Assignment 8, due November 18

Corrections: Fixed question 2: the value function formula (2) got a complete overhall. The garbled $x_\Delta X$ became the intended $x + \Delta X$. Those who know LaTeX will know that this is a very easy typo to make. Those who can read know it *should* have been an easy typo for me to spot.

1. (*Ornstein Unlenbeck backwards and forwards*) The notes for Week 7 give a solution of the backward equation for an Ornstein Uhlenbeck process. This exercise examines that solution from different points of view.

   (a) The backward equation may be rewritten in a way that makes it more intuitive to mathematicians and physicists with experience working with physical diffusion processes. The *time to go* variable is $s = T - t$. Write the backward equation for $dX_t = a(X_t)dt + b(X_t)dW_t$ in terms of x and s. Show that the final condition in the t variable becomes an initial condition in the s variable. Show that the solution to this equation, for all positive values of s, gives the solution to the backward equation for all final times $T > 0$. Use the time to go variable for the rest of this set of exercise 1. It makes the algebra a little simpler.

   (b) Find the ansatz solution for the case $dX_t = \gamma X_t dt + \sigma dW_t$ and $\gamma > 0$. This corresponds to the problem in the notes, with $-\gamma$ for γ. The system in the notes would be stable without noise. This system would be unstable even without noise. The algebra for the two cases is similar, but the qualitative behavior is different. Show that $f(x,s) \to 0$ as $s \to \infty$ exponentially fast. Interpret this by thinking about how likely it is for X_T to be close to zero in the unstable system for large T.

   (c) Consider the stable Ornstein Uhlenbeck problem $dX_t = -\gamma X_t dt + \sigma dW_t$. Suppose $X_0 = 0$. Use Ito's lemma for general stochastic processes to find the equation for $v(t) = \mathrm{E}\left[X_t^2\right]$. Show that the limiting variance exists: $v(t) \to v_\infty$ as $t \to \infty$. Find an explicit formula for v_∞. Hint: calculate $dv(t) = d\,\mathrm{E}\left[X_t^2\right] = \mathrm{E}\left[d\left(X_t\right)^2\right]$.

   (d) Suppose $X \sim \mathcal{N}(0, v_\infty)$. Calculate $\mathrm{E}\left[e^{-x^2/2}\right]$. When you are done, substitute the formula for v_∞ and get the expectation as a function of γ and σ. Show that this is the limit as $T \to \infty$ of the solution we got from the backward equation in Week 7. Explain why this should be true.

2. (*Brownian motion with killing*) Let X_t be a Brownian motion with $dX_t = dW_t$ starting at some point X_0. Suppose $V(x) \geq 0$ is a given function, called the *local killing rate*. (These Halloween like terms are standard for this problem. It is an in-homogeneous *birth death* process, with birth left out for simplicity.) There is a *killing time* τ that is an inhomogeneous exponential:

$$\mathrm{P}(\tau \in [t, t+dt] \mid \tau \geq t) = V(X_t)dt \ . \tag{1}$$

Models like this have many applications. One example is the diffusion of neutrons in a heterogeneous material with different absorption rate in different places. Define a value function $f(x,t)$ by

$$f(x,t) = \mathrm{P}(\tau > T \mid \tau > t \text{ and } X_t = x) \ . \tag{2}$$

(a) Use the tower property to derive a PDE satisfied by f. One way to do this is to compare paths that start at x at time $t = 0$ to paths that start at $x + \Delta X$ at time Δt. There is a probability $V(x)dt$ to get killed in this starting interval.

(b) What is the initial or final condition that completely determines f?

(c) Find an explicit solution to this PDE with initial/final conditions for the killing rate function $V(x) = x^2/2$. Use the ansatz method with an ansatz that is somewhat Gaussian.

(d) Which starting points have the highest survival probability according to your explicit solution? Explain why this should be true.

(e) Give a mathematical proof of the following theorem: If $V(x) \geq 0$ for all x and $V(x) = 0$ only for $x = 0$ and $V(x)$ is a continuous function of x with $V(x) \to \infty$ as $x \to \infty$, then $f(x,t) \leq Ce^{-mt}$ for some fixed positive m and any x. Hint: One way is to show that there is a $\alpha > 0$ so that $\mathrm{P}(\tau > t+1 \mid \tau > t) \leq 1 - \alpha$. The hard part is that there may be points where $V(0) = 0$, to $\alpha > 0$ depends on the probability of the particle with $X_t = 0$ wandering away from x.

3. (*Another view of Ito's lemma for diffusions*). Suppose X_t is a diffusion process with drift $\mathrm{E}[dX_t \mid \mathcal{F}_t] = a_X(X_t)dt$ and infinitesimal variance $\mathrm{E}\Big[(dX_t)^2 \mid \mathcal{F}_t\Big] = \mu_X(X_t)dt$. Suppose $Y_t = \phi(X_t)$, for some appropriate function ϕ. Assume that ϕ is one-to-one, which means there is a unique y for any x and a unique x for any y. The inverse function is $x = \phi^{-1}(y)$. You can understand the effect of this change of variables either using Ito's lemma or by doing a change of variable in the backward equation. The results should be the same.

   (a) The infinitesimal mean and variance of Y are $\mathrm{E}[dY_t \mid \mathcal{F}_t] = a_Y(Y_t)dt$, and $\mathrm{E}\Big[(dY_t)^2 \mid \mathcal{F}_t\Big] = \mu_Y(Y_t)dt$. Use Ito's lemma to find formulas for $a_Y(y)$ and $\mu_Y(y)$ in terms of $a_X(x)$ and $\mu_X(x)$ and ϕ' and ϕ''. Show that Y_t is a diffusion process.

   (b) Write the backward equation for $f(x,t) = \mathrm{E}_{x,t}[V(X_T)]$. Let $g(y,t)$ be the value function for the Y diffusion process $g(y,t) = \mathrm{E}_{y,t}\big[V(\phi^{-1}(Y_T))\big]$. Show that $g(\phi(x),t) = f(x,t)$. Write the backward equation for f in the x variable and the backward equation for g in the y variable. Use the ordinary chain rule to show that the change of variable $y = \phi(x)$ transforms the backward equation for g into the backward equation for f.

(c) Consider the case when S_t is a geometric Brownian motion with $dS_t = rS_t dt + \sigma S_2 dW_t$. Write the backward equation for $f(s,t) = \mathrm{E}_{s,t}[V(S_T)]$. Use ordinary calculus (the chain rule from ordinary multivariate calculus) to express this equation in the new variable $x = \log(s)$. Show that this is the backward equation for the diffusion process $X_t = \log(S_t)$.