt \ .
\end{aligned}$$

1.4 Ito's lemma

Ito's lemma is the tool we use for practical computations involving diffusion processes. It is a stochastic calculus version of the chain rule of ordinary calculus. If X_t is an Ito process, we form another process as a function of X_t and t:

$$Y_t = f(X_t, t) \ . \tag{9}$$

eq:fX

If X_t is an Ito process, then Y_t is one too. We calculate dY_t by Taylor expansion, keeping terms of order dt or larger. The result is

$$dY_t = df(X_t,t) = \partial_x f(X_t,t)dX_t + \frac{1}{2}\partial_x^2 f(X_t,t)\left(dX_t\right)^2 + \partial_t f(X_t,t)dt \ .$$

This is transformed, under the informal "Ito rule" to

$$dY_t = df(X_t,t) = \partial_x f(X_t,t)dX_t + \frac{1}{2}\partial_x^2 f(X_t,t)\mu_t dt + \partial_t f(X_t,t)dt \; . \tag{10}$$

eq:il

This makes Y_t an Ito process with

$$\begin{aligned} \mathrm{E}[\, dY_t \mid \mathcal{F}_t] &= \partial_x f(X_t,t)\, \mathrm{E}[\, dX_t \mid \mathcal{F}_t] + \frac{1}{2}\partial_x^2 f(X_t,t)\mu_t dt \\ &= \left(\partial_x f(X_t,t)a_t + \frac{1}{2}\partial_x^2 f(X_t,t)\mu_t\right) dt \;, \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\Big[\, (dY_t)^2 \mid \mathcal{F}_t\Big] &= (\partial_x f(X_t,t))^2 \; \mathrm{E}\Big[\, (dX_t)^2 \mid \mathcal{F}_t\Big] \\ &= (\partial_x f(X_t,t))^2 \; \mu_t dt \; . \end{aligned}$$

The theorem that is Ito's lemma is that the integrals of the left and right sides of (eq:il 10) are the same. The integral of the left side is the total change in f in time t. Therefore, we prove Ito's lemma by proving the integral identity

$$f(X_t,t) - f(X_0,0) = \int_0^t \partial_x f(X_s,s)dX_s + \int_0^t \left(\partial_t f(X_s,s) + \frac{1}{2}\partial_x^2 f(X_s,s)\right) ds \tag{11}$$

eq:ili

The main part of the proof is a calculation that shows that the fluctuations in $(dX_t)^2$ are indeed negligible in the limit $\Delta t \to 0$.

Use our standard notation: $f_j = f(X_{t_j},t_j)$, and $X_j = X_{t_j}$, and $\Delta X_j = X_{j+1} - X_j$. We use a telescoping representation followed by Taylor expansion

$$\begin{aligned} f(X_t,t) - f(x_0,0) &\approx \sum_{t_j<t} [f_{j+1} - f_j] \\ &= \sum_{t_j<t} [f(X_j + \Delta X_j, t_j + \Delta t) - f(X_j,t_j)] \\ &= \sum_{t_j<t} \left[\partial_x f(X_j,t_j)\Delta X_j + \tfrac{1}{2}\partial_x^2 f(X_j,t_j)\Delta X_j^2 + \partial_t f(X_j,t_j)\Delta t\right] \\ &\quad + \sum_{t_j<t} \left[O\left(|\Delta X_j|^3\right) + O\left(|\Delta X_j|\,\Delta t\right) + O\left(\Delta t^2\right)\right] \\ &= S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \; . \end{aligned}$$

The numbering of the terms is the same as last week. We go through them one by one, leaving the hardest one, S_2, for last.

