# Assignment 9, due November 25

**Corrections** (typos in problem 1 fixed.)

1. (*Hard to reverse forward operator*) You can use eigenvalues of generators to figure see that it is hard to run the forward equation backwards. You can do this with the Fourier transform, but this exercise does it with the eigenvalues and eigenvectors associated with linear Gaussian processes. Some of these calculations seem remarkable. These results were discovered by trial and error before the slick but opaque approach here was discovered.

   Consider the Ornstein Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t \; .$$

   We are going to find the eigenvalues and "eigenvectors" of the operator in the forward equation $\partial_t u = L^* u$, where $L$ is the generator of the Ornstein Uhlenbeck process. *Eigenfunctions* play the role of eigenvectors. An eigenfunction and eigenvalue pair is a function $v_n(x)$ and eigenvalue, $\lambda_n$ so that

$$L^* v_n(x) = \lambda_n v_n(x) \; . \tag{1}$$

   If we solve the forward equation with initial data $u(x,0) = v_n(x)$, the solution is clearly $u(x,t) = e^{\lambda t} v_n(x)$. In particular, if $\lambda_n > 0$, then the solution grows as time increases, while the solution decays if $\lambda_n < 0$. We can hope to solve the general *initial value problem* using eigenfunctions and eigenvalues. This would mean writing the initial data as a linear combination of eigenfunctions

$$u(x,0) = \sum_n a_n v_n(x) \; .$$

   Then the solution would be the sum of the individual eigenfunction solutions

$$u(x,t) = \sum_n a_n e^{\lambda_n t} v_n(x) \; . \tag{2}$$

   We don't expect solutions of the forward equation, which can represent probability densities, to grow exponentially as $t \to \infty$. Therefore we expect not to find eigenvalues with $\lambda_n > 0$. Throughout this exercise, we assume that $\sigma^2 = 2\gamma$. This simplifies the eigenfunction formulas.

   (a) Show that $v_0(x) = e^{-x^2/2}$ is an eigenfunction with eigenvalue $\lambda_0 = 0$. Hint: There is a traditional way to do this, but it will be better for the rest of this exercise if you use the fact that $xe^{-x^2/2} = -\partial_x e^{-x^2/2}$.

   (b) Show that for any integer $n$, and any sufficiently differentiable function $w(x)$, we have $\partial_x^n (xw(x)) = x\partial_x^n w(x) + n\partial_x^{n-1} w(x)$.

   (c) Show that $\partial_x (xv_n(x)) = \partial_x^{n+1} e^{-x^2/2} + c_n v_n(x)$. Use this to show that $v_n$ is an eigenfunction of $L^*$ with eigenvalue $\lambda_n = -\gamma n$. Hint: rearrange the terms in part 1b.

(d) Show that if $u(x,0)$ is a probability density, then $u(x,t) \to \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ exponentially fast as $t \to \infty$. Hint: it converges to something exponentially fast because of the eigenvalues. That something must be a probability density if $u(x,0)$ is a probability density because the forward equation is for probability densities.

(e) Show that running the forward equation backwards is unstable in the following way. Suppose

$$u(x,T) = \sum_{n=0}^{\infty} a_n v_n(x)$$

is given. Show that if $u(x,t)$ exists at times $t < T$, it is given by

$$u(x,t) = \sum_{n=0}^{\infty} e^{n(T-t)} a_n v_x(x) \ .$$

Conclude that the backward process is exponentially unstable with arbitrarily large exponents.

(f) Show that if $v_n(x) = \partial_x^n e^{-x^2/2}$, then $v_n$ has the form $v_n(x) = H_n(x)e^{-x^2/2}$, where $H_n(x)$ is a polynomial of degree $n$. (This polynomial is what mathematicians call $H_n$ the *Hermite* polynomial of degree $n$. Physicists have their Hermite polynomials too, but the formula for them is a little different.)

(g) If $L$ is an $n \times n$ matrix, then the eigenvalues of $L$ are the same as the eigenvalues of $L^*$. Show that this is true for the $L$ of this problem. Hint: the eigenfunctions of $L$ are $g_n(x) = H_n(x)$. You may want to use the definition of part (1f) in the form of the*Rogrigues* formula: $H_n(x) = e^{x^2/2}\partial_x^n e^{-x^2/2}$.

(h) Show, formally, that as $T \to \infty$, the value function $f(x,0) = \mathrm{E}_{x,0}[V(X_T)] \to$ *const* as $T \to \infty$. We showed this in an example using the ansatz method two weeks ago. You may assume that

$$V(x) = \sum_{n=0}^{\infty} a_n H_n(x) \ .$$

(i) Repeat with some modifications the argument of part (1e) to show that running the backward equation forward is exponentially unstable with an arbitrarily large exponent.

2. (*Adjoint in a different pairing*) Consider the pairing

$$\langle u, f \rangle_w = \int_{-\infty}^{\infty} u(x) f(x) w(x)\, dx \ .$$

Let $L_w^*$ be the adjoint of $L$ with respect to this pairing. Suppose $L$ is the generator of the Ornstein Uhlenbeck process, and that $\sigma^2 = 2\gamma$, and that $w(x) = e^{-x^2/2}$. Show that $L_w^* = L$.

3. Let $X_t$ be the continuous time process $X_t = rN_t - r\lambda t$, where $N_t$ is a Poisson arrival process with rate $\lambda$. This process drifts down at speed $r\lambda$. It also has Poisson arrivals of size $r$. An arrival comes in time $dt$ with probability $\lambda dt$. Write the generator, $L$, for this process. Show explicitly that $Lx = 0$ (that is, $Lf = 0$ if $f(x) = x$). In general, show that a continuous time Markov process is a martingale if and only if $Lx = 0$.

4. Suppose $dX_t = \mu X_t dt + \sigma X_t dW_t$. We have the formula $X_t = X_0 e^{\sigma W_t + (\mu - \sigma^2/2)t}$. Use this formula to get a formula for $u(x, t)$, which is the probability density of $X_t$ under the condition that $X_0 = 0$. Write the generator, $L$, for this process. Show by explicit calculation that your formula for $u(x, t)$ satisfies the forward equation.