Assignment 4, due October 7

Corrections: (none yet)

1. (*Brownian motion with reflection*) A *reflecting Brownian motion*, with a *reflecting barrier* at $x = a$, is a stochastic process that never crosses a and does not stick to a. For $X_t \neq a$, X_t acts like a Brownian motion. Suppose $X_0 = 0$ and $a > 0$. A reflecting Brownian motion has a probability density, $X_t \sim u(x,t)$, that satisfies the heat equation if $x < a$, and has

$$\int_{-\infty}^{a} u(x,t)\, dx = 1 \; . \tag{1}$$

   (a) The conservation formula (1) implies a boundary condition that u satisfies at $x = a$. What is this condition? Hint: What must the probability flux be at $x = a$? This boundary condition is called a *reflecting* boundary condition. For `wikipedia` lovers, it is also called a *Neumann* boundary condition.

   (b) Suppose that $v(x)$ is symmetric about the point a, which is the condition $v(a - x) = v(a + x)$ for all x. Show that if if v is a smooth function of x, then v satisfies the boundary condition from part a.

   (c) Adapt the method of images from this week's material to write a formula for the $u(x,t)$ that satisfies the correct initial condition for $X_0 = 0$ and boundary condition at $a > 0$. It is closely related to the formula from class.

   (d) Write a formula for $m_t = \mathrm{E}[X_t]$ for reflecting Brownian motion. The *cumulative normal* distribution is $N(z) = \mathrm{P}(Z \leq z)$, when $Z \sim \mathcal{N}(0,1)$. Derive a formula for m_t in terms of this and other explicit functions. Verify that m_t is exponentially small for small t. Verify that $m_t \to -\infty$ as $t \to \infty$ and scales as $t^{1/2}$.

   (e) It is argued (possibly later in this course, or the book *Stochastic Integrals* by Henry McKean) that a reflecting Brownian motion is kept inside the allowed region $\{x \leq a\}$ by a rightward force at the reflecting boundary. This force is different from zero only when $X_t = a$. The force is just strong enough to prevent $X_t > a$. This picture suggests that the total force is proportional to the total time spent at the boundary. Since only the boundary force has a preferred direction, if $X_0 = 0$, it may be that

$$\mathrm{E}[X_t] = \mathrm{E}\left[\int_0^t F_s ds\right] \; ,$$

both sides being negative. Since the force only acts when $X_t = a$, it may be plausible that $\mathrm{E}[F_s] = -C\,u(a,s)$. Verify that this picture is true, at least as far as the formula

$$m_t = -C \int_0^t u(x,t)\,dt \; .$$

Find $C > 0$.

2. (*Kolmogorov reflection principle*) Let X_n be a discrete time *symmetric* random walk on the integers, positive and negative. The random walk is symmetric if $\mathrm{P}(x \to x+1) = \mathrm{P}(x \to x-1)$, and it has some probability not to move: $\mathrm{P}(x \to x) = 1 = 2\,\mathrm{P}(x \to x-1) > 0$. Suppose the walk starts with $X_0 = 0$. Let $H_a(t) = \mathrm{P}(X_n = a \text{ for some } n \leq t)$ be the hitting probability for this discrete process. Show that if $a > 0$, then

$$\mathrm{P}(H_a(t)) = \mathrm{P}(X_t = a) + 2\mathrm{P}(X_t > a) \; . \tag{2}$$

Hint: The discrete time version of the argument from class is rigorous.

3. (*backward equation*) Let X_t be a standard Brownian motion starting from $X_0 = 0$. Let $\tau = \min\{t \text{ so that } |X_t| = 1\}$. Find the expected hitting time $\mathrm{E}[\tau]$. Hint:

   (a) Suppose $V(x,t)$ is a running time reward function and the total reward starting from x at time t is

   $$\int_t^\tau V(X_s,s)\,ds \; .$$

   There the process starts with $X_t = s$, and τ is the first hitting time after t, and $|x| \leq 1$. Define the value function for this to be

   $$f(x,t) = \mathrm{E}_{x,t}\left[\int_t^\tau V(X_s,s)\,ds\right] \; .$$

   Figure out the PDE that f satisfies.

   (b) The case $V(x,t) = 1$ gives the expected hitting time. There is a subtlety here that we need to show $\mathrm{E}[\tau] < \infty$. The assignment for a future week will show that there is a $x > 0$ so that $\mathrm{P}_{x,0}(\tau > t) \leq e^{-ct}$. You may assume that $\mathrm{E}[\tau] < \infty$ for this exercise.

4. (*ill posedness*) This exercise shows that it is impossible to run the forward equation backwards or the backward equation forwards. These are more theoretical than most exercises.

   (a) Show that if $g(x)$ and $w(x)$ are two functions with $|g(x)\| \leq M$ for all x, and

   $$\int_{-\infty}^{\infty} |w(x)|\;dx = 1 \; ,$$

then

$$\int_{-\infty}^{\infty} g(x)w(x)\,dx \le M\ .$$

(b) Show that if $u(x,t)$ satisfies the heat equation $\partial_t u = \frac{1}{2}\partial_x^2 u$, with $u(x,0)$ being a probability density, then

$$|\partial_x u(x,t)| \le \frac{\sqrt{e}}{t}\ .$$

for all $t > 0$ and all x. Hint: use (3) from Week 4 notes; differentiate under the integral sign; use the similarity variable $z = x/\sqrt{t}$ and see that $\partial_x \left[t^{1/2} e^{-x^2/(2t)} \right] = -t^{-1} z e^{-z^2/2}$.

(c) Show that if $w(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and $t > 2$ (say), then there is no probability density $u(x,0)$ so that $u(x,t) = w(x)$. Conclude that it is not possible to "run the heat equation backwards". That "problem" is not *well posed*.

5. (*Computing*) Download the file `coding.pdf`. It contains guidelines for coding. Please follow them from now on. There are some links on the `resources` page of the course web site that have more material on programming practice. If you watch a good programmer, you will see her or him always following a set of programming guidelines. Ultimately they will save you time in the computing assignments.

   The material for this week contains the PDF

$$M_t = \max_{0 \le s \le t} X_s$$

   and a formula for

$$S_{t,a}(x)dx = \mathrm{P}(x \le X_t \le x + dx \mid X_s < a \text{ for } 0 \le s \le t)$$

   You made histograms of these distributions last week. This week, put the exact formulas on the graphs to see whether they agree. Play with parameters to see how good a fit you can get in a reasonable amount of computer time.