The first one is

$$S_1 = \sum_{t_j<t} \partial_x f(X_j,t_j)\Delta X_j \to \int_0^t \partial_x f(X_s,s)\, dX_s \quad \text{as } m \to \infty\text{, almost surely} \; .$$

That is the Ito integral that was defined in Section sec:ii 1.3. The third term is

$$S_3 = \sum_{t_j < t} \partial_t f(X_j, t_j)\Delta t \to \int_0^t \partial_t f(X_s, s)\, ds \quad \text{as } m \to \infty \; .$$

People do not feel the need to say "almost surely" when it's an ordinary Riemann sum converging to an ordinary integral. The first error term is S_4. Our Borel Cantelli argument shows that the error terms go to zero almost surely as $m \to \infty$. For example, using familiar arguments,

$$\mathrm{E}[\, S_4] \leq C \sum_{t_j<t} \mathrm{E}\Big[\, |\Delta X|^3 \Big] \leq C \sum_{t_j<t} \Delta t^{3/2} = Ct\Delta t^{1/2} = C_t 2^{-m/2} \; .$$

The sum over m is finite.

Finally, the Ito term:

$$\begin{aligned} S_2 &= \tfrac{1}{2} \sum_{t_j<t} \partial_x^2 f(X_j,t_j)\mu(X_j)\Delta t + \tfrac{1}{2} \sum_{t_j<t} \partial_x^2 f(X_j,t_j) \left[\Delta X_j^2 - \mu(X_j)\Delta t \right] \\ &= S_{2,1} + S_{2,2} \; . \end{aligned}$$

The first sum, $S_{2,1}$, converges to an integral that is the last remaining part of (eq:il1 ??). The second sum goes to zero almost surely as $m \to \infty$, but the argument is more complicated than it was for Brownian motion. Denote a generic term in $S_{2,2}$ as

$$R_j = \partial_x^2 f(X_j,t_j) \left[\Delta X_j^2 - \mu(X_j)\Delta t \right] \; .$$

With this, $S_{2,2} = \sum R_j$, and

$$\mathrm{E}\big[\, S_{2,2}^2 \big] = \sum_{t_j<t} \sum_{t_k<t} \mathrm{E}[\, R_j R_k] \; .$$

The diagonal part of this sum is

$$\sum_{t_j<t} \mathrm{E}\big[\, R_j^2 \big] \; .$$

But $R_j^2 \leq C\left(\Delta X_j^4 + \Delta t^2\right)$, so the diagonal sum is OK. The off diagonal sum was exactly zero in the Brownian motion case because there was no $O(\Delta t^2)$ on the right of (eq:iv1 ??). The off diagonal sum is

$$2 \sum_{t_k<t} \left[\sum_{t_k<t_j<t} \mathrm{E}[\, R_j R_k] \right] \; .$$

The inner sum is on the order of Δt, because

$$\mathrm{E}[\, R_j R_k] = \mathrm{E}[\, \mathrm{E}[\, R_j \mid \mathcal{F}_j]\, R_k] \leq O(\Delta t^2)\, |R_k| \, ,$$

so

$$\sum_{t_k<t_j<t} \mathrm{E}[\,R_j R_k] \leq \left[\sum_{t_j>t_k} O(\Delta t^2) \right] |R_k| \leq C_t \Delta t \; |R_k| \ .$$

You can see from the definition that $\mathrm{E}[\,|R_k|] = O(\Delta t)$. Therefore, the outer sum is bounded by

$$2 \sum_{t_k<t} C_t O(\Delta t^2) = C_t O(\Delta t) \leq C_t 2^{-m} \ .$$

This is what Borel and Cantelli need to show $S_{2,2} \to 0$ almost surely.

1.5 Geometric Brownian motion

Geometric Brownian motion is the solution of the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t \ . \tag{12}$$ eq:gbm

Ito's lemma gives a quick route to the solution formula we had before. If (12) [eq:gbm] were an ODE, the solution would be an exponential. This suggests a log transformation

$$X_t = \log(S_t) \ . \tag{13}$$ eq:lt

We use the general version of Ito's lemma to calculate dX_t. This is a substitution of the form (9) [eq:fX] with X_t for S_t, and X_t for Y_t, and $\log(s)$ for $f(x,t)$. The required derivatives are

$$\log(s) \xrightarrow{\partial_s} \frac{1}{s} \xrightarrow{\partial_s} \frac{-1}{s^2} \quad , \quad \log(s) \xrightarrow{\partial_t} 0 \ .$$

The quadratic variation is $(dS_t)^2 = \sigma^2 S_t^2 dt$. This information in the Ito differential formula (10) [eq:il] gives,

$$\begin{aligned} dX_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 \\ &= \frac{1}{S_t} (\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t \ . \end{aligned}$$

This integrates to

$$X_t = X_0 + \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \ .$$

We undo the log transformation using $S_0 = e^{X_0}$ and $S_t = e^{X_t}$:

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t} \ . \tag{14}$$ eq:gbms

This is the solution formula we had before.

Trading strategies and replication

The Ito calculus can be used to model and design stock trading strategies. One example is the “constant fraction” strategy, in which the investor keeps a constant fraction of her assets in the “risky asset” (the stock) and the rest in the “risk free asset”, a bank account. In this model, the risky asset is a stock whose price is a geometric Brownian motion. If you own n shares of stock, in time dt you receive or pay ndS_t. This assumes that n is constant during this small time interval. The amount of money invested in the stock is $A_t = n_t S_t$. If n is constant during the dt time interval, then

$$\begin{aligned} dA_t &= n_t dS_t \\ &= n_t(\mu S_t dt + \sigma S_t dW_t) \\ &= \mu A_t dt + \sigma A_t dW_t \ . \end{aligned} \tag{15}$$

eq:dA

The number of shares, n, does not have to be an integer.

This model assumes trading (buying or selling stock) is *frictionless*. This means that at any time you can buy or sell any amount of stock at the price S_t per share. The model has no other trading costs (broker fees, exchange fees, bid/ask spread, etc.). At time t, you first decide how much of your money you want to invest in stock, which is A_t. Then you watch the market for time dt and receive dA_t given by (15). You “receive” a negative amount if $dA_t < 0$. Then, at time $t + dt$ you have the chance to *rebalance*, which means choose a different number of shares $n_{t+dt} \neq n_t$. This does not change your total wealth, only your *asset allocation*. The total wealth at time t is Z_t. The amount in the stock is A_t and the amount in the bank is $Z_t - A_t$. We assume that money in the bank earns a *risk free rate*, which is a constant r. In time dt, the change in the bank account is $(Z - A)rdt$. This is called risk free because it is known at time t.

A *trading strategy* is a way to choose A_t in terms of Z_t. A simple way it to invest a fixed fraction of your total wealth in the risky asset. That means $A_t = mZ_t$ with a fixed constant m (“m” is for Merton, who showed that such strategies are optimal in some sense.). If you follow this strategy, then

$$\begin{aligned} dZ_t &= (\text{bank account change}) + (\text{stock account change}) \\ &= (Z_t - A_t)rdt + \mu A_t dt + \sigma A_t dW_t \\ dZ_t &= (r + (\mu - r)m)\, Z_t dt + m\sigma Z_t dW_t \ . \end{aligned} \tag{16}$$

eq:dZ

We see that Z_t is a geometric Brownian motion with expected return $\mu_m = r + (\mu - r)m$ and volatility $\sigma_m = m\sigma$.

We conclude the following. The dynamic fixed-proportion trading strategy produces a stock price process with expected return $\mu_m = r + (\mu - r)m$ and volatility σ_m. If the original μ, σ stock exists in the market, we can *replicate* the more general μ_m, σ_m stock using a dynamic trading strategy.

Other dynamic trading strategies can replicate more complicated financial instruments. The *European style call option* is one example. This is a financial contract that pays $V(S_t)$ at time T, where $v(s) = \max(0, s - K)$. You can think of this as the right to buy the stock at time T for price K. If the stock is worth more than K at that time, you buy it at K then sell it at it's true price S_T and keep the difference. If the stock is worth less than K, you do nothing and get nothing. The option expires *out of the money.*

The dynamic replication strategy says to buy $\Delta(S_t, t)$ shares of stock at time t and put the rest of your money in the bank.

1.6 Multi-component processes

Multi-component processes behave like scalar processes, but with slightly more complicated algebra. If $x \in \mathbb{R}^n$, a function $f(x)$ has a Taylor expansion

$$f(x+\Delta x) - f(x) = \sum_{j=1}^{n} \partial_{x_j} f(x) \Delta x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k + O(|\Delta x|^3) \; .$$

The second derivative part is

$$\frac{1}{2} \sum_{j=1}^{n} \partial_{x_j}^2 f(x) \Delta x_j^2 + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1, k\neq j}^{n} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k \tag{17}$$

eq:d2

Mixed partial derivative terms are double counted in the second sum. It is a mathematical fact that the order of partial differentiation does not matter:

$$\partial_{x_k} \left(\partial_{x_j} f(x) \right) = \partial_{x_j} \left(\partial_{x_k} f(x) \right) \; .$$

Therefore, the off diagonal $j \neq k$ terms satisfy

$$\tfrac{1}{2} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k + \tfrac{1}{2} \partial_{x_k} \partial_{x_j} f(x) \Delta x_k \Delta x_j = \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k \; .$$

Therefore, the second derivative sum may be written in the possibly more familiar way without the $\frac{1}{2}$ in the mixed derivative sum

$$\frac{1}{2} \sum_{j=1}^{n} \partial_{x_j}^2 f(x) \Delta x_j^2 + \sum_{j=1}^{n} \sum_{k=j+1}^{n} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k \; . \tag{18}$$

eq:d2p

I often prefer the redundant form (17) [eq:d2] because it is easier to express in the *Einstein summation convention.* Einstein found himself writing multiple sums over multiple indices (often more than two) and decided to leave out the summation symbol whenever possible. There are various forms of the convention, but here I propose the rule: if an index appears twice or more, you are supposed to sum over that index. For example

$$\sum_{j=1}^{n} \partial_{x_j} f(x) \Delta x_j = \partial_{x_j} f(x) \Delta x_j \; .$$

A bigger example is

$$\frac{1}{2} \sum_{j=1}^{n} \partial_{x_j}^2 f(x) \Delta x_j^2 + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1, k\neq j}^{n} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k = \tfrac{1}{2} \partial_{x_j} \partial_{x_k} f(x) \Delta x_j \Delta x_k \; .$$

The sum on the right contains both diagonal ($j = k$) and off diagonal ($j \neq k$) terms. You may use other compact notations, such as $\text{grad} f = \nabla f$ and $\partial_{x_j} f(x) \Delta x_j = \nabla f(x) \cdot \Delta x$.

The *Laplace operator* is important for functions of Brownian motion. It is defined, with or without the summation convention, by

$$\triangle f(x) = \partial_{x_j} \partial_{x_j} f = \sum_{j=1}^{n} \partial_{x_j}^2 f(x) \; .$$

You will see that the Laplacian symbol $\triangle$ is slightly different from the capital Greek Δ. Handwritten versions will be identical. You will have to guess whether it's $\triangle f$ (the Laplacian of f), or Δf (the change in f).

As an example, here is the Laplacian of the Euclidean distance function

$$\begin{aligned}
|x| \quad &= \quad \left(x_1^2 + \cdots + x_n^2\right)^{1/2} \\
&\xrightarrow{\partial_{x_j}} \frac{1}{2} \left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2x_j \\
&= \quad \left(x_1^2 + \cdots + x_n^2\right)^{-1/2} x_j \\
&\xrightarrow{\partial_{x_j}} \frac{-1}{2} \left(x_1^2 + \cdots + x_n^2\right)^{-3/2} (2x_j)\, x_j + \left(x_1^2 + \cdots + x_n^2\right)^{-1/2} \\
&= \quad \frac{1}{|x|} \left(1 - \frac{x_j^2}{|x|^2}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\triangle |x| &= \sum_{j=1}^{n} \partial_{x_j}^2 |x| \\
&= \frac{n}{|x|} - \sum_{j=1}^{n} \frac{x_j^2}{|x|^3} \\
\triangle |x| &= \frac{n-1}{|x|}
\end{aligned} \qquad (19) \quad \boxed{\texttt{eq:ld}}$